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FIXED POINTS
FOR NON-EXPANSIVE SET-VALUED MAPPINGS

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Let E be a Banach space and $F : E \rightrightarrows E$ be a 1-Lipschitz set-valued mapping with closed convex non-empty values. We study the set of fixed points $\mathrm{Fix}(F) = \{x \in E : x \in F(x)\}$ and provide in any space E with $\dim(E) \geq 2$ an example of such a mapping F such that $\mathrm{Fix}(F)$ is not connected.

1. Introduction

In this paper we are concerned with set-valued mappings from a Banach space E into itself having closed convex values. We will consider only mappings F which are 1-Lipschitz for the Hausdorff distance d_H on the set $\mathscr{F}(E)$ of non-empty closed subsets of E. Recall that, for A and B in $\mathscr{F}(E)$:

$$d_H(A,B) = \max\big(\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\big)$$

so $d(z,F(y)) \leq \|x-y\|$ for all $x,y \in E$ and $z \in F(x)$.

For such a mapping F we will be essentially interested in the set $\mathrm{Fix}(F) = \{x \in E : x \in F(x)\}$ of fixed points of F which is clearly closed in E. Of course it can happen that $\mathrm{Fix}(F) = \emptyset$, for example if $F(x) = \{x+a\}$ where a is a fixed non-zero vector in E.

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The case of multivalued contraction mappings, i.e. the case where F is q-Lipschitz for a $q < 1$ was extensively studied for long (see [2], [1], [3], [4]) and many properties of structure or conservation for the set $\mathrm{Fix}(F)$ of fixed points were shown. For example:

i. $\mathrm{Fix}(F) \neq \emptyset$ (even for non convex-valued mappings). ([2], [1])

ii. $\mathrm{Fix}(F)$ is an absolute retract, in particular it is path-connected. ([3])

iii. $\mathrm{Fix}(F)$ is not a singleton if all values $F(x)$ have several points ([4] for the case where $q < \frac{1}{2}$ or E is a Hilbert space, [5] for the general case)

iv. $\mathrm{Fix}(F)$ is bounded if so are all values $F(x)$ for $x \in E$ (or even for only one $x \in E$). ([5])

v. $\mathrm{Fix}(F)$ is compact if so are all values $F(x)$ for $x \in E$.([5])

We shall show in this paper that most of these results disappear when the Lipschitz constant q of F (which is < 1 if F is a contraction mapping) is only assumed to be ≤ 1 and $\dim(E) \geq 2$.

In the sequel we will call *quasi-contraction* any 1-Lipschitz set-valued mapping from E to E with (non-empty) closed convex values.

Clearly properties (iv) and (v) become false already in the trivial example where $E = \mathbb{R}$ and $F(x) = \{x\}$ since $F(x)$ is then always single-valued, and a fortiori compact and bounded though $\mathrm{Fix}(F) = \mathbb{R}$ is unbounded. We provide in section 4 an example of quasi-contraction in a Hilbert space for which property (iii) does not hold. Concerning property (ii) and namely the connectedness of the set of fixed points of a quasi-contraction, the main part of this paper consists in proving that it does not hold in general.

After studying in section 2 the very simple case where E has dimension 1, we look in section 3 at the case where $F(x)$ is single-valued. It turns out that if E is finite-dimensional we can prove that $\mathrm{Fix}(F)$ is connected but that this is no more true for infinite-dimensional spaces.

The remainder of the paper is devoted to show that in every normed space of dimension at least 2 one can construct a quasi-contraction having a non-connected set of fixed points. In section 5 we provide such a construction for the 2-dimensional euclidean space, and generalize it to every 2-dimensional smooth normed space in section 7. The general case is dealt in sections 6 and 7.

For any two points a and b in a normed space E we will denote by $[a,b] \subset E$ the segment with endpoints a and b, it is the set $\{(1-t)a + tb : t \in [0,1]\}$. The following simple lemma will be of constant use troughout the paper.

Lemma 1.1. *Let E be a normed space, a, b, a', b' be points of E. Then*

$$d_H([a,b],[a',b']) \leq \max(\|a-a'\|,\|b-b'\|)\,.$$

Proof. If $w \in [a,b]$ we have $w = ta+(1-t)b$ for some $t \in [0,1]$ hence

$$\begin{aligned} d(w,[a',b']) &\leq d\big(ta+(1-t)b, ta'+(1-t)b'\big) \\ &= \|t\big((a-a'\big)+(1-t)\big(b-b'\big)\| \\ &\leq t\|a-a'\|+(1-t)\|b-b'\| \\ &\leq \max(\|a-a'\|,\|b-b'\|) \end{aligned}$$

whence it follows that $\sup_{w\in[a,b]} d(w,[a',b']) \leq \max(\|a-a'\|,\|b-b'\|)$, hence that $d_H([a,b],[a',b']) \leq \max(\|a-a'\|,\|b-b'\|)$. □

2. The case of dimension 1

Proposition 2.1. *Let F be a quasi-contraction from $\mathbb{R}$ to $\mathbb{R}$ (the values of F are closed intervals). Then* $\mathrm{Fix}(F)$ *is either empty or a closed interval. In particular* $\mathrm{Fix}(F)$ *is connected.*

Proof. Since F is a quasi-contraction, it is easy to see that there are two mappings a and b from $\mathbb{R}$ to $\bar{\mathbb{R}} = \mathbb{R}\cup\{-\infty,+\infty\}$ such that $F(x) = \mathbb{R}\cap[a(x),b(x)]$ where $a(x) = -\infty$ for all x or $a(x) > -\infty$ for all x (and $b(x) = +\infty$ for all x or $b(x) < +\infty$ for all x). If $-\infty < a(x) \leq a(y)$ we have $d(a(x),F(y)) = |a(y)-a(x)|$ hence $d_H(F(x),F(y)) \geq |a(y)-a(x)|$ and similarly $d_H(F(x),F(y)) \geq |b(y)-b(x)|$ if $b(x) < +\infty$. So by Lemma 1.1

$$d_H(F(x),F(y)) = \max(|a(x)-a(y)|,|b(x)-b(y)|)$$

whence it follows that both a and b are 1-Lipschitz or constantly infinite.

Suppose towards a contradiction that there exist $x_0, x_0' \in \mathrm{Fix}(F)$, $x_0' < x < x_0$ and $x \notin \mathrm{Fix}(F)$. Then we have $b(x) = b(x_0) = +\infty$ or $b(x_0) \geq x_0$, hence

$$b(x) \geq b(x_0) - |x-x_0| \geq x_0-(x_0-x) = x$$

Thus since $x \notin \mathrm{Fix}(F)$ we necessarily get $a(x) > x$, so $a(x) > -\infty$, and since $x_0' < x$,

$$a(x_0') \geq a(x) - |x-x_0'| > x-(x-x_0') = x_0'$$

hence $x_0' \notin F(x_0)$, a contradiction. This shows that $\mathrm{Fix}(F)$ is an interval. □

3. The case of functions

In this section we consider single-valued quasi-contractions F, which we identify with 1-Lipschitz functions f by $F(x) = \{f(x)\}$. More generally we will study the case where H is a closed convex subset of the Banach space E and $f : H \to H$ is 1-Lipschitz.

Proposition 3.1. *Let E be a strictly convex normed space, $H \subset E$ be closed and convex, $f : H \to H$ be 1-Lipschitz. Then* $\mathrm{Fix}(F)$ *is convex, possibly empty.*

Proof. Clearly, if $H = E$, $a \in E$ is not zero and f is the translation $x \mapsto x + a$, f is an isometry and $\mathrm{Fix}(f) = \emptyset$.

If $f : H \to H$ is 1-Lipschitz and u, v are two distinct points of $\mathrm{Fix}(f)$ then for all $t \in]0,1[$ the points $x_t = tu + (1-t)v$ and $y_t = f(x_t)$ satisfy

$$\|y_t - u\| = \|f(x_t) - f(u)\| \le \|x_t - u\| = (1-t)\|u - v\|$$
$$\|y_t - v\| = \|f(x_t) - f(v)\| \le \|x_t - v\| = t\|u - v\|$$

thus $\|u - v\| \le \|y_t - u\| + \|y_t - v\| \le \|u - v\|$, whence $y_t \in [u, v]$ because E is strictly convex, and $y_t - v = s(u - v)$ for some $s \in [0,1]$. Then since $s\|u - v\| = \|y_t - v\| = t\|u - v\|$, we conclude that $s = t$ and $y_t = x_t$, hence that $x_t \in \mathrm{Fix}(F)$. □

If E is not strictly convex, the previous result does not hold any more. For example if $E = \mathbb{R}^2$ equipped with the norm $u = (x, y) \mapsto \|u\|_\infty = \max(|x|, |y|)$, the function $f : (x, y) \mapsto (x, \sin x)$ is 1-Lipschitz: indeed

$$\|f(x,y) - f(x',y')\| = \max(|x - x'|, |\sin x - \sin x'|) = |x - x'| \le \|(x,y) - (x',y')\|$$

and $\mathrm{Fix}(f) = \{(x, \sin x) : x \in \mathbb{R}\}$ is connected, but not convex. So far it is unclear, even in a finite-dimensional space, whether a 1-Lipschitz mapping could have a non-connected set of fixed points. Nevertheless we shall see later on, in Theorems 3.3, 3.4 and Corollary 7.7 what really happens.

Lemma 3.2. *Let H be a non-empty convex compact subset of a finite-dimensional space E and $f : H \to H$ be a 1-Lipschitz function. Then the set* $\mathrm{Fix}(f)$ *is compact connected and non-empty.*

Proof. It follows readily from Brouwer's theorem that $\mathrm{Fix}(f)$ is non-empty. And it is closed in H hence compact. For the connectedness we proceed by induction on the dimension of E, or more precisely on the dimension $\delta(H)$ of the linear subspace A_H generated by $H - H$. If $\delta(H) = 1$ then $A_H \approx \mathbb{R}$ is strictly convex and it follows from Proposition 3.1 that $\mathrm{Fix}(f)$ is convex, hence connected.

Assume that the statement of the lemma holds for all compact convex K such that $\delta(K) < n$, that H is a compact convex subset of E such that $\delta(H) = n$ and that a_0 and a_1 are two distinct fixed points of the 1-Lipschitz function $f : H \to H$. By translation invariance we can and do assume that $0 \in H$ and H spans E. Choose by Hahn-Banach's theorem a linear functional $\xi \in E^*$ of norm 1 such that $\langle \xi, a_1 - a_0 \rangle = \|a_1 - a_0\|$.

For all $t \in [0,1]$ consider the set

$$H_t = \{x \in H : \|x - a_0\| \le t\|a_1 - a_0\| \text{ and } \|x - a_1\| \le (1-t)\|a_1 - a_0\|\}$$

which is convex and compact. Then $f(H_t) \subset H_t$: indeed if $x \in H_t$, $y = f(x) \in H$ and

$$\begin{cases} \|y - a_0\| = \|f(x) - f(a_0)\| \le \|x - a_0\| \le t\|a_1 - a_0\| \\ \|y - a_1\| = \|f(x) - f(a_1)\| \le \|x - a_1\| \le (1-t)\|a_1 - a_0\|. \end{cases}$$

Moreover for $x \in H_t$,

$$\begin{cases} \langle \xi, x - a_0 \rangle \le \|\xi\|.\|x - a_0\| \le t\|a_1 - a_0\| \\ \langle \xi, a_1 - x \rangle \le \|\xi\|.\|x - a_1\| \le (1-t)\|a_1 - a_0\| \end{cases}$$

hence

$$\begin{aligned} 0 &= \langle \xi, a_1 - a_0 \rangle - \|a_1 - a_0\| = \langle \xi, x - a_0 \rangle + \langle \xi, a_1 - x \rangle - \|a_1 - a_0\| \\ &= \big(\langle \xi, x - a_0 \rangle - t\|a_1 - a_0\|\big) + \big(\langle \xi, a_1 - x \rangle - (1-t)\|a_1 - a_0\|\big) \end{aligned}$$

and

$$0 \le t\|a_1 - a_0\| - \langle \xi, x - a_0 \rangle = \big(\langle \xi, a_1 - x \rangle - (1-t)\|a_1 - a_0\|\big) \le 0$$

from what we deduce that

$$\langle \xi, x - a_0 \rangle = t\|a_1 - a_0\| \text{ and } \langle \xi, a_1 - x \rangle = (1-t))\|a_1 - a_0\|,$$

hence that $\langle \xi, x \rangle = \theta := \langle \xi, a_0 \rangle + t\|a_1 - a_0\|$. Thus this shows that H_t is included in the affine hyperplane $V_\theta = \{x \in E : \langle \xi, x \rangle = \theta\}$ for which $\delta(V_\theta) < \dim(E) = n$. It follows then from the induction hypothesis that $H_t \cap \mathrm{Fix}(f) = \mathrm{Fix}(f_{|H_t})$ is compact connected and non-empty.

Assume that $\mathrm{Fix}(f)$ is not connected. So it would exist two disjoint compact subsets A_0 and A_1 of H such that $\mathrm{Fix}(f) \subset A_0 \cup A_1$ and two points a_0 and a_1 with $a_i \in A_i$. For $t \in [0,1]$, let H_t be the set intoduced above which corresponds to the points a_0, a_1. Then, from what precedes, for all $t \in [0,1]$, $\mathrm{Fix}(f_{|H_t}) \subset A_0 \cup A_1$ ($i \in \{0,1\}$) and, by connectedness of $\mathrm{Fix}(f_{|H_t})$, $\mathrm{Fix}(f_{|H_t}) \subset A_i$ for some i. Then the sets $T_i = \{t \in [0,1] : \mathrm{Fix}(f_{|H_t}) \subset A_i\}$ form a partition of $[0,1]$. Moreover, note that $0 \in T_0$ and $1 \in T_1$. Thus, if we prove that T_0 and T_1 are also closed, we will get a contradiction which will complete the proof of the connectedness of $\mathrm{Fix}(f)$.

Let (t_n) be a sequence in T_0 which converges to t^*; there exists for all n a point $x_n \in A_0 \cap \mathrm{Fix}(f_{H_{t_n}})$. Since $H \cap B(a_0, \|a_1 - a_0\|)$ is compact and $H_{t_n} \subset H \cap B(a_0, \|a_1 - a_0\|)$, up to passing to a subsequence we can assume that (x_n) converges to some point $x^* \in A_0$. We have

$$\|x^* - a_0\| = \lim_n \|x_n - a_0\| \leq \limsup_n t_n \|a_1 - a_0\| = t^* \|a_1 - a_0\|$$

and similarly $\|x^* - a_1\| \leq (1-t)\|a_1 - a_0\|$.

Thus $x^* \in H_{t^*}$. Moreover $\|f(x^*) - x^*\| = \lim \|f(x_n) - x_n\| = 0$. It follows that $x^* \in \mathrm{Fix}(f_{H_{t^*}}) \cap A_0$, hence that $\mathrm{Fix}(f_{H_{t^*}}) \cap A_0 \neq \emptyset$ and that $t^* \in T_0$. By the same argument one can show that T_1 is closed. This completes the proof of the connectedness of $\mathrm{Fix}(f)$, hence this of Lemma 3.2. □

Theorem 3.3. *Let E be a normed finite-dimensional space, $H \subset E$ be a closed convex subset and $f : H \to H$ be a 1-Lipschitz function. Then the set of fixed points of f is connected.*

Proof. It is enough to consider the case where $\mathrm{Fix}(f)$ is non-empty. Then let $a \in \mathrm{Fix}(f)$. For all integer $n \geq 1$ the set $H_n = \{x \in H : \|x - a\| \leq n\}$ is compact convex non-empty and stable under f. So it follows from Lemma 3.2 that $\mathrm{Fix}(f_{|H_n}) = \mathrm{Fix}(f) \cap H_n$ is connected and contains a. Then $\bigcup_n \mathrm{Fix}(f_{H_n}) = \mathrm{Fix}(f)$ is connected. □

We now show that for infinite-dimensional spaces E there is no particular topological property of the sets $\mathrm{Fix}(f)$ for 1-Lipschitz functions $f : E \to E$. Indeed :

Theorem 3.4. *Let X be a complete metric space. Then there exist a Banach space E and a 1-Lipschitz function $f : E \to E$ such that* $\mathrm{Fix}(f)$ *is isometric to X. Moreover is X is separable the space E can be chosen separable.*

Proof. Remark first that since $\mathrm{Fix}(f)$ is closed in E hence complete, the completeness of X is a necessary condition.

It is well-known that any metric space X can be isometrically embedded into a Banach space. For example if $a \in X$ and D is a dense subset of X the function $\psi : x \mapsto \Big(d(x,y) - d(a,y)\Big)_{y \in D}$ is an isometry from X to a subset of the space ℓ^∞_D. And if X is separable, the space $\overline{\mathrm{span}}(\psi(X))$ is a separable Banach space.

Recall that c_0 denotes the Banach space of all real sequences converging to 0 equipped with the norm : $u = (x_n) \mapsto \|u\| = \sup_n |x_n|$ and denote $\underline{0}$ the null sequence in c_0. Choose an isometric embedding $j : X \to W$ for some Banach space W and define $H = j(X) \subset W$ and $E = W \times c_0$ equipped with the norm $(w,u) \mapsto \max(\|w\|, \|u\|)$.

For $w \in W$ and $u \in c_0$ define $f(w,u) = (w,v)$ where $v = (v_n) \in c_0$ is defined by

$$v_n = \begin{cases} d(w,H) & \text{if } n = 0 \\ u_{n-1} & \text{if } n > 0 \end{cases}$$

Claim 3.5. The function f is 1-Lipschitz and even an isometry.

Proof. We have

$$\begin{aligned} \|f(w,u) - f(w',u')\| &= \max\big(\|w-w'\|, |d(w,H) - d(w',H)|, \sup_{n\geq 1} |u_{n-1} - u'_{n-1}|\big) \\ &= \max\big(\|w-w'\|, |d(w,H) - d(w',H)|, \|u-u'\|\big) \\ &= \max\big(\|w-w'\|, \|u-u'\|\big) = \|(w,u) - (w',u')\| \end{aligned}$$

since $|d(w,H) - d(w',H)| \leq \|w - w'\|$. ◇

Claim 3.6. $\mathrm{Fix}(f) = H \times \{\underline{0}\}$.

Proof. It is clear that if $w \in H$ then $f(w,\underline{0}) = (w,\underline{0})$.

Conversely, if $u = (x_n)$ and (w,u) is a fixed point of f we have $x_0 = d(w,H)$ and $x_n = x_{n-1}$ for all $n \geq 1$. Thus u is the constant sequence with value $d(w,H)$, which does not belong to c_0 if $d(w,H) \neq 0$. So $w \in H$ and $u = \underline{0}$. ◇

It follows from previous claim that the function $x \mapsto (j(x), \underline{0})$ is an isometry from X onto $\mathrm{Fix}(f)$. □

4. Uniqueness of fixed points

We provide in this section an example of a quasi-contraction F on a Hilbert space H such that $F(x)$ is a singleton for no $x \in H$ but $\mathrm{Fix}(F)$ is a singleton. And this shows that Property (*iii*) in the Introduction does not hold in general for quasi-contractions.

Theorem 4.1. *There exists a quasi-contraction F on a Hilbert space H such that* $\mathrm{diam}(F(x)) = 1$ *for all $x \in H$ but* $\mathrm{Fix}(F)$ *is a singleton.*

Proof. Let H be the Hilbert space ℓ^2, $S : H \to H$ be the isometric mapping defined by $x = (x_n)_{n\geq 0} \mapsto y = (y_n)_{n\geq 0}$ where $y_0 = 0$ and $y_n = x_{n-1}$ for $n > 0$. Let $u = (u_n) \in H$ be the unit vector such that $u_0 = 1$ and $u_n = 0$ for $n > 0$, and $\underline{0}$ be the null vector of H.

For $x \in H$ define $F(x)$ as the segment $[S(x), S(x)+u]$ whose diameter is 1. We claim that F is a quasi-contraction. Indeed by Lemma 1.1:

$$\begin{aligned} d_H(F(y),F(x)) &\le \max\big(\|S(y)-S(x)\|, \|(S(y)+u)-(S(x)+u)\|\big) \\ &= \|S(y)-S(x)\| = \|S(y-x)\| = \|y-x\|. \end{aligned}$$

If $x^* = (x_n^*)$ is a fixed point of F there exists some $t \in [0,1]$ such that

$$x^* = (1-t)S(x^*) + t(S(x^*)+u) = S(x^*) + tu$$

so $x_0^* = tu_0 = t$ and $x_n^* = x_{n-1}^*$ for $n > 0$. This implies that the sequence $(x_n) \in \ell^2$ has to be constant with the value t, which is possible only with $t = 0$, and $x^* = \underline{0}$. And this shows that $\mathrm{Fix}(F) = \{\underline{0}\}$ is a singleton. □

5. The 2-dimensional euclidean space

The aim of this section is to construct a quasi-contraction on the 2-dimensional euclidean space $\mathbb{R}^2$ whose set of fixed points is not connected. It follows from Proposition 2.1 that such a construction cannot be achieved in a 1-dimensional space, and from Proposition 3.1 that it is impossible with a single-valued quasi-contraction.

Consider the points $x_0 = (-1,0)$ and $x_1 = (1,0)$ of the euclidean space $\mathbb{R}^2$ and the symmetry $S : (u,v) \mapsto (-u,v)$ of $\mathbb{R}^2$ exchanging x_0 and x_1. We want to define two 1-Lipschitz mappings α and β from $\mathbb{R}^2$ to itself such that $\alpha(x_0) = x_0$ and $S\circ\beta(z) = \alpha\circ S(z)$ for all z. In particular this implies $\beta(x_1) = S\circ\alpha(x_0) = S(x_0) = x_1$. We define then the set-valued mapping F by $F(z) = [\alpha(z), \beta(z)]$, which is clearly convex and closed.

Lemma 5.1. *If the mapping α is 1-Lipschitz then F is a quasi-contraction.*

Proof. Since S is an isometry it is clear that $\beta = S\circ\alpha\circ S$ is 1-Lipschitz too. Then by Lemma 1.1, if z and z' are in $\mathbb{R}^2$

$$d_H(F(z), F(z')) \le \max(\|\alpha(z) - \alpha(z')\|, \|\beta(z) - \beta(z')\|) \le \|z - z'\|,$$

the wanted inequality. □

Fix $\varepsilon \in]0,1]$ and define the function $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = \begin{cases} \varepsilon & \text{if } t \le 0 \\ t + \varepsilon \mathrm{e}^{-t} & \text{if } t \ge 0 \end{cases}$$

Claim 5.2. The function φ is 1-Lipschitz and has no fixed point on $\mathbb{R}$.

Proof. It is immediately checked that φ is continuous, derivable on $\mathbb{R}\setminus\{0\}$, that $\varphi'(t)=0$ for $t<0$ and $\varphi'(t)=1-\varepsilon e^{-t}\geq 0$ if $t>0$. Since $|\varphi'(t)|\leq 1$ for $t\neq 0$, the function φ is 1-Lipschitz. It is non-decreasing with values in $[\varepsilon,+\infty[$ and if t^* were a fixed point of φ we would have $t^*=\varphi(t^*)\geq\varepsilon>0$ and $t^*=\varphi(t^*)=t^*+\varepsilon e^{-t^*}$ hence $e^{-t^*}=0$, that is impossible. ◇

Define now the function α on $X=\{x_0,x_1\}\cup(\{0\}\times\mathbb{R})$ by

$$\begin{cases} \alpha(x_0)=x_0 \\ \alpha(x_1)=(0,\varepsilon) \\ \alpha(0,v)=\left(-\frac{1}{2},\varphi(v)\right) \end{cases}$$

Lemma 5.3. *It is possible to choose $\varepsilon\in]0,1]$ such that this function α be 1-Lipschitz on X.*

Proof. For $v\in\mathbb{R}$ denote $y_v=(0,v)\in X$. We have to prove that for some convenient $\varepsilon>0$:

i. $\|\alpha(x_0)-\alpha(x_1)\|^2\leq\|x_0-x_1\|^2=4$,

ii. $\forall v\in\mathbb{R}$, $\|\alpha(x_0)-\alpha(y_v)\|^2\leq\|x_0-y_v\|^2$,

iii. $\forall v\in\mathbb{R}$, $\|\alpha(x_1)-\alpha(y_v)\|^2\leq\|x_1-y_v\|^2$,

iv. $\forall v,w\in\mathbb{R}$, $\|\alpha(y_v)-\alpha(y_w)\|\leq\|y_v-y_w\|$.

For (i) we must have $1+\varepsilon^2\leq 4$, that is true since $\varepsilon\leq 1<\sqrt{3}$.

For (ii) we must have

$$\left(-1+\frac{1}{2}\right)^2+\varphi(v)^2\leq 1+v^2$$

it is $\frac{1}{4}+\varphi(v)^2\leq 1+v^2$. And since $\varphi(v)^2=\varepsilon^2$ if $v\leq 0$ and if $v\geq 0$:

$$\begin{aligned}\varphi(v)^2&=(v+\varepsilon e^{-v})^2=v^2+\varepsilon^2e^{-2v}+2\varepsilon ve^{-v}\leq v^2+\varepsilon^2+2\varepsilon\sup_{t\geq 0}te^{-t}\\ &=v^2+\varepsilon^2+2e^{-1}\varepsilon\leq v^2+\varepsilon^2+\varepsilon\end{aligned}$$

we must have $\frac{1}{4}+v^2+\varepsilon^2+\varepsilon\leq 1+v^2$, that holds as soon as $\varepsilon^2+\varepsilon\leq\frac{3}{4}$, hence whenever $0<\varepsilon\leq\frac{1}{2}$.

For (iii) we must have $\left(\frac{1}{2}\right)^2 + (\varphi(v)-\varepsilon)^2 \le 1+v^2$. And since $\varphi(v) \ge \varepsilon$ we have $(\varphi(v)-\varepsilon)^2 \le \varphi(v)^2$. We have seen that if ε is chosen in $]0,\frac{1}{2}]$ then for all v: $\frac{1}{4}+\varphi(v)^2 \le 1+v^2$, so a fortiori $\frac{1}{4}+(\varphi(v)-\varepsilon)^2 \le 1+v^2$.

Finally for (iv), we have to show that

$$\|\alpha(y_v)-\alpha(y_w)\| = |\varphi(v)-\varphi(w)| \le \|y_v - y_w\| = |v-w|$$

but this follows immediately from Claim 5.2.

Taking $\varepsilon = \frac{1}{2}$ completes the proof of Lemma 5.3. □

Using Kirszbraun-Valentine's Theorem, we can extend the function α into a 1-Lipschitz function (still denoted by α) from $\mathbb{R}^2$ to $\mathbb{R}^2$, and then define $\beta = S\circ\alpha\circ S$, wich is 1-Lipschitz too.

Theorem 5.4. *The set-valued mapping* $F : z \mapsto [\alpha(z),\beta(z)]$ *is 1-Lipschitz, but the set* $\mathrm{Fix}(F)$ *of its fixed points is not connected.*

Proof. That F be 1-Lipschitz follows from Lemma 5.1. Since $x_0 = \alpha(x_0) \in F(x_0)$ we have $x_0 \in \mathrm{Fix}(F)$ and since $x_1 = \beta(x_1) \in F(x_1)$ we have $x_1 \in \mathrm{Fix}(F)$. Hence $\{x_0,x_1\} \subset \mathrm{Fix}(F)$.

We now show that $(\{0\}\times\mathbb{R}) \cap \mathrm{Fix}(F) = \emptyset$. Indeed if there were some $y_v = (0,v)$ in $\mathrm{Fix}(F)$ we should have $y_v \in \mathrm{conv}(\alpha(y_v),\beta(y_v))$. Since $\alpha(y_v) = \left(-\frac{1}{2},\varphi(v)\right)$ and $\beta(y_v) = \left(\frac{1}{2},\varphi(v)\right)$ we would get

$$(0,v) = y_v \in \mathrm{conv}(\alpha(y_v),\beta(y_v)) = [-\frac{1}{2},\frac{1}{2}] \times \{\varphi(v)\}$$

hence $\varphi(v) = v$, in contradiction with Claim 5.2.

It follows that the two disjoint open subsets $W_0 = \{(x,y) \in \mathrm{Fix}(F) : x < 0\}$ and $W_1 = \{(x,y) \in \mathrm{Fix}(F) : x > 0\}$ of $\mathrm{Fix}(F)$ are both non-empty and cover $\mathrm{Fix}(F)$. Thus $\mathrm{Fix}(F)$ is not connected. □

6. The non-smooth case

It is also possible to give a simple example in any normed space E whose dual space E^* is not strictly convex (in particular if the norm of E itself is not smooth) of a quasi-contraction whose set of fixed points is not connected.

It E^* is not strictly convex there are two non-zero vectors u and v of E^* such that $\|u\| = \|u+v\| = \|u-v\| = 1$. Define then the real function h on E by

$$h(x) = \langle u,x\rangle + \sin^2(\langle v,x\rangle)$$

Lemma 6.1. *The function h is 1-Lipschitz.*

Proof. In fact h is of class $\mathscr{C}^1$ and its differential at x is $h'(x) = u + \sin(2\langle v, x\rangle).v$. The convex function $\nu : t \mapsto \|u + tv\|$ satisfies $\nu(-1) = \nu(1) = 1$ hence $\nu(t) \le 1$ for $t \in [-1, 1]$. It follows that $\|h'(x)\| \le 1$ for all x hence that h is 1-Lipschitz. $\square$

Lemma 6.2. *The set-valued mapping $P : \mathbb{R} \rightrightarrows E$ defined by $P(t) = \{y \in E : \langle u, y\rangle \ge t\}$ is 1-Lipschitz and takes closed convex non-empty values.*

Proof. It is clear that $P(x)$ is convex closed and non-empty. Notice that if $t \le t'$ then we have $P(t') \subset P(t)$, so $d_H(P(t), P(t')) = \sup_{y \in P(t)} d(y, P(t'))$. If $y \in P(t)$ and $\varepsilon > 0$ we can find some $z \in E$ with $\|z\| \le 1 + \varepsilon$ and $\langle u, z\rangle = 1$.

Then $y' = y + (t' - t)z$ satisfies $\langle u, y'\rangle = \langle u, y\rangle + (t' - t) \ge t'$, hence $y' \in P(t')$ and $\|y - y'\| \le (1 + \varepsilon)(t' - t)$. So $d(y, P(t')) \le t' - t$ and P is 1-Lipschitz. $\square$

Theorem 6.3. *Let E be a normed space. Assume that the norm on E^* is not strictly convex. Then there exists a quasi-contraction $F : E \rightrightarrows E$ with closed convex values such that* $\mathrm{Fix}(F)$ *is not connected.*

Proof. Take h and P as in previous Lemma, and define $F = P \circ h$ which is clearly 1-Lipschitz since so are P and h. If $x \in \mathrm{Fix}(F)$ we must have

$$\langle u, x\rangle \ge h(x) = \langle u, x\rangle + \sin^2(\langle v, x\rangle)$$

hence $\sin(\langle v, x\rangle) = 0$, that implies $\langle v, x\rangle = k\pi$ for some integer $k \in \mathbb{Z}$. If $a \in E$ satisfies $\langle v, a\rangle = 1$ (and such points exist since $v \neq 0$) we get

$$\mathrm{Fix}(F) = \bigcup_{k \in \mathbb{Z}} (k.a + \ker v)$$

which is the discrete union of a countable family of pairwise disjoint closed hyperplanes, hence it cannot be connected. $\square$

7. The smooth case

We now want to extend Theorem 5.4 to every normed space E of dimension 2. It follows from Theorem 6.3 that one can assume the norm of E is smooth. Recall that a basis $(e_1, e_2, \ldots, e_n)$ of a finite-dimensional normed space E is called an *Auerbach basis* of E if $\|e_j\| = 1$ for all $j = 1, 2, \ldots, n$ and moreover $\|e_j^*\| = 1$ for all $j = 1, 2, \ldots, n$ where $(e_1^*, e_2^*, \ldots, e_n^*)$ is the dual basis of E^* (what means $\langle e_j^*, e_k\rangle = \delta_j^k$).

Lemma 7.1. *If E is a 2-dimensional normed space with smooth norm, there exists an Auerbach basis (e_1, e_2) of E such that $\|e_2 + te_1\| > 1$ for all real $t \neq 0$.*

Proof. Let B be the unit ball of E. It is a well-known fact that if the determinant function $\Delta : (u,v) \mapsto u \wedge v$ attains at (x,y) its supremum on $B \times B$ then (x,y) is an Auerbach basis. It can be easily seen that the converse is not true: the canonical basis (e_1,e_2) of ℓ_2^∞ satisfies $\Delta(e_1,e_2) = 1$ though e_1+e_2 and e_2-e_1 have norm 1 and $\Delta(e_1+e_2,e_2-e_1) = 2$.

Assume that (e_1,e_2) is such an "extremal" Auerbach basis. If there is some $t \neq 0$ such that $e_2+te_1 \in B$ then we have $e_1 \wedge e_2 > 0$ and for all $s > 0$

$$((1-\frac{s}{2})e_1 - \frac{s}{t}e_2) \wedge (e_2+te_1) = (1-\frac{s}{2}+s)\, e_1 \wedge e_2 = (1+\frac{s}{2})\, e_1 \wedge e_2 > e_1 \wedge e_2$$

what shows that $z_s = (1-\frac{s}{2})e_1 - \frac{s}{t}e_2 \notin B$: indeed if not the basis

$$(z_s,e_2') = (z_s,e_2+te_1)$$

would satisfy $\Delta(z_s,e_2') > \Delta(e_1,e_2)$ with $(z_s,e_2') \in B \times B$.

For $s < 0$ we have $\|z_s\| \geq \langle e_1^*, z_s\rangle = 1 - \frac{s}{2} > 1$. It follows that $\|z_s\| \geq 1$ for all $s \in \mathbb{R}$. Denote $u^* = e_1^* - \frac{t}{2}e_2^*$. If $u \in \{v : \langle u^*,v\rangle = 1\}$ we have $\langle u^*, u-e_1\rangle = 0$ so $u = z_s$ for some $s \in \mathbb{R}$, hence $\|u\| \geq 1$. This shows that $\|u^*\| \leq 1$. Then $\|e_1^*\| = 1$, $\|u^*\| \leq 1$ and

$$1 \geq \|\lambda u^* + (1-\lambda)e_1^*\| = \|e_1^* - \lambda\frac{t}{2}e_2^*\| \geq \langle e_1^* - \lambda\frac{t}{2}e_2^*, e_1\rangle = 1$$

for $\lambda \in [0,1]$, what shows that the norm of E^* is not strictly convex, in contradiction with the hypothesis of smoothness of E. □

Lemma 7.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a continuous positive function such that $f(0) \leq 1$. Then there exists a convex non-increasing positive and 1-Lipschitz function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ satisfying $\varphi(x) \leq f(x)$ for all $x \geq 0$.*

Proof. For all $\alpha > 0$ set $\tilde{f}(\alpha) = \inf_{0\leq t\leq 2\alpha} f(t)$ which is positive by compactness of $[0,2\alpha]$. And the affine decreasing function

$$f_\alpha : x \mapsto \tilde{f}(2\alpha)(1-\frac{x}{2\alpha})$$

satisfies $f_\alpha(x) \leq 0 < f(x)$ for $x \geq 2\alpha$, $f_\alpha(x) \leq \tilde{f}(2\alpha) \leq f(x)$ for $0 \leq x \leq 2\alpha$ and $f_\alpha(\alpha) = \frac{1}{2}\tilde{f}(2\alpha) > 0$. It follows that $\varphi : x \mapsto \sup_{\alpha\geq 1/2} f_\alpha(x)$ is convex, non-increasing, everywhere positive on $[\frac{1}{2},+\infty[$ hence a fortiori on $\mathbb{R}^+$, and that $\varphi \leq f$.

Finally since the function f_α is $\frac{\tilde{f}(2\alpha)}{2\alpha}$-Lipschitz the function φ is λ-Lipschitz for $\lambda = \sup_{\alpha\geq 1/2}\frac{\tilde{f}(2\alpha)}{2\alpha} = \tilde{f}(1) \leq f(0) \leq 1$. □

Lemma 7.3. *If the basis (e_1, e_2) of E is as in Lemma 7.1 there exists a positive convex non-decreasing 1-Lipschitz function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ such that for all $x \in \mathbb{R}^+$ the inequality*

$$|x| + \varphi(|x|) \le \|e_1 + xe_2\|$$

holds true.

Proof. Consider the function $f_+ : x \mapsto \|e_1 + xe_2\| - x$ on $\mathbb{R}^+$. Since (e_1, e_2) is an Auerbach basis we have $\|e_1 + xe_2\| \ge \|xe_2\| = x$, hence $f_+(x) \ge 0$. And if we had $f_+(x) = 0$ for some $x \in \mathbb{R}^+$ we would have $x = \|e_1 + xe_2\| \ge 1$ hence $1 = \|e_2 + \frac{1}{x}e_1\| > \|e_2\| = 1$ since by hypothesis $\|e_2 + se_1\| > 1$ for all $s \ne 0$. It follows that f_+ is positive. And in the same way one sees that the function $f_- : x \mapsto \|e_1 - xe_2\| - x$ is positive on $\mathbb{R}^+$. Moreover $f = \min(f_+, f_-)$ satisfies $f(0) = 1$.

Applying then Lemma 7.2 to f we get a positive convex non-decreasing 1-Lipschitz function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ such that $|x| + \varphi(|x|) \le \|e_1 + xe_2\|$ for all $x \in \mathbb{R}$. □

Still assuming the basis (e_1, e_2) of E satisfies the condition of Lemma 7.1, we define the closed convex set H by

$$H = \{x \in E : \langle e_1^*, x\rangle \in [-1, 1] \text{ and } \langle e_2^*, x\rangle \ge 0\}$$

Claim 7.4. There exists a 1-Lipschitz retraction p from E to H.

Proof. The function $p_1 : (x, y) \mapsto \big(x, \max(y, 0)\big)$ is 1-Lipschitz: indeed if u and v belong to E, $p_1(u) - p_1(v)$ is a convex combination of the vectors $u - v$ and $\langle e_1^*, u - v\rangle e_1$ which have both a norm at most $\|u - v\|$. Thus $\|p_1(u) - p_1(v)\| \le \|u - v\|$. Moreover p_1 is the identity mapping on H and $p_1(E) \subset \mathbb{R} \times \mathbb{R}^+$.

In the same way the mapping $p_2 : (x, y) \mapsto (\max(-1, \min(1, x)), y)$ is the identity on H and is 1-Lipschitz since when u and v belong to E, $p_2(u) - p_2(v)$ is a convex combination of the vectors $u - v$ and $\langle e_2^*, u - v\rangle e_2$ which have both a norm at most $\|u - v\|$. Moreover $p_2(\mathbb{R} \times \mathbb{R}^+) \subset H$.

Then $p = p_2 \circ p_1$ is the identity on H, is 1-Lipschitz and satisfies $p(E) \subset H$, so is a 1-Lipschitz retraction on H. ◇

Theorem 7.5. *If E is a 2-dimensional normed space with smooth norm, there exists on E a quasi-contraction F such that* $\mathrm{Fix}(F)$ *is not connected.*

Proof. Choose the basis (e_1, e_2), the function φ and the set H as above. We will define two 1-Lipschitz functions α and β from H to H and set $F(x) = [\alpha(x), \beta(x)]$ which will be a quasi-contraction by Lemma 1.1.

In order to ensure $\text{Fix}(F)$ is not connected we want to have $\alpha(-e_1+te_2) = -e_1+te_2$, $\beta(e_1+te_2) = e_1+te_2$, and $te_2 \notin \text{Fix}(F)$ for all $t \geq 0$, so $\{-1,1\} \times \mathbb{R}^+ \subset \text{Fix}(F)$ and $(\{0\} \times \mathbb{R}^+) \cap \text{Fix}(F) = \emptyset$.

Let φ be as in Lemma 7.3 and define the function $\alpha : H_0 = \{-1,0\} \times \mathbb{R}^+ \to E$ by :

$$\alpha(u) = -e_1 + \lambda(u)e_2$$

where $\lambda : H_0 \to \mathbb{R}^+$ is defined by $\lambda(-e_1+te_2) = t$ and $\lambda(te_2) = t+\varphi(t)$. In particular $\alpha(H_0) \subset H$, $\alpha(-e_1) = -e_1$ and $\alpha(ye_2) = (-1, y+\varphi(y))$.

Claim 7.6. The function λ is 1-Lipschitz from H_0 to $\mathbb{R}^+$.

Proof. Denote $a_t = (-1,t)$ and $c_t = (0,t)$. We have to prove the following inequalities, for s and $t \geq 0$:

i. $|\lambda(a_s) - \lambda(a_t)| \leq \|a_s - a_t\|$,

ii. $|\lambda(c_s) - \lambda(c_t)| \leq \|c_s - c_t\|$,

iii. $|\lambda(a_s) - \lambda(c_t)| \leq \|a_s - c_t\|$.

For (i) we have $\|\lambda(a_s) - \lambda(a_t)\| = |s-t| = \|a_s - a_t\|$.

For (ii) we have $\|c_s - c_t\| = \|(s-t)e_2\| = |s-t|$ and

$$\|\lambda(c_s) - \lambda(c_t)\| = |(t+\varphi(t)) - (s+\varphi(s))| = |s+\varphi(s) - t - \varphi(t)|$$

Without loss of generality we can assume $s \leq t$; so we have $\varphi(t) \leq \varphi(s)$ and $s+\varphi(s) \leq t+\varphi(t)$ since $s \mapsto s+\varphi(s)$ is non-decreasing, so

$$|s+\varphi(s) - t - \varphi(t)| = t - s - (\varphi(s) - \varphi(t)) \leq t - s = |s-t| = \|c_s - c_t\|$$

Finally for (iii) we have

$$\begin{aligned} |\lambda(a_s) - \lambda(c_t)| &= |s - (t+\varphi(t))| = |s - \varphi(t) - t| \\ &\leq |s-t| + \varphi(t) \leq |s-t| + \varphi(|s-t|) \end{aligned}$$

since $t \geq |s-t|$ and by Lemma 7.3 applied to $x = t-s$:

$$\|a_s - c_t\| = \|-e_1 + se_2 - te_2\| = \|e_1 + (t-s)e_2\| \geq |s-t| + \varphi(|s-t|)$$

hence $|\lambda(a_s) - \lambda(c_t)| \leq |s-t| + \varphi(|s-t|) \leq \|a_s - c_t\|$. ◇

It is well-known that any 1-Lipschitz function λ defined on a subset H_0 of the metric space H can be extended into a 1-Lipschitz function $\tilde{\lambda}$ on H by the formula

$$\tilde{\lambda}(x) = \inf_{y \in H_0} \Big(\lambda(y) + d(x,y) \Big)$$

which yields a non-negative function $\tilde{\lambda}$ if $\lambda \geq 0$. Define then the function α on H by $\alpha(u) = -e_1 + \tilde{\lambda}(u)e_2$. We clearly have for u and v in H :

$$\|a(u) - a(v)\| = \|\big(\tilde{\lambda}(u) - \tilde{\lambda}(v)\big)e_2\| = |\tilde{\lambda}(u) - \tilde{\lambda}(v)| \leq \|u - v\|$$

Replacing $-e_1$ by e_1 we can define in the same way a 1-Lipschitz function $\beta : H \to H$ such that $\beta(e_1 + te_2) = e_1 + te_2$ and $\beta(te_2) = e_1 + (t + \varphi(t))e_2$. Then the set-valued mapping F defined on H by $F(u) = [\alpha(u), \beta(u)]$ takes closed convex values. It is 1-Lipschitz because by Lemma 1.1:

$$d_H(F(u), F(v)) \leq \max(\|\alpha(u) - \alpha(v)\|, \|\beta(u) - \beta(v)\|) \leq \|u - v\|.$$

By definition we have $\alpha(-1,t) = (-1,t)$ and $\beta(1,t) = (1,t)$; so all points of $\{-1,1\} \times \mathbb{R}^+$ are fixed points for F.

Conversely for $t \geq 0$ we have $F(0,t) = [-1,1] \times \{t + \varphi(t)\}$ and this shows that $(0,t) \notin \mathrm{Fix}(F)$ since $\varphi(t) \neq 0$, hence that the two non-empty open subsets $W_0 = \{u = (x,y) \in \mathrm{Fix}(F) : x < 0\}$ and $W_1 = \{u = (x,y) \in \mathrm{Fix}(F) : x > 0\}$ of $\mathrm{Fix}(F)$ form a partition of $\mathrm{Fix}(F)$. Thus $\mathrm{Fix}(F)$ is not connected. And since $G = F \circ p$ satisfies $\mathrm{Fix}(G) = \mathrm{Fix}(F)$, we have just constructed a quasi-contraction on E whose set of fixed points is not connected. And this completes the proof of Theorem 7.5. □

Corollary 7.7. *If the normed space E has dimension at least 2, there exists on E a quasi-contraction G such that* $\mathrm{Fix}(G)$ *is not connected.*

Proof. Notice that following Theorem 2.1 the condition "$\dim(E) \geq 2$" is necessary and that following Theorem 3.3 such a quasi-contraction cannot be single-valued if E is finite-dimensional.

Take a closed linear subspace E_0 of codimension 2 and denote π the canonical projection onto the quotient space E/E_0. Recall that the norm on E/E_0 is given by $\|y\| = \inf\{\|x\| : x \in \pi^{-1}(y)\}$.

It follows from Theorems 7.5 and 6.3 that there exists on the 2-dimensional space E/E_0 a quasi-contraction F with closed convex values such that $\mathrm{Fix}(F)$ is not connected. Define then for $x \in E$: $G(x) = \pi^{-1}(F(\pi(x)))$ which is clearly a non-empty closed convex subset of E. And

$$x \in \mathrm{Fix}(G) \iff x \in G(x) \iff \pi(x) \in F(\pi(x)) \iff \pi(x) \in \mathrm{Fix}(F)$$

so $\mathrm{Fix}(G) = \pi^{-1}(\mathrm{Fix}(F))$, and $\pi(\mathrm{Fix}(G)) = \mathrm{Fix}(F)$ since π is onto.

If $\mathrm{Fix}(G)$ were connected so would be $\mathrm{Fix}(F) = \pi(\mathrm{Fix}(G))$. Thus $\mathrm{Fix}(G))$ is not connected. It remains to show that G is 1-Lipschitz. And this follows immediately from the facts that F is 1-Lipschitz and that the mapping $T \mapsto \pi^{-1}(T)$ is 1-Lipschitz from $\mathscr{F}(E/E_0)$ to $\mathscr{F}(E)$. Indeed :

$$\begin{aligned} d(x,\pi^{-1}(T)) &= \inf_{t\in T}\inf_{y\in t}\|x-y\| = \inf_{t\in T}\inf_{u\in E_0, y\in t}\|(x-u)-y)\| \\ &= \inf_{t\in T}\|\pi(x)-t\| = d(\pi(x),T) \end{aligned}$$

whence $d_H(\pi^{-1}(S),\pi^{-1}(T)) = d_H(S,T)$. □

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