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A CLASSIFICATION OF THE PLANE SIMILARITY TRANSFORMATIONS

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Introduction.

Geometric transformations have been included in school programs since long now. However, being this teaching missing experience, school books introduce it by very different presentations that often omit the appropriate links to other subjects. This affecting didactics, which not always meet expectations ([7], [8]). Besides, more in general, books focus more on isometric transformations and disregard similarity and even more affine ones.

This paper approaches the problem of introducing similarity transformations of a plane into itself starting from the affine transformations.

There are several ways of defining similarity transformations as affine transformations satisfying one of the following characterizations ([1], [5], [7]):

1. the ratio of the lengths of any two line segments is equal to the ratio of the length of their images;
2. corresponding angles are equal;
3. the image of a circle is a circle;
4. the transformation can be represented as the product of a homothetic transformation and an isometric transformation.

In this paper, starting from characterization 3, we give a unified approach to the plane similarity transformations, so as to get a complete classification of

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the similarity transformations and of the isometric transformations. The method we use is based only on elementary concepts of linear algebra and allows us to obtain nine disjoint classes of similarity transformations, and for each of which we give a geometric interpretation.

The argument developed here can be opportunely included in mathematics didactics courses and, more generally, into those courses aimed to future mathematic teachers.

1. Similarity transformations and fixed points.

The classification of similarity transformations of the plane is usually made by searching for fixed points and is made up of four disjoint classes:

- similarity transformations in which each point is fixed: *the identity transformation*;

- similarity transformations with a straight line of fixed points and no other fixed point: *reflection in a straight line* (inverse similarity transformations that are isometric transformations);

- similarity transformations with only one fixed point: *Rotations* (direct similarity transformations that are isometric transformations), *homothetic transformations* and *spiral similarities*[1] (direct similarity transformations that are not isometric transformations), *anti-homothetic transformations*[2] (inverse similarity transformations that are not isometric transformations);

- similarity transformations without fixed points: *Translations* (direct similarity transformations that are isometric transformations) *glide reflections*[3] (inverse similarity transformations that are isometric transformations).

In particular note that since all the similarity transformations that are not isometric transformations (*homothetic transformations, spiral similarities, anti-homothetic transformations*) have only one fixed point they all are in a same class. So, in order to distinguish one from the other, it is necessary to use other properties.

The classification of plane similarity transformations obtained in this paper is done by searching for fixed straight lines and fixed circles. Looking for fixed straight lines we obtain a classification of similarity transformations into seven

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[1] Spiral similarities are the product of a homothetic transformation and a rotation transformation with the same centre of the homothetic transformation.

[2] Anti-homothetic transformations are the product of a reflection in a straight line and a homothetic transformation with the centre on the straight line.

[3] Glide reflections are the product of a reflection in a straight line and a translation along the same straight line.

disjoint classes (n. 3); looking for fixed circles we get a classification in nine disjoint classes (n. 4). The complete results are given in n. 5 where we also note some geometric properties and algebraic characteristics of the different similarity transformations.

2. Affine transformations and similarity transformations.

It is known that an *affine transformation* is a one-one mapping α of a plane π onto itself such that the images of straight lines are straight lines ([1], [4]).

Given a system of Cartesian coordinates in the plane, the equations of the affine transformations are the following:

$$\begin{cases} x' = a_{11}x + a_{12}y + a \\ y' = a_{21}x + a_{22}y + b \end{cases}$$

with $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

An affine transformation is said to be a *similarity transformation* if it maps circles in circles.

The equations of similarity transformations, that can be found from those of the affine transformations by imposing the previous characterization, must satisfy the following conditions:

$$\begin{cases} a_{11}^2 + a_{21}^2 = a_{12}^2 + a_{22}^2 = k^2 \\ a_{11}a_{12} + a_{21}a_{22} = 0 \end{cases}$$

So, if $a_{11} = k\cos\theta$ and $a_{21} = k\sin\theta$, with $k > 0$, we have:

$$\begin{cases} x' = k(\cos\theta x \mp \sin\theta y) + a \\ y' = k(\sin\theta x \pm \cos\theta y) + b \end{cases} \qquad k > 0 \tag{1}$$

Note that if an affine transformation maps a given circle into a circle then it maps any circle into a circle.

3. Fixed straight lines in a similarity transformation.

Let ω be a similarity transformation of equations (1). We look for the fixed straight lines of ω.

Let r' be a straight line of equation:

$\alpha x' + \beta y' + \gamma = 0$.

By ω it becomes:

$$\alpha[k(\cos\theta x \mp \sin\theta y) + a] + \beta[k(\sin\theta x \pm \cos\theta y) + b] + \gamma = 0$$

which is the straight line r of equation:

$$[\alpha k \cos\theta + \beta k \sin\theta]x \mp [\alpha k \sin\theta - \beta k \cos\theta]y + \alpha a + \beta b + \gamma = 0.$$

The straight lines r and r' are the same line if and only if the following matrix has rank 1:

$$\begin{Vmatrix} \alpha & \beta & \gamma \\ k(\alpha\cos\theta + \beta\sin\theta) & \mp k(\alpha\sin\theta - \beta\cos\theta) & \alpha a + \beta b + \gamma \end{Vmatrix}$$

This happens when:

$$(2) \qquad \begin{cases} k(\alpha^2 + \beta^2)\sin\theta = 0 \\ \alpha^2 a + \alpha\beta b + \alpha\gamma(1 - k\cos\theta) - \beta\gamma k \sin\theta = 0 \\ \alpha\beta a + \beta^2 b + \alpha\gamma k \sin\theta + \beta\gamma(1 - k\cos\theta) = 0 \end{cases}$$

or

$$(3) \qquad \begin{cases} k(\alpha^2 \sin\theta - 2\alpha\beta\cos\theta - \beta^2\sin\theta) = 0 \\ \alpha^2 a + \alpha\beta b + \alpha\gamma(1 - k\cos\theta) - \beta\gamma k \sin\theta = 0 \\ \alpha\beta a + \beta^2 b - \alpha\gamma k \sin\theta + \beta\gamma(1 + k\cos\theta) = 0 \end{cases}$$

according to the top or the bottom sign.

Let us solve the system (2). Since $k > 0$ and $\alpha^2 + \beta^2 > 0$, from the first equation of (2) it follows that $\sin\theta = 0$, from which either $\cos\theta = 1$ or $\cos\theta = -1$.

For all the values of θ such that $\sin\theta \neq 0$ the system (2) does not have any solution, and for these values of θ there are not fixed straight lines.

Therefore, the similarity transformations without fixed straight lines are those having equations:

$$\begin{cases} x' = k(\cos\theta x - \sin\theta y) + a \\ y' = k(\sin\theta x + \cos\theta y) + b \end{cases}$$

with $k > 0$ and $\sin\theta \neq 0$.

If $\sin\theta = 0$, the solutions of the system (2) are the following.

For $\cos\theta = -1$:

i)
$$\alpha = \alpha,\ \beta = \beta,\ \gamma = -\frac{1}{1+k}(\alpha a + \beta b)$$

The equations of the similarity transformations corresponding to this solution are:

$$\begin{cases} x' = -kx + a \\ y' = -ky + b \end{cases} \quad k > 0$$

and these have as fixed straight lines all the lines of the following proper pencil: [4]

$$\alpha\left(x - \frac{a}{1+k}\right) + \beta\left(y - \frac{b}{1+k}\right) = 0$$

For $\cos\theta = 1$:

ii) for $k \neq 1$:

$$\alpha = \alpha, \quad \beta = \beta, \quad \gamma = -\frac{1}{1-k}(\alpha a + \beta b)$$

iii) for $k = 1$, a and b not both equal to zero:

$$\alpha = \rho b, \quad \beta = -\rho a, \quad \gamma = \rho h$$

iv) for $k = 1$, a and b both equal to zero:

$$\alpha = \alpha, \quad \beta = \beta, \quad \gamma = \gamma.$$

The equations of the similarity transformations corresponding to this solution are:

- for the solution *ii*):

$$\begin{cases} x' = kx + a \\ y' = ky + b \end{cases} \quad k \neq 1$$

and these have as fixed straight lines all the lines of the following proper pencil:

$$\alpha\left(x - \frac{a}{1-k}\right) + \beta\left(y - \frac{b}{1-k}\right) = 0$$

[4] A proper pencil is the set of all straight lines through a point.

- for the solution iii):

$$\begin{cases} x' = x + a \\ y' = y + b \end{cases}$$

and these have as fixed straight lines all the lines of the following improper pencil [5]

$$bx - ay + h = 0$$

- for the solution iv):

$$\begin{cases} x' = x \\ y' = y \end{cases}$$

and this similarity transformation is the identity transformation; in this case all the straight lines are fixed.

Let us solve the system (3) considering two cases: $\sin\theta = 0$ and $\sin\theta \neq 0$.

1^{st} case: $/sin\theta = 0$.
System (3) becomes:

$$\begin{cases} \alpha\beta\cos\theta = 0 \\ \alpha[\alpha a + \beta b + \gamma(1 - k\cos\theta)] = 0 \\ \beta[\alpha a + \beta b + \gamma(1 + k\cos\theta)] = 0 \end{cases}$$

Since $\cos\theta$ is either 1 or -1, the previous system is one of the two following systems:

$$(3') \qquad \begin{cases} \alpha\beta = 0 \\ \alpha[\alpha a + \beta b + \gamma(1 - k)] = 0 \\ \beta[\alpha a + \beta b + \gamma(1 + k)] = 0 \end{cases}$$

$$(3'') \qquad \begin{cases} \alpha\beta = 0 \\ \alpha[\alpha a + \beta b + \gamma(1 + k)] = 0 \\ \beta[\alpha a + \beta b + \gamma(1 - k)] = 0 \end{cases}$$

The solutions of the system $(3')$ are:

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[5] An improper pencil is the set of all straight lines parallel to a given one.

$j)$ for $k \neq 1$:

$$\begin{cases} \alpha_1 = 0, & \beta_1 = \beta, & \gamma_1 = -\dfrac{\beta b}{1+k} \\ \alpha_2 = \alpha, & \beta_2 = 0, & \gamma_2 = -\dfrac{\alpha a}{1-k} \end{cases}$$

$jj)$ for $k = 1$ and $a = 0$:

$$\begin{cases} \alpha_1 = 0, & \beta_1 = \beta, & \gamma_1 = -\dfrac{\beta b}{2} \\ \alpha_3 = \alpha, & \beta_3 = 0, & \gamma_3 = \gamma \end{cases}$$

$jjj)$ for $k = 1$ and $a \neq 0$:

$$\alpha_1 = 0, \qquad \beta_1 = \beta, \qquad \gamma_1 = -\frac{\beta b}{2}$$

and those of the system (3″)

$l)$ for $k \neq 1$:

$$\begin{cases} \alpha_1 = \alpha, & \beta_1 = 0, & \gamma_1 = -\dfrac{\alpha a}{1+k} \\ \alpha_2 = 0, & \beta_2 = \beta, & \gamma_2 = -\dfrac{\beta b}{1-k} \end{cases}$$

$ll)$ for $k = 1$ and $b = 0$:

$$\begin{cases} \alpha_1 = \alpha, & \beta_1 = 0, & \gamma_1 = -\dfrac{\alpha a}{2} \\ \alpha_3 = 0, & \beta_3 = \beta, & \gamma_3 = \gamma \end{cases}$$

$lll)$ for $k = 1$ and $b \neq 0$:

$$\alpha_1 = 0, \qquad \beta_1 = \beta, \qquad \gamma_1 = -\frac{\alpha a}{2}$$

<u>2^{st} case</u>: $\sin\theta \neq 0$.

Since $k > 0$, the first equation of the system (3) can be written as follows:

$$(\alpha \sin\theta - \beta(\cos\theta - 1))(\alpha \sin\theta - \beta(\cos\theta + 1))$$

so system (3) becomes:

$$\begin{cases} \alpha\sin\theta-(\cos\theta\mp 1)\beta=0\\ \alpha^2a+\alpha\beta b+\alpha\gamma(1-k\cos\theta)-\beta\gamma k\sin\theta=0\\ \alpha\beta a+\beta^2b-\alpha\gamma k\sin\theta+\beta\gamma(1+k\cos\theta)=0 \end{cases}$$

From the first equation of this system it follows:

$$\alpha=\rho(\cos\theta\mp 1)\qquad \beta=\rho\sin\theta$$

and so the other two equations are both:

$$\rho((\cos\theta\mp 1)a+\sin\theta b)+(1\pm k)\gamma=0$$

Therefore, the solutions of the system ($3'''$) are:

i) for $k\neq 1$:

$$\begin{cases} \alpha_1=\rho(\cos\theta-1), \qquad \beta_1=\rho\sin\theta, \qquad \gamma_1=-\rho\dfrac{(\cos\theta-1)a+\sin\theta b}{1+k}\\ \alpha_2=\rho(\cos\theta+1), \qquad \beta_2=\rho\sin\theta, \qquad \gamma_2=-\rho\dfrac{(\cos\theta+1)a+\sin\theta b}{1-k} \end{cases}$$

ii) for $k=1$ and $(\cos\theta+1)a+\sin\theta b=0$:

$$\begin{cases} \alpha_1=\rho(\cos\theta-1), \qquad \beta_1=\rho\sin\theta, \qquad \gamma_1=-\rho\dfrac{(\cos\theta-1)a+\sin\theta b}{2}\\ \alpha_3=\rho(\cos\theta+1), \qquad \beta_3=\rho\sin\theta, \qquad \gamma_3=\rho\gamma \end{cases}$$

iii) for $k=1$ and $(\cos\theta+1)a+\sin\theta b\neq 0$:

$$\alpha_1=\rho(\cos\theta-1), \qquad \beta_1=\rho\sin\theta, \qquad \gamma_1=-\rho\frac{(\cos\theta-1)a+\sin\theta b}{2}$$

Because of the solutions of the systems 3’, 3”, 3”’, each similarity transformation ω has:

1) two fixed straight lines (solutions j, l, i):

$$\alpha_1x+\beta_1y+\gamma_1=0$$

$$\alpha_2x+\beta_2y+\gamma_2=0$$

that are perpendicular because $\alpha_1\alpha_2 + \beta_1\beta_2 = 0$.

2) one fixed straight line and an improper pencil of fixed straight lines (solutions jj, ll, ii):

$$\alpha_1 x + \beta_1 y + \gamma_1 = 0$$

$$\alpha_3 x + \beta_3 y + \gamma_3 = 0$$

the straight lines of the pencil are perpendicular to the straight line because $\alpha_1\alpha_3 + \beta_1\beta_3 = 0$.

3) only one fixed straight line (solutions jjj, lll, iii):

$$\alpha_1 x + \beta_1 y + \gamma_1 = 0$$

Thus we obtain the following classification of the plane similarity transformations into seven disjoint classes:
- similarity transformations in which each straight line is fixed;
- similarity transformations with one fixed straight line and a pencil of fixed straight lines perpendicular to the straight line;
- similarity transformations with an improper pencil of fixed straight lines;
- similarity transformations with a proper pencil of fixed straight lines;
- similarity transformations with two perpendicular fixed straight lines;
- similarity transformations with only one fixed straight line;
- similarity transformations without fixed straight lines.

4. Fixed circles in a similarity transformation.

We look now for the fixed circles in a similarity transformation and we proceed as we did when we looked for fixed straight lines.

Let ω be a similarity transformation of equations (1) and let:

$$(x' - m)^2 + (y' - n)^2 = h \quad h > 0$$

be the equation of a circle c'.

The circle c', according to ω, comes from:

$$[k(\cos\theta x \mp \sin\theta y) + a - m]^2 + [k(\sin\theta x \pm \cos\theta y) + b - n]^2 = h$$

that is the circle c of equation:

$$k^2(x^2 + y^2) + 2k[(a - m)\cos\theta +$$

$+\ (b-n)\sin\theta]x \mp 2k[(a-m)\sin\theta-(b-n)\cos\theta]y+(a-m)^2+(b-n)^2 = h$

The circles c and c' are the same circle if and only if the following matrix has rank 1:

$$\left\| \begin{matrix} 1 & -m & -n & m^2+n^2-h \\ k^2 & \begin{matrix} k[(a-m)\cos\theta+ \\ +(b-n)\sin\theta] \end{matrix} & \begin{matrix} \mp k[(a-m)\sin\theta- \\ -(b-n)\cos\theta] \end{matrix} & (a-m)^2+(b-n)^2-h \end{matrix} \right\|$$

This happens when the three minors extracted from this matrix, bordering the element of the first line first column (the number 1), are equal to zero.
We obtain then the following system:

$$\begin{cases} k[(a-m)\cos\theta+(b-n)\sin\theta] = -k^2 m \\ \mp k[(a-m)\sin\theta-(b-n)\cos\theta] = -k^2 n \\ (a-m)^2+(b-n)^2-h = k^2(m^2+n^2-h) \end{cases} \tag{4}$$

Squaring the first two equations and adding them we obtain:

$$(a-m)^2+(b-n)^2 = k^2(m^2+n^2)$$

and from the third it follows:

$$h(k^2-1) = 0$$

Therefore $k=1$ and system (4) becomes:

$$\begin{cases} (\cos\theta-1)m+\sin\theta n = a\cos\theta+b\sin\theta \\ \sin\theta m-(\cos\theta\mp 1)n = a\sin\theta-b\cos\theta \end{cases}$$

We get then the following four solutions; the first two when we choose the top sign, the other two when we choose the bottom sign:

1) for $k=1$, $\cos\theta=1$ $a=0$ and $b=0$:

$$m=m,\quad n=n,\quad h=h$$

2) for $k=1$ and $\cos\theta\neq 1$:

$$m=\frac{a(1-\cos\theta)-b\sin\theta}{2(1-\cos\theta)},\quad n=\frac{a\sin\theta+b(1-\cos\theta)}{2(1-\cos\theta)},\quad h=h$$

3) for $k = 1$, $\cos\theta \neq 1$, $a = 0$ and $b \neq 0$:

$$m = m, \quad n = \frac{b}{2}, \quad h = h$$

4) for $k = 1$, $\cos\theta \neq 1$, $a = 0$ and $a\sin\theta + (1-\cos\theta)b = 0$:

$$m = \frac{\sin\theta n + a}{1-\cos\theta}, \quad n = n, \quad h = h$$

Therefore:

Solution 1) identifies the identity transformation.

Solution 2) identifies similarity transformations that have the following concentric circles as fixed circles:

$$\left(x - \frac{a(1-\cos\theta) - b\sin\theta}{2(1-\cos\theta)}\right)^2 + \left(y - \frac{a\sin\theta + b(1-\cos\theta)}{2(1-\cos\theta)}\right)^2 = h$$

with any $h > 0$. These similarity transformations, when $\cos\theta = -1$, are among those that have a proper pencil of fixed straight lines, precisely those for which $\cos\theta = -1$ and $k = 1$, while all the other similarity transformations, precisely those for which $k = 1$ and $\sin\theta \neq 0$, are among those that do not have fixed straight lines.

Solutions 3) and 4) determine circles of any radius and variable centre in order to describe a straight line; these similarity transformations are exactly those with one fixed straight line and an improper pencil of fixed straight lines that are perpendicular to the straight line.

5. Classification of the similarity transformations.

Because of the results obtained in n. 3 and n. 4, we get the following classification that partitions the set of similarity transformations into nine classes:

– similarity transformations in which each straight line is fixed and each circle is fixed: *identity transformation*;

– similarity transformations with one fixed straight line r and the not-proper pencil of fixed straight lines perpendicular to r; these similarity transformations have also all the circles with centre on r that are fixed: *reflections in a straight line*;

– similarity transformations with a not-proper pencil of fixed straight lines and without fixed circles: *translations*;

– similarity transformations with one proper pencil of fixed straight lines and the circles with centre in centre of the pencil that are fixed: *reflections in a point*;

– similarity transformations with one proper pencil of fixed straight lines and without fixed circles: *homothetic transformations*;

– similarity transformations with two perpendicular fixed straight lines and without fixed circles: *anti-homothetic transformations*;

– similarity transformations with only one fixed straight line and without fixed circles: *glide reflections*;

– similarity transformations without fixed straight lines and with one pencil of fixed concentric circles: *rotations*;

– similarity transformations without fixed straight lines and circles: *spiral similarities*.

The following are the equations of the similarity transformations for each of the previous nine classes:

The identity transformation

$$\begin{cases} x = x' \\ y = y' \end{cases}$$

Reflections about a straight line

$$\begin{cases} x' = \cos\theta x + \sin\theta y + \sigma(\cos\theta - 1) \\ y' = \sin\theta x - \cos\theta y + \sigma \sin\theta \end{cases} \qquad (\theta \neq 0)$$

and

$$\begin{cases} x' = x \\ y' = -y + b \end{cases}$$

Translations

$$\begin{cases} x' = x + a \\ y' = y + b \end{cases} \qquad \textit{(a and b not both equal to zero)}$$

Reflections in a point

$$\begin{cases} x' = -x + a \\ y' = -y + b \end{cases}$$

Homothetic transformations

$$\begin{cases} x' = \pm kx + a \\ y' = \pm ky + b \end{cases} \quad (k \neq 1)$$

Anti-homothetic transformations

$$\begin{cases} x' = k(\cos\theta x + \sin\theta y) + a \\ y' = k(\sin\theta x - \cos\theta y) + b \end{cases} \quad (k \neq 1)$$

Glide reflections

$$\begin{cases} x' = \cos\theta x + \sin\theta y + a \\ y' = \sin\theta x - \cos\theta y + b \end{cases}$$

$$(\theta \neq 0 \textit{ and } a\sin\theta + b(1-\cos\theta) \neq 0 \textit{ or } \theta = 0 \textit{ and } a \neq 0)$$

Rotations

$$\begin{cases} x' = \cos\theta x - \sin\theta y + a \\ y' = \sin\theta x - \cos\theta y + b \end{cases} \quad (\sin\theta \neq 0)$$

Spiral similarities

$$\begin{cases} x' = k(\cos\theta x - \sin\theta y) + a \\ y' = k(\sin\theta x - \cos\theta y) + b \end{cases} \quad (k \neq 1,\ \sin\theta \neq 0)$$

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