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A SHORT PROOF FOR A DETERMINANTAL FORMULA FOR GENERALIZED FIBONACCI NUMBERS

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The aim of this note is to provide a short and elegant proof for a recent determinantal formula for generalized Fibonacci numbers. The attractiveness of the proof presented here is its elementary nature.

1. Preliminaries

The study of sequences generated by the homogeneous linear second order difference equation with constant coefficients

$$u_{n+1} = a\,u_n + b\,u_{n-1}\,, \quad \text{for } n \geqslant 1\,, \tag{1}$$

with certain initial conditions, goes back to the beginning of 1960s with the analysis of the algebraic properties of (u_n) [2, 7, 8]. Many relevant number sequences are obtained from (1), namely the Fibonacci numbers, setting $a = b = u_1 = 1$ and $u_0 = 0$.

It is well-known that (1) can be represented by the determinant of the Jacobi matrix

$$T_n = \begin{pmatrix} a & -1 & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & b & a \end{pmatrix}_{n\times n} \tag{2}$$

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together with the specialisation of the initial conditions, namely, $\det T_0 = 1$ and $\det T_1 = a$. From the well-established theory of orthogonal polynomials (see, e.g., the standard reference [3]), the determinant of T_n can be given by (cf. e.g. [6])

$$\det T_n = (-i\sqrt{b})^n U_n\left(\frac{a\,i}{2\sqrt{b}}\right),$$

where $\{U_n(x)\}_{n\geqslant 0}$ are the Chebyshev polynomials of second kind, i.e., the orthogonal polynomials satisfying the three-term recurrence relations

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad \text{for all } n = 1,2,\dots ,$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$. The main explicit formula for the Chebyshev polynomials of second kind is

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi), \tag{3}$$

for all $n = 0,1,2\dots$. While (3) is more common to find in the orthogonal polynomials theory, there are other explicit representations and relations for $U_n(x)$. Among them, the most frequent to find in number theory are

$$U_n(x) = \frac{\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}}{2\sqrt{x^2-1}},$$

an immediate consequence of de Moivre's formula, or

$$U_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}$$

which can be found for example in [1, (22.3.7)]. As stated in [5, p.187], many of them are paraphrases of trigonometric identities and derivations from (3). Nonetheless, here no explicit formula for $U_n(x)$ is required for our aims.

Now, the Fibonacci numbers can be obtained directly from (cf. e.g. [4])

$$\det\begin{pmatrix} 1 & -1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & 1 & 1 \end{pmatrix}_{n\times n},$$

with $a = b = 1$, $u_0 = 0$ and $u_1 = 1$ in (2). This means that the nth Fibonacci number F_n can be given by (cf. [2, 7])

$$F_n = (-i)^{n-1} U_{n-1}\left(\frac{i}{2}\right).$$

We observe that the determinant of a tridiagonal matrix is known in the literature as a *continuant* (cf. [10]). The terminology "tridiagonal determinant" is however inaccurate.

2. A determinantal formula

Recently in [9], Qi and Guo using intricate techniques proved that

$$u_n = \frac{1}{n!}\begin{vmatrix} \binom{1}{0}a & -1 & & & \\ 2\binom{2}{0}b & \binom{2}{1}a & -1 & & \\ & 2\binom{3}{1}b & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 2\binom{n}{n-2}b & \binom{n}{n-1}a \end{vmatrix}. \tag{4}$$

Using the multilinearity of the determinant, our purpose here is to provide a simple proof for (4). Indeed,

$$\begin{vmatrix} \binom{1}{0}a & -1 & & & \\ 2\binom{2}{0}b & \binom{2}{1}a & -1 & & \\ & 2\binom{3}{1}b & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & 2\binom{n}{n-2}b & \binom{n}{n-1}a \end{vmatrix} =$$

$$= \begin{vmatrix} 1\cdot a & -1 & & & \\ 2\cdot 1\cdot b & 2\cdot a & -1 & & \\ & 3\cdot 2\cdot b & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & n(n-1)\cdot b & n\cdot a \end{vmatrix}$$

$$= \begin{vmatrix} 1\cdot a & -1 & & & \\ 2\cdot b & 2\cdot a & -2 & & \\ & 3\cdot b & \ddots & \ddots & \\ & & \ddots & \ddots & -(n-1) \\ & & & n\cdot b & n\cdot a \end{vmatrix}$$

$$= n!\begin{vmatrix} a & -1 & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & b & a \end{vmatrix}_{n\times n}$$

$$= \quad n!\,u_n\,,$$

for any positive integer n.

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