LE MATEMATICHE
Vol. LXXIV (2019) – Issue II, pp. 369–395
doi: 10.4418/2019.74.2.10

ULTRA-RELATIVISTIC LIMIT OF EXTENDED THERMODYNAMICS OF RAREFIED POLYATOMIC GAS

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The aim of this paper is to evaluate the ultra-relativistic limit of a recent causal theory proposed for polyatomic dissipative relativistic gas. The explicit expression of characteristic velocities of the hyperbolic system is found in term of the degree of freedom of the molecule and is compared with the one of monatomic gas.

1. Introduction

In [1] a causal hyperbolic relativistic model of rarefied gas with internal structure (polyatomic) was presented. The model was completed in [2] with explicit expression for the production terms using a new relativistic BGK model [3]. The differential system contains a constant parameter a

$$a = \frac{D-5}{2},$$

where $D = 3 + f^i$ is related to the degrees of freedom of a molecule given by the sum of the space dimension 3 for the translational motion and the contribution from the internal degrees of freedom $f^i \geq 0$ due to the internal motion (rotation

Received on July 22, 2019

AMS 2010 Subject Classification: 76Y05, 82C35, 82C40, 76P99
Keywords: Extended thermodynamics, Relativistic fluids
This paper was supported by GNFM of INdAM; one of the authors (S. Pennisi) thanks also for the support of the project GESTA - Fondazione Banco di Sardegna.

and vibration). For monatomic gases $D=3$ and $a=-1$. The singular limit of the theory, for $a \to -1$, gives the ET theory of monatomic gases of Liu-Müller-Ruggeri (LMR) [4]; this property has been proved in [5].
In the paper [1] it was also considered the classical limit obtained when the ratio

$$\gamma = \frac{mc^2}{k_B T}$$

is very large (m is the particle mass, c the light velocity, k_B the Boltzmann constant and T the absolute temperature), and was proved that the differential system converges to the corresponding classical ET theory of polyatomic gases [6], [7].
Instead it was not considered the opposite limit when $\gamma \to 0$ corresponding to the so-called ultra-relativistic limit. In this limit the bodies are so extremely hot that the mean kinetic energy of particles surpasses the rest energy or the mass is extremely small. This limit has been considered in [8] but only for the equilibrium model with only 5 moments, i.e., for Euler polyatomic gases. Now, in section 2, we take the limit for the dissipative 14 moments model which was considered in [1].
In section 3 we find the characteristic velocities in our ultra-relativistic limit and obtain that they are continuous functions of a.

2. The ultra-relativistic limit for dissipative polyatomic gas

In [1] the authors consider as field variables the quantities

$$\begin{aligned} &V^\alpha(x^\mu) \quad - \quad \text{particle, particle flux vector}\,, \\ &T^{\alpha\beta}(x^\mu) \quad - \quad \text{energy momentum tensor}\,. \end{aligned}$$

For the determination of the 14 state variables one needs the field equations i.e., the conservation laws of particle number and energy momentum and the extended balance law of fluxes,

$$\partial_\alpha V^\alpha = 0 \quad , \quad \partial_\alpha T^{\alpha\beta} = 0 \quad , \quad \partial_\alpha A^{\alpha<\beta\gamma>} = I^{<\beta\gamma>}\,,$$

where $\partial_\alpha = \partial/\partial x^\alpha$ with x^α being the space-time coordinates $\alpha = 0,1,2,3$. It is assumed that $T^{\alpha\beta}$, $A^{\alpha\beta\gamma}$ and $I^{\beta\gamma}$ are completely symmetric tensors and $<\cdots>$ denotes the traceless part of a tensor. Moreover, the fields $(V^\alpha, T^{\alpha\beta})$ are expressed in terms of the usual physical variables through the decomposition:

$$V^\alpha = nmU^\alpha, \qquad T^{\alpha\beta} = t^{<\alpha\beta>_3} + (p+\pi)h^{\alpha\beta} + \frac{2}{c^2}U^{(\alpha}q^{\beta)} + \frac{e}{c^2}U^\alpha U^\beta, \quad (1)$$

where U^α is the four-velocity ($U^\alpha U_\alpha = c^2$), n is the number density, p is the pressure, $h^{\alpha\beta}$ is the projector tensor:

$$h^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2} U^\alpha U^\beta ,$$

$g^{\alpha\beta} = diag(1, -1, -1, -1)$ being the metric tensor, e is the energy, π is the dynamical pressure, the symbol $< \cdots >_3$ denotes the 3-dimensional traceless part of a tensor, i.e., $t^{<\alpha\beta>_3} = T^{\mu\nu}\left(h^\alpha_\mu h^\beta_\nu - \frac{1}{3} h^{\alpha\beta} h_{\mu\nu}\right)$ is the viscous deviatoric stress, and $q^\alpha = -h^\alpha_\mu U_\nu T^{\mu\nu}$ is the heat flux. In this case we can take n, T, U^α, $t^{<\alpha\beta>_3}$, π, q^α as independent variables.
The expression of $A^{\alpha\beta\gamma}$ has been found in [1] and we want now to take its ultra relativistic limit. We prove here the following:

Theorem 2.1. The ultra relativistic limit of the triple tensor $A^{\alpha\beta\gamma}$ is:

$$A^{\alpha\beta\gamma} = -\frac{2\tilde{h}_2 + \gamma\frac{\partial \tilde{h}_2}{\partial\gamma}}{\tilde{h}_0} mnU^\alpha U^\beta U^\gamma + 3\frac{\tilde{h}_2}{\tilde{h}_0} mnc^2 h^{(\alpha\beta}U^{\gamma)} - \tag{2}$$

$$-\frac{3}{c^2}\frac{\tilde{N}_1}{D_1}\pi U^\alpha U^\beta U^\gamma - 3\frac{\tilde{N}_{11}}{D_1}\pi h^{(\alpha\beta}U^{\gamma)} +$$

$$+\frac{3}{c^2}\frac{\tilde{N}_3}{D_2} q^{(\alpha}U^\beta U^{\gamma)} + \frac{3}{5}\frac{\tilde{N}_{31}}{D_2} q^{(\alpha}h^{\beta\gamma)} + 3\tilde{C}_5 t^{(<\alpha\beta>_3}U^{\gamma)} ,$$

with

$$D_1 = \begin{vmatrix} -\tilde{h}_0 & \frac{\partial \tilde{h}_0}{\partial\gamma} & \tilde{h}_2 + \gamma\frac{\partial\tilde{h}_2}{\partial\gamma} \\ \frac{\partial \tilde{h}_0}{\partial\gamma} & -\frac{\partial^2 \tilde{h}_0}{\partial\gamma^2} & -\frac{\partial}{\partial\gamma}\left(\tilde{h}_2 + \gamma\frac{\partial\tilde{h}_2}{\partial\gamma}\right) \\ -\frac{\tilde{h}_0}{\gamma} & -\frac{1}{\gamma^2}\left(\tilde{h}_0 - \gamma\frac{\partial\tilde{h}_0}{\partial\gamma}\right) & \frac{\partial\tilde{h}_2}{\partial\gamma} - \frac{5}{3}\frac{\tilde{h}_2}{\gamma} \end{vmatrix} ,$$

$$\tilde{N}_1 = \begin{vmatrix} -\tilde{h}_0 & \frac{\partial \tilde{h}_0}{\partial\gamma} & \tilde{h}_2 + \gamma\frac{\partial\tilde{h}_2}{\partial\gamma} \\ \frac{\partial \tilde{h}_0}{\partial\gamma} & -\frac{\partial^2 \tilde{h}_0}{\partial\gamma^2} & -\frac{\partial}{\partial\gamma}\left(\tilde{h}_2 + \gamma\frac{\partial\tilde{h}_2}{\partial\gamma}\right) \\ -\frac{1}{3}\left(2\tilde{h}_2 + \gamma\frac{\partial\tilde{h}_2}{\partial\gamma}\right) & \frac{\partial\tilde{h}_2}{\partial\gamma} + \frac{1}{3}\gamma\frac{\partial^2\tilde{h}_2}{\partial\gamma^2} & \frac{1}{45}\left(6\tilde{h}_5 + 6\gamma\frac{\partial\tilde{h}_5}{\partial\gamma} + \gamma^2\frac{\partial^2\tilde{h}_5}{\partial\gamma^2}\right) \end{vmatrix} ,$$

$$\tilde{N}_{11} = \begin{vmatrix} -\tilde{h}_0 & \frac{\partial \tilde{h}_0}{\partial\gamma} & \tilde{h}_2 + \gamma\frac{\partial\tilde{h}_2}{\partial\gamma} \\ \frac{\partial \tilde{h}_0}{\partial\gamma} & -\frac{\partial^2 \tilde{h}_0}{\partial\gamma^2} & -\frac{\partial}{\partial\gamma}\left(\tilde{h}_2 + \gamma\frac{\partial\tilde{h}_2}{\partial\gamma}\right) \\ \tilde{h}_2 & -\frac{\partial\tilde{h}_2}{\partial\gamma} & -\frac{1}{45}\left(\tilde{h}_5 + 3\gamma\frac{\partial\tilde{h}_5}{\partial\gamma}\right) \end{vmatrix} ,$$

$$D_2 = \begin{vmatrix} \frac{\tilde h_0}{\gamma} & 2\,\tilde h_2 \\ \frac{\partial\left(\frac{\tilde h_0}{\gamma}\right)}{\partial \gamma} & 2\,\frac{\partial \tilde h_2}{\partial \gamma} \end{vmatrix} \qquad \tilde N_3 = \begin{vmatrix} \frac{\tilde h_0}{\gamma} & 2\,\tilde h_2 \\ \frac{\partial \tilde h_2}{\partial \gamma} & \frac{2}{15}\left(2\,\tilde h_5 + \gamma\,\frac{\partial \tilde h_5}{\partial \gamma}\right) \end{vmatrix}$$

$$\tilde N_{31} = \begin{vmatrix} \frac{\tilde h_0}{\gamma} & 2\,\tilde h_2 \\ -5\,\frac{\tilde h_2}{\gamma} & -\frac{2}{3}\,\tilde h_5 \end{vmatrix}, \quad \tilde C_5 = \frac{1}{15}\,\frac{\tilde h_5}{\tilde h_2}\,\gamma .$$

$$\tilde h_0 = \begin{cases} \gamma^{-3}\,\Gamma(2-a) & \text{if} \quad a<2\,, \\ -\gamma^{-3}\,\ln\gamma & \text{if} \quad a=2\,, \\ \gamma^{-a-1}\,R_{-a} & \text{if} \quad 2<a\,. \end{cases} \tag{3}$$

$$\tilde h_2 = \begin{cases} \frac{1}{3}\,\gamma^{-5}\,\Gamma(3-a)(a+5) & \text{if} \quad a<3\,, \\ -\frac{8}{3}\,\gamma^{-5}\,\ln\gamma & \text{if} \quad a=3\,, \\ \gamma^{-a-2}\,2\,\frac{a+1}{a-3}\,R_{-1-a} & \text{if} \quad 3<a\,. \end{cases}$$

$$\tilde h_5 = \begin{cases} \gamma^{-7}\,\Gamma(4-a)\,(a+4)(a+11) & \text{if} \quad a<4\,, \\ -120\,\gamma^{-7}\,\ln\gamma & \text{if} \quad a=4\,, \\ 4(a+1)(a+2)\,\gamma^{-a-3}\,(R_{2-a}-2R_{-a}+R_{-2-a}) = 60\,\frac{(a+2)}{(a-1)}\gamma^{-a-3}\,R_{2-a} & \text{if} \quad a>4\,. \end{cases}$$

In these expressions the numbers R_k appear and they are defined by

$$R_k = \lim_{\gamma\to 0} \bar R_k \quad \text{with} \quad \bar R_k = \int_1^{+\infty} e^{-\gamma y}\sqrt{y^2-1}\,y^k\,dy\,. \tag{4}$$

Their expressions are reported in Appendix A.
To prove our theorem we note firstly that, from the calculations reported in [1], it is evident that the closure is determined by the 4-vector

$$\begin{aligned} & h'^{\alpha} = -k_B c \int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{\chi}{k_B}} \quad p^{\alpha}\mathcal{I}^a\,d\vec P\,d\mathcal{I} \quad \text{with} \\ & \chi = m\lambda + \left(1+\frac{\mathcal{I}}{mc^2}\right)\lambda_\beta p^\beta + \frac{1}{m}\left(1+\frac{2\mathcal{I}}{mc^2}\right)\Sigma_{\beta\gamma}p^\beta p^\gamma\,. \end{aligned} \tag{5}$$

The closure is

$$V^{\alpha} = \frac{\partial h'^{\alpha}}{\partial \lambda} \quad , \quad T^{\alpha\beta} = \frac{\partial h'^{\alpha}}{\partial \lambda_\beta} \quad , \quad A^{\alpha\beta\gamma} = \frac{\partial h'^{\alpha}}{\partial \Sigma_{\beta\gamma}}\,. \tag{6}$$

Now Taylor's expansion of $h^{'\alpha}$ at second order with respect to the state where $\Sigma_{\beta\gamma} = 0$ is

$$h^{'\alpha} = h_0^{'\alpha} + h_1^{'\alpha\beta\gamma}\Sigma_{\beta\gamma} + \frac{1}{2}h_2^{'\alpha\beta\gamma\mu\nu}\Sigma_{\beta\gamma}\Sigma_{\mu\nu}$$

with

$$\begin{aligned}
h_0^{'\alpha} &= -k_B c \int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[m\lambda+\left(1+\frac{\mathcal{I}}{mc^2}\right)\lambda_\beta p^\beta\right]} p^\alpha \mathcal{I}^a \, d\vec{P}\, d\mathcal{I}, \\
h_1^{'\alpha\beta\gamma} &= \frac{c}{m}\int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[m\lambda+\left(1+\frac{\mathcal{I}}{mc^2}\right)\lambda_\beta p^\beta\right]} p^\alpha p^\beta p^\gamma \left(1+\frac{2\mathcal{I}}{mc^2}\right)\mathcal{I}^a \, d\vec{P}\, d\mathcal{I}, \\
h_2^{'\alpha\beta\gamma\mu\nu} &= \frac{-c}{m^2 k_B}\int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[m\lambda+\left(1+\frac{\mathcal{I}}{mc^2}\right)\lambda_\beta p^\beta\right]} p^\alpha p^\beta p^\gamma p^\mu p^\nu \left(1+\frac{2\mathcal{I}}{mc^2}\right)^2 \mathcal{I}^a \, d\vec{P}\, d\mathcal{I},
\end{aligned} \tag{7}$$

which are integrable, as it can be seen in [9].

Let us now calculate these integrals. To this end we define

$$\gamma = \frac{mc}{k_B}\sqrt{\lambda_\beta\lambda^\beta} \quad , \quad l_\beta = \frac{mc^2}{k_B\gamma}\lambda_\beta$$

from which it follows

$$\frac{\partial\gamma}{\partial\lambda_\beta} = \frac{m}{k_B}l^\beta \quad , \quad \frac{\partial l_\beta}{\partial\lambda_\gamma} = -\frac{mc^2}{\gamma k_B}h_\beta^\gamma \quad , \quad l^\beta l_\beta = c^2 .$$

Now, from the Representation Theorems, we have that

$$\begin{aligned}
h_0^{'\alpha} &= h_0^{'}(\lambda,\gamma)\, l^\alpha, \\
h_1^{'\alpha\beta\gamma} &= \frac{1}{c^2}h_1^{'}(\lambda,\gamma)\, l^\alpha l^\beta l^\gamma + 3h_2^{'}(\lambda,\gamma)\, h^{(\alpha\beta}l^{\gamma)}, \\
h_2^{'\alpha\beta\gamma\mu\nu} &= \frac{1}{c^4}h_3^{'}(\lambda,\gamma)\, l^\alpha l^\beta l^\gamma l^\mu l^\nu + \frac{1}{c^2}h_4^{'}(\lambda,\gamma)\, h^{(\alpha\beta}l^\gamma l^\mu l^{\nu)} + h_5^{'}(\lambda,\gamma)\, h^{(\alpha\beta}h^{\gamma\mu}l^{\nu)} .
\end{aligned} \tag{8}$$

Thanks to these expressions, it is easier to calculate the above integrals and we find, in particular,

$$\begin{aligned}
h_0^{'} &= -4\pi k_B m^3 c^3 e^{-1-\frac{m\lambda}{k_B}} \int_0^{+\infty} J_{2,1}^* \mathcal{I}^a \, d\mathcal{I}, \\
h_2^{'} &= \frac{4}{3}\pi m^4 c^5 e^{-1-\frac{m\lambda}{k_B}} \int_0^{+\infty} J_{4,1}^* \left(1+\frac{2\mathcal{I}}{mc^2}\right)\mathcal{I}^a \, d\mathcal{I}, \\
h_5^{'} &= \frac{-4\pi m^5 c^7}{k_B} e^{-1-\frac{m\lambda}{k_B}} \int_0^{+\infty} J_{6,1}^* \left(1+\frac{2\mathcal{I}}{mc^2}\right)^2 \mathcal{I}^a \, d\mathcal{I}, \\
&\text{where} \quad J_{m,n}(\gamma) = \int_0^{+\infty} e^{-\gamma\cosh s}\sinh^m s\cosh^n s\, ds \quad , \\
&J_{m,n}^* \quad \text{is} \quad J_{m,n}(\gamma) \quad \text{with} \quad \gamma \quad \text{replaced by} \quad \gamma^* = \gamma\left(1+\frac{\mathcal{I}}{mc^2}\right).
\end{aligned} \tag{9}$$

In effect here γ stands for γ_E and l^μ stands for $l_E^\mu = U^\mu$ to keep the notation simple. The calculations to solve these integrals are the same as used in [1]; so we avoid here to furnish further details. It is not necessary to find the expressions of $h_1^{'}$, $h_3^{'}$ and $h_4^{'}$ because we see in (7) that $h_1^{'\alpha\beta\gamma}$ and $h_2^{'\alpha\beta\gamma\mu\nu}$ are symmetric with their derivatives with respect to λ_α; so we can apply the results of sect. IV of [10] and find that

$$h_1^{'} = -2h_2^{'} - \gamma\frac{\partial h_2^{'}}{\partial\gamma}\,,$$
$$h_4^{'} = -\frac{4}{3}h_5^{'} - \frac{2}{3}\gamma\frac{\partial h_5^{'}}{\partial\gamma} \quad , \quad h_3^{'} = -\frac{2}{5}h_4^{'} - \frac{1}{10}\gamma\frac{\partial h_4^{'}}{\partial\gamma}\,. \tag{10}$$

Obviously, we can also find directly the expressions of $h_1^{'}$, $h_3^{'}$ and $h_4^{'}$ and obtain the same results, thanks to the properties of $J_{m,n}(\gamma)$.
Now we want to take the non ultra-relativistic limit of the expressions (9) but, first of all it is better to clarify what this limit means. It is not a simple limit, otherwise the independent variables γ disappears when it tends to zero and we have only 13 independent variables instead of 14. Instead of this, we do the following considerations: If a given function $F(\gamma)$ can be written as $F(\gamma) = F_1(\gamma) + F_2(\gamma)$ with $\lim_{\gamma\to 0}\frac{F_2(\gamma)}{F_1(\gamma)} = 0$, we say that $F_1(\gamma)$ is the leading term of $F(\gamma)$ and substitute $F(\gamma)$ with $F_1(\gamma)$. In other words, we neglect terms which, in the limit for γ going to zero, are of less order with respect to the leading term. We apply this procedure and obtain that in the ultra-relativistic limit the leading terms of $h_0^{'}$, $h_2^{'}$, $h_5^{'}$ are respectively

$$h_0^{'} = -4\pi k_B m^3 c^3 (mc^2)^{a+1}\Gamma(a+1)e^{-1-\frac{m\lambda}{k_B}}\,\tilde{h}_0(\gamma)\,,$$
$$h_2^{'} = 4\pi m^4 c^5 (mc^2)^{a+1}\Gamma(a+1)e^{-1-\frac{m\lambda}{k_B}}\,\tilde{h}_2(\gamma)\,, \tag{11}$$
$$h_5^{'} = -\frac{4\pi m^5 c^7}{k_B}(mc^2)^{a+1}\Gamma(a+1)e^{-1-\frac{m\lambda}{k_B}}\,\tilde{h}_5(\gamma)\,,$$

with $\tilde{h}_0$, $\tilde{h}_2$, $\tilde{h}_0$ given by the above reported eqs. (3). In particular, for the expression of $\tilde{h}_2$ for $a > 3$ we have used the property

$$R_{-1-a} = \frac{a-3}{a}R_{1-a} \quad \text{for} \quad a > 3\,,$$

which is proved in $(31)_2$ of appendix A. If we write this expression with $a-1$ instead of a and with $a+1$ instead of a, we obtain respectively

$$R_{-a} = \frac{a-4}{a-1}R_{2-a} \quad \text{for} \quad a > 4\,,$$
$$R_{-2-a} = \frac{a-2}{a+1}R_{-a} = \frac{a-2}{a+1}\,\frac{a-4}{a-1}R_{2-a} \quad \text{for} \quad a > 4\,,$$

and we have taken into account these results in the above expression of $\tilde{h}_5$.
After having determined the 4-potential, we are now ready to find the closure our balance equations in terms of the physical variables.

2.1. The closure at equilibrium

The equations (6) at equilibrium become

$$mnU^{\alpha} = 4\pi m^4 c^3 (mc^2)^{a+1}\Gamma(a+1)e^{-1-\frac{m\lambda_E}{k_B}}\,\tilde{h}_0(\gamma_E)\,l_E^{\alpha}\,, \tag{12}$$

$$ph^{\alpha\beta} + \frac{e}{c^2}U^{\alpha}U^{\beta} = 4\pi m^4 c^5 (mc^2)^{a+1}\Gamma(a+1)e^{-1-\frac{m\lambda_E}{k_B}}\,\frac{1}{\gamma_E}\left(\tilde{h}_0\,h^{\alpha\beta} - \gamma_E\frac{\partial\tilde{h}_0}{\partial\gamma}\,\frac{1}{c^2}\,l_E^{\alpha}l_E^{\beta}\right),$$

$$A_E^{\alpha\beta\gamma} = 4\pi m^4 c^5 (mc^2)^{a+1}\Gamma(a+1)e^{-1-\frac{m\lambda_E}{k_B}}\left[-\frac{1}{c^2}\left(2\,\tilde{h}_2 + \gamma_E\frac{\partial\tilde{h}_2}{\partial\gamma}\right)l_E^{\alpha}\,l_E^{\beta}\,l_E^{\gamma} + 3\tilde{h}_2\,h^{(\alpha\beta}\,l_E^{\gamma)}\right].$$

The modulus of the first one of these equations gives

$$4\pi m^4 c^3 (mc^2)^{a+1}\Gamma(a+1)e^{-1-\frac{m\lambda_E}{k_B}}\,\tilde{h}_0 = mn\,, \tag{13}$$

$$\text{and there remains} \qquad l_E^{\alpha} = U^{\alpha}\,.$$

After that, $(12)_2$ gives

$$p = \frac{mc^2}{\gamma_E}\,n = n\,k_B\,T\,,$$

$$e = -\frac{1}{\tilde{h}_0}\,\frac{\partial\tilde{h}_0}{\partial\gamma}\,mn\,c^2$$

where we have used $(13)_1$. By using (3) we now have

$$e = \begin{cases} 3\,n\,k_B\,T & \text{if} \quad a < 2 \\ 3\,n\,k_B\,T\left(1 - \frac{1}{3}\,\frac{\gamma^2}{\ln\gamma}\right) & \text{if} \quad a = 2 \\ (a+1)\,n\,k_B\,T & \text{if} \quad a > 2 \end{cases},$$

from which Theorem 2 of sect. V in [8] follows immediately. Finally, $(12)_3$ gives

$$A_E^{\alpha\beta\gamma} = -\frac{2\,\tilde{h}_2 + \gamma\frac{\partial\tilde{h}_2}{\partial\gamma}}{\tilde{h}_0}\,mnU^{\alpha}U^{\beta}U^{\gamma} + 3\,\frac{\tilde{h}_2}{\tilde{h}_0}\,mn\,c^2\,h^{(\alpha\beta}U^{\gamma)} = \tilde{A}_1^0\,U^{\alpha}U^{\beta}U^{\gamma} + 3\tilde{A}_{11}^0\,h^{(\alpha\beta}U^{\gamma)}$$

with

$$\tilde{A}_1^0 = \begin{cases} (2-a)\,(a+5)\,\frac{n k_B^2 T^2}{m c^4} & \text{if} \quad a<2 \\ \frac{7}{-\ln\gamma}\,\frac{n k_B^2 T^2}{m c^4} & \text{if} \quad a=2 \\ \frac{\Gamma(3-a)}{R_{-a}}\,(a+5)\,\gamma^{a-2}\,\frac{n k_B^2 T^2}{m c^4} & \text{if} \quad 2<a<3 \\ \frac{-8\ln\gamma}{R_{-3}}\,\frac{n k_B T}{c^2} & \text{if} \quad a=3 \\ 2(a+1)\,\frac{R_{1-a}}{R_{-a}}\,\frac{n k_B T}{c^2} & \text{if} \quad a>3 \end{cases}$$

$$\tilde{A}_{11}^0 = \begin{cases} \frac{1}{3}\tilde{A}_1^0 c^2 & \text{if} \quad a \le 3 \\ 2\,\frac{(a+1)}{a}\,\frac{R_{1-a}}{R_{-a}}\,n k_B T & \text{if} \quad a>3 \end{cases} .$$

2.2. First order deviation from equilibrium

The first order deviation from equilibrium of equations (6) is

$$\left(\frac{\partial^2 h_0^{'\alpha}}{\partial \lambda^2}\right)_E (\lambda-\lambda_E) + \left(\frac{\partial^2 h_0^{'\alpha}}{\partial \lambda\, \partial \lambda_\theta}\right)_E (\lambda_\theta - \lambda_\theta^E) + \left(\frac{\partial h_1^{'\alpha\mu\nu}}{\partial \lambda}\right)_E \Sigma_{\mu\nu} = V^\alpha - V_E^\alpha = 0,$$

$$\left(\frac{\partial^2 h_0^{'\alpha}}{\partial \lambda_\beta\, \partial \lambda}\right)_E (\lambda-\lambda_E) + \left(\frac{\partial^2 h_0^{'\alpha}}{\partial \lambda_\beta\, \partial \lambda_\theta}\right)_E (\lambda_\theta - \lambda_\theta^E) + \left(\frac{\partial h_1^{'\alpha\mu\nu}}{\partial \lambda_\beta}\right)_E \Sigma_{\mu\nu} = \tag{14}$$
$$= \pi h^{\alpha\beta} + \frac{2}{c^2} q^{(\alpha} U^{\beta)} + t^{<\alpha\beta>_3},$$

$$\left(\frac{\partial h_1^{'\alpha\beta\gamma}}{\partial \lambda}\right)_E (\lambda-\lambda_E) + \left(\frac{\partial h_1^{'\alpha\beta\gamma}}{\partial \lambda_\theta}\right)_E (\lambda_\theta - \lambda_\theta^E) + h_{2E}^{'\alpha\beta\gamma\mu\nu} \Sigma_{\mu\nu} = A^{\alpha\beta\gamma} - A_E^{\alpha\beta\gamma}.$$

From the first two of these relations we obtain now $\lambda - \lambda_E$, $\lambda_\theta - \lambda_\theta^E$, $\Sigma_{\mu\nu}$ and substitute them in the third one, obtaining in this way the requested closure. By using (8) , (10) and (11), eqs. (14) become

$$-\tilde{h}_0 U^\alpha (\lambda-\lambda_E)+\frac{\partial \tilde{h}_0}{\partial\gamma} U^\alpha U^\theta (\lambda_\theta-\lambda^E_\theta)-\frac{\tilde{h}_0}{\gamma} c^2 h^{\alpha\theta}(\lambda_\theta-\lambda^E_\theta)+$$
$$\left(\tilde{h}_2+\gamma\frac{\partial \tilde{h}_2}{\partial\gamma}\right) U^\alpha U^\mu U^\nu \Sigma_{\mu\nu}-2\tilde{h}_2 h^{\alpha\mu} U^\nu \Sigma_{\mu\nu}=0,$$

$$\left(\frac{\partial \tilde{h}_0}{\partial\gamma} U^\alpha U^\beta-\frac{c^2}{\gamma}\tilde{h}_0 h^{\alpha\beta}\right)(\lambda-\lambda_E)-\left[\frac{\partial^2 \tilde{h}_0}{\partial\gamma^2} U^\alpha U^\beta+\frac{c^2}{\gamma^2} h^{\alpha\beta}\left(\tilde{h}_0-\gamma\frac{\partial \tilde{h}_0}{\partial\gamma}\right)\right] U^\mu (\lambda_\mu-\lambda^E_\mu)+$$

$$+2\frac{c^2}{\gamma^2}\left(-\tilde{h}_0+\gamma\frac{\partial \tilde{h}_0}{\partial\gamma}\right) U^{(\alpha} h^{\beta)\mu}(\lambda_\mu-\lambda^E_\mu)-\frac{\partial(\tilde{h}_2+\gamma\frac{\partial \tilde{h}_2}{\partial\gamma})}{\partial\gamma} U^\alpha U^\beta (U^\mu U^\nu \Sigma_{\mu\nu})+$$

$$\frac{\partial \tilde{h}_2}{\partial\gamma} c^2 h^{\alpha\beta}(U^\mu U^\nu \Sigma_{\mu\nu})+4\frac{\partial \tilde{h}_2}{\partial\gamma} c^2 U^{(\alpha} h^{\beta)\mu} U^\nu \Sigma_{\mu\nu}-\frac{c^4}{\gamma}\tilde{h}_2\left(2h^{\alpha\mu}h^{\nu\beta}+\frac{1}{c^2} U^\mu U^\nu h^{\alpha\beta}\right)\Sigma_{\mu\nu}=$$
$$\frac{1}{m^2 n}\left(\pi h^{\alpha\beta}+\frac{2}{c^2} q^{(\alpha}U^{\beta)}+t^{<\alpha\beta>_3}\right), \tag{15}$$

$$A^{\alpha\beta\gamma}-A^{\alpha\beta\gamma}_E=\left\{\left[(2\tilde{h}_2+\gamma\frac{\partial \tilde{h}_2}{\partial\gamma}) U^\alpha U^\beta U^\gamma-3c^2\tilde{h}_2 h^{(\alpha\beta}U^{\gamma)}\right](\lambda-\lambda_E)+\right.$$

$$+\left[\left(-3\frac{\partial \tilde{h}_2}{\partial\gamma}-\gamma\frac{\partial^2 \tilde{h}_2}{\partial\gamma^2}\right) U^\alpha U^\beta U^\gamma+3c^2\frac{\partial \tilde{h}_2}{\partial\gamma} h^{(\alpha\beta}U^{\gamma)}\right] U^\mu (\lambda_\mu-\lambda^E_\mu)+$$

$$+\frac{c^2}{\gamma}\left[3\gamma\frac{\partial \tilde{h}_2}{\partial\gamma} U^{(\alpha}U^\beta h^{\gamma)\mu}-3c^2\tilde{h}_2 h^{(\alpha\beta}h^{\gamma)\mu}\right](\lambda_\mu-\lambda^E_\mu)+$$

$$+\left(-\frac{2}{5}\tilde{h}_5-\frac{2}{5}\gamma\frac{\partial \tilde{h}_5}{\partial\gamma}-\frac{1}{15}\gamma^2\frac{\partial^2 \tilde{h}_5}{\partial\gamma^2}\right) U^\alpha U^\beta U^\gamma (U^\mu U^\nu \Sigma_{\mu\nu})+$$

$$+\frac{3}{5}c^2\left(\frac{4}{3}\tilde{h}_5+\frac{2}{3}\gamma\frac{\partial \tilde{h}_5}{\partial\gamma}\right) h^{\mu(\alpha}U^\beta U^{\gamma)} U^\nu \Sigma_{\mu\nu}+\frac{1}{5}c^2\left(\tilde{h}_5+\gamma\frac{\partial \tilde{h}_5}{\partial\gamma}\right) h^{(\alpha\beta}U^{\gamma)} U^\nu U^\nu \Sigma_{\mu\nu}-$$

$$\left.\frac{2}{5}c^4\tilde{h}_5 h^{\mu(\alpha}h^{\beta\gamma)} U^\nu \Sigma_{\mu\nu}-\frac{2}{5}c^4\tilde{h}_5 U^{(\alpha}h^{<\beta}_\mu h^{\gamma>_3)}_\nu \Sigma^{\mu\nu}-\frac{2}{15}c^2\tilde{h}_5 U^{(\alpha}h^{\beta\gamma)}(U^\mu U^\nu \Sigma_{\mu\nu})\right\}\frac{m^2 n}{\tilde{h}_0 k_B}.$$

Now eq. $(15)_1$ contracted with U_α and with h^δ_β gives respectively the following eqs. $(16)_1$ and $(17)_1$; eq. $(15)_2$ contracted with $U_\alpha U_\beta$, with $h_{\alpha\beta}$, with $U_\alpha h^\delta_\beta$ and with $h_\alpha^{<\delta} h_\beta^{\theta>_3}$ gives respectively the following eqs. $(16)_{2,3}$, $(17)_2$ and (18):

$$-\tilde{h}_0(\lambda-\lambda_E)+\frac{\partial\tilde{h}_0}{\partial\gamma}U^\theta(\lambda_\theta-\lambda_\theta^E)+\left(\tilde{h}_2+\gamma\frac{\partial\tilde{h}_2}{\partial\gamma}\right)U^\mu U^\nu\Sigma_{\mu\nu}=0, \tag{16}$$

$$\frac{\partial\tilde{h}_0}{\partial\gamma}(\lambda-\lambda_E)-\frac{\partial^2\tilde{h}_0}{\partial\gamma^2}U^\mu(\lambda_\mu-\lambda_\mu^E)-\frac{\partial(\tilde{h}_2+\gamma\frac{\partial\tilde{h}_2}{\partial\gamma})}{\partial\gamma}(U^\mu U^\nu\Sigma_{\mu\nu})=0,$$

$$-\frac{1}{\gamma}\tilde{h}_0(\lambda-\lambda_E)-\frac{1}{\gamma^2}\left(\tilde{h}_0-\gamma\frac{\partial\tilde{h}_0}{\partial\gamma}\right)U^\mu(\lambda_\mu-\lambda_\mu^E)+\frac{1}{\gamma}\left(\gamma\frac{\partial\tilde{h}_2}{\partial\gamma}-\frac{5}{3}\tilde{h}_2\right)(U^\mu U^\nu\Sigma_{\mu\nu})=$$
$$\frac{\tilde{h}_0k_B}{m^2c^2n}\pi,$$

$$\frac{\tilde{h}_0}{\gamma}c^2h^{\delta\theta}(\lambda_\theta-\lambda_\theta^E)+2\tilde{h}_2h^{\delta\mu}U^\nu\Sigma_{\mu\nu}=0, \tag{17}$$

$$\frac{1}{\gamma^2}\left(-\tilde{h}_0+\gamma\frac{\partial\tilde{h}_0}{\partial\gamma}\right)h^{\delta\mu}(\lambda_\mu-\lambda_\mu^E)+2\frac{\partial\tilde{h}_2}{\partial\gamma}h^{\delta\mu}U^\nu\Sigma_{\mu\nu}=\frac{\tilde{h}_0k_B}{m^2c^4n}q^\delta,$$

$$-2\tilde{h}_2h^{\mu<\delta}h^{\theta>_3\nu}\Sigma_{\mu\nu}=\frac{\tilde{h}_0k_B}{m^2c^4n}t^{<\delta\theta>_3}. \tag{18}$$

Now we obtain the expressions of $(\lambda-\lambda_E)$, $U^\theta(\lambda_\theta-\lambda_\theta^E)$, $U^\mu U^\nu\Sigma_{\mu\nu}$ from (16), $h^{\delta\theta}(\lambda_\theta-\lambda_\theta^E)$ and $h^{\delta\mu}U^\nu\Sigma_{\mu\nu}$ from (17), $h^{\mu<\delta}h^{\theta>_3\nu}\Sigma_{\mu\nu}$ from (18); after that, we substitute them in $(15)_3$ so obtaining

$$A^{\alpha\beta\gamma}-A_E^{\alpha\beta\gamma}=-\frac{3}{c^2}\frac{\tilde{N}_1}{D_1}\pi U^\alpha U^\beta U^\gamma-3\frac{\tilde{N}_{11}}{D_1}\pi h^{(\alpha\beta}U^{\gamma)}+$$

$$+\frac{3}{c^2}\frac{\tilde{N}_3}{D_2}q^{(\alpha}U^\beta U^{\gamma)}+\frac{3}{5}\frac{\tilde{N}_{31}}{D_2}q^{(\alpha}h^{\beta\gamma)}+3\tilde{C}_5t^{(<\alpha\beta>_3}U^{\gamma)},$$

with D_1, $\tilde{N}_1$, $\tilde{N}_{11}$, D_2, $\tilde{N}_3$, $\tilde{N}_{31}$, $\tilde{C}_5$ reported above in $(2)_{2-8}$.
In fact, from (16) we get

$$\lambda-\lambda_E=\frac{M_1}{D_1}\frac{\tilde{h}_0k_B}{m^2c^2n}\pi,\quad U^\theta(\lambda_\theta-\lambda_\theta^E)=\frac{M_2}{D_1}\frac{\tilde{h}_0k_B}{m^2c^2n}\pi,$$
$$U^\mu U^\nu\Sigma_{\mu\nu}=\frac{M_3}{D_1}\frac{\tilde{h}_0k_B}{m^2c^2n}\pi,$$

where M_1, M_2 and M_3 are the algebraic complements of the third line of D_1.
Similarly, from (17) the following expression follows

$$h^{\delta\theta}(\lambda_\theta-\lambda_\theta^E)=\frac{\tilde{M}_1}{D_2}\frac{\tilde{h}_0k_B}{m^2c^4n}q^\delta,\quad h^{\delta\mu}U^\nu\Sigma_{\mu\nu}=\frac{\tilde{M}_2}{D_2}\frac{\tilde{h}_0k_B}{m^2c^4n}q^\delta,$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are the algebraic complements of the second line of D_2.
Finally, from (18) we get

$$h^{\mu<\delta}h^{\theta>_3\nu}\Sigma_{\mu\nu} = -\frac{1}{2}\gamma\frac{\tilde{h}_0}{\tilde{h}_2}\frac{k_B}{m^2c^4n}t^{<\delta\theta>_3}\,.$$

By using these results, we find with a simple substitution the closure of our balance equations in terms of the physical variables and this is reported above in the theorem expressed by (2).
By using (3), we can find the leading terms of our closure. We may also calculate the wave speeds and we do this in the next section, distinguishing some sub-cases according to different values of a.

3. Characteristic velocities in the ultra-relativistic limit

Let μ be the characteristic velocities of the differential system with unit of measure chosen so that the light speed is $c = 1$. We find their values according to the following cases:

3.1. Case $a < 4$:

We find

$$\mu = 0 \quad \text{with multiplicity}\,6$$
$$\mu_1^2 = \frac{1}{5} \quad \text{with multiplicity}\,2$$
$$\mu_2^2 = \frac{1}{E} \quad \text{with multiplicity}\,1 \quad \text{and with} \quad E = \begin{cases} 3 & \text{if} \quad a \leq 2 \\ a+1 & \text{if} \quad 2 < a < 4 \end{cases}$$
$$\mu_3^2 = \frac{3}{5} \quad \text{with multiplicity}\,1\,.$$

3.2. Case $a = 4$:

We find

$$\mu = 0 \quad \text{with multiplicty}\,6$$
$$\mu_1^2 = \mu_2^2 = \frac{1}{5} \quad \text{with multiplicty}\,3$$
$$\mu_3^2 = \frac{3}{5} \quad \text{with multiplicty}\,1\,.$$

3.3. Case $a > 4$:

We find

$$\mu = 0 \quad \text{with multiplicty}\, 6$$
$$\mu_1^2 = \frac{1}{a+1} + \frac{(a-4)\,(R_{1-a})^2}{(a+2)^2[a^2\,(R_{-a})^2 - (a+1)(a-4)(R_{1-a})^2]} \qquad \text{with multiplicity}\, 2\,, \tag{19}$$

and $\mu_2^2 < \mu_3^2$ the solutions of the equation

$$\begin{vmatrix}
1 & a+1 & 1 & 2\,\frac{(a+1)^2}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 4\,\frac{a+1}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 0 \\
a+1 & (a+1)(a+2) & (a+2) & 2\,\frac{(a+1)^2(a+2)}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 4\,\frac{(a+1)(a+2)}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 0 \\
1 & a+2 & (a+2)\,\mu^2 & \frac{2}{3}\,\frac{(a+1((3a+11))}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 4\,\frac{(a+1)(a+2)}{a-3}\,\frac{R_{-1-a}}{R_{-a}}\,\mu^2 & -\frac{4}{3}\,\frac{a+1}{a-3}\,\frac{R_{-1-a}}{R_{-a}} \\
2\,\frac{(a+1)^2}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 2\,\frac{(a+1)^2(a+2)}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & \frac{2}{3}\,\frac{(a+1)(3a+11)}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & \frac{4}{3}\,\frac{a+2}{a-4}\,(3a^2+6a+8) & \frac{8}{3}\,\frac{a+2}{a-4}\,(3a+8) & 0 \\
4\,\frac{a+1}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 4\,\frac{(a+1)(a+2)}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 4\,\frac{(a+1)(a+2)}{a-3}\,\frac{R_{-1-a}}{R_{-a}}\,\mu^2 & \frac{8}{3}\,\frac{a+2}{a-4}\,(3a+8) & 16\,\frac{(a+1)(a+2)}{a-4}\,\mu^2 & -\frac{16}{3}\,\frac{a+2}{a-4} \\
0 & 0 & -\frac{4}{3}\,\frac{a+1}{a-3}\,\frac{R_{-1-a}}{R_{-a}} & 0 & -\frac{16}{3}\,\frac{a+2}{a-4} & \frac{4}{3}\,\frac{a+2}{a-4}
\end{vmatrix} = 0\,. \tag{20}$$

The graph of the $\mu^2 \neq 0$ is reported in the Figure 1.

These equations show that the eigenvalues do not depend on γ; moreover, they depend on a and on the parameter $\frac{R_{1-a}}{R_{-a}}$. Through numerical calculus, we have confirmed that they are real; more than that, we have obtained that they do not exceed the speed of light.

In any case, we see that there is continuity with respect to a in the limit for $a \to 4$ because (19) has limit $\frac{1}{5}$, while for the equation (20), we can multiply its last 3 columns by $(a-4)$ and, after that, take the limit. In this way this equation

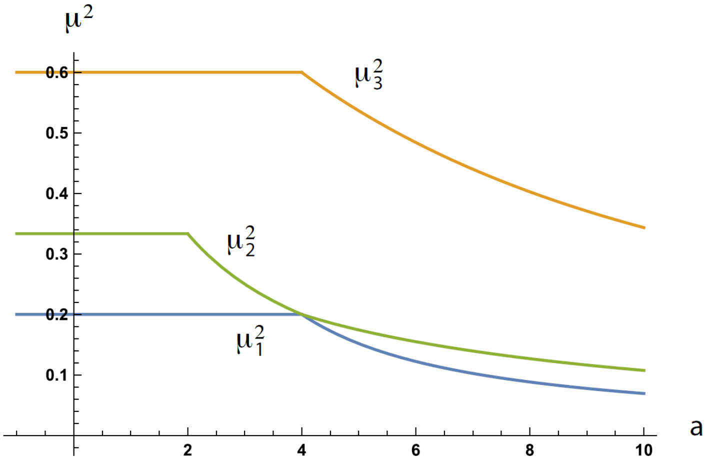

Figure 1: Square of characteristic velocities as function of a.

splits into two equations, that is,

$$\begin{vmatrix} 1 & 5 & 1 \\ 5 & 30 & 6 \\ 1 & 6 & 6\mu^2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} 640 & 320 & 0 \\ 320 & 480\mu^2 & -32 \\ 0 & -32 & 8 \end{vmatrix} = 0\,;$$

the first one of these equations has solution $\mu^2 = \frac{1}{5}$ and the second one has solution $\mu^2 = \frac{3}{5}$, as for the case $a = 4$.

3.4. Proof of the above results

The equations for calculating the wave velocities μ associated to our balance equation are independent from the variables used; so we may also use the Lagrange Multipliers as independent variables and, in this way, they are

$$\left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right) \delta V^\alpha = 0 \quad , \quad \left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right) \delta T^{\alpha\beta} = 0$$
$$\left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right) \delta A^{\alpha<\beta\gamma>} = 0\,,$$

with $n_\alpha n^\alpha = -1$, $n_\alpha l^\alpha = 0$. They can be expressed as

$$\left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right)\left[\frac{\partial^2 h'^\alpha}{\partial \lambda^2}\,\delta\lambda + \frac{\partial^2 h'^\alpha}{\partial \lambda\,\partial \lambda_\theta}\,\delta\lambda_\theta + \frac{\partial^2 h'^\alpha}{\partial \lambda\,\partial \Sigma_{\mu\nu}}\,\delta\Sigma_{\mu\nu}\right] = 0\,, \tag{21}$$

$$\left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right)\left[\frac{\partial^2 h'^\alpha}{\partial \lambda_\beta\,\partial \lambda}\,\delta\lambda + \frac{\partial^2 h'^\alpha}{\partial \lambda_\beta\,\partial \lambda_\theta}\,\delta\lambda_\theta + \frac{\partial^2 h'^\alpha}{\partial \lambda_\beta\,\partial \Sigma_{\mu\nu}}\,\delta\Sigma_{\mu\nu}\right] = 0\,,$$

$$\left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right)\left[\frac{\partial^2 h'^\alpha}{\partial \Sigma_{\beta\gamma}\partial \lambda}\,\delta\lambda + \frac{\partial^2 h'^\alpha}{\partial \Sigma_{\beta\gamma}\partial \lambda_\theta}\,\delta\lambda_\theta + \frac{\partial^2 h'^\alpha}{\partial \Sigma_{\beta\gamma}\partial \Sigma_{\mu\nu}}\,\delta\Sigma_{\mu\nu}\right] = 0\,.$$

It is convenient, for the subsequent calculations, to consider the quadratic form

$$\begin{aligned} Q = &\left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right)\Bigg[\frac{\partial^2 h'^\alpha}{\partial \lambda^2}\,(\delta\lambda)^2 + 2\,\frac{\partial^2 h'^\alpha}{\partial \lambda\,\partial \lambda_\theta}\,\delta\lambda\,\delta\lambda_\theta + 2\,\frac{\partial^2 h'^\alpha}{\partial \lambda\,\partial \Sigma_{\mu\nu}}\,\delta\lambda\,\delta\Sigma_{\mu\nu} + \\ &+ \frac{\partial^2 h'^\alpha}{\partial \lambda_\beta\,\partial \lambda_\theta}\,\delta\lambda_\beta\,\delta\lambda_\theta + 2\,\frac{\partial^2 h'^\alpha}{\partial \lambda_\beta\,\partial \Sigma_{\mu\nu}}\,\delta\lambda_\beta\,\delta\Sigma_{\mu\nu} + \frac{\partial^2 h'^\alpha}{\partial \Sigma_{\beta\gamma}\partial \Sigma_{\mu\nu}}\,\delta\Sigma_{\beta\gamma}\delta\Sigma_{\mu\nu}\Bigg] = \\ &= \left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right)\left[\delta\lambda\,\delta\left(\frac{\partial h'^\alpha}{\partial \lambda}\right) + \delta\lambda_\beta\,\delta\left(\frac{\partial h'^\alpha}{\partial \lambda_\beta}\right) + \delta\Sigma_{\beta\gamma}\delta\left(\frac{\partial h'^\alpha}{\partial \Sigma_{\beta\gamma}}\right)\right] = \\ &= \left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right)\delta\lambda_A\,\delta\left(\frac{\partial h'^\alpha}{\partial \lambda_A}\right). \end{aligned}$$

It follows that eqs. (21) can be written simply as

$$\frac{1}{2}\,\frac{\partial Q}{\partial(\delta\lambda)} = 0 \quad , \quad \frac{1}{2}\,\frac{\partial Q}{\partial(\delta\lambda_\beta)} = 0 \quad , \quad \frac{1}{2}\,\frac{\partial Q}{\partial(\delta\Sigma_{\beta\gamma})} = 0\,.$$

By using (5), we see that

$$\delta\left(\frac{\partial h'^\alpha}{\partial \lambda_A}\right) = c\,\delta\left(\int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\quad \frac{\partial \chi}{\partial \lambda_A}\,p^\alpha\,\mathcal{I}^a\,d\vec{P}\,d\mathcal{I}\right).$$

Since $\frac{\partial\chi}{\partial\lambda_A}$ is independent of λ_B, we deduce that

$$\delta\left(\frac{\partial h'^\alpha}{\partial \lambda_A}\right) = \frac{-c}{k_B}\left(\int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\quad \frac{\partial \chi}{\partial \lambda_A}\,\frac{\partial \chi}{\partial \lambda_B}\,p^\alpha\,\mathcal{I}^a\,d\vec{P}\,d\mathcal{I}\right)\delta\,\lambda_B\,.$$

So the above expression of Q becomes

$$\begin{aligned} Q &= \left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right)\frac{-c}{k_B}\left(\int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\quad \frac{\partial \chi}{\partial \lambda_A}\,\frac{\partial \chi}{\partial \lambda_B}\,p^\alpha\,\mathcal{I}^a\,d\vec{P}\,d\mathcal{I}\right)\delta\,\lambda_A\,\delta\,\lambda_B = \\ &= \left(n_\alpha - \frac{\mu}{c}\, l_\alpha\right)\frac{-c}{k_B}\left[\int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{\chi}{k_B}}\quad \left(\frac{\partial \chi}{\partial \lambda_A}\,\delta\,\lambda_A\right)^2 p^\alpha\,\mathcal{I}^a\,d\vec{P}\,d\mathcal{I}\right]. \end{aligned}$$

By calculating the coefficients of $\delta\lambda_A$ and $\delta\lambda_B$ at equilibrium, it follows

$$Q = \left(n_\alpha - \frac{\mu}{c}\,l_\alpha\right)\frac{-c}{k_B}\left[\int_{\Re^3}\int_0^{+\infty} e^{-1-\frac{1}{k_B}\left[m\lambda+\left(1+\frac{\mathcal{I}}{mc^2}\right)\lambda_\beta p^\beta\right]} \quad \left(\frac{\partial\chi}{\partial\lambda_A}\,\delta\lambda_A\right)^2 p^\alpha\,\mathcal{I}^a\,d\vec{P}\,d\mathcal{I}\right].$$

From this result it is evident the convexity requirement and, consequently, hyperbolicity of the field equations. By using the expression $(5)_2$ of χ, it follows

$$Q = \left(n_\alpha - \frac{\mu}{c}\,l_\alpha\right)\left[\frac{\partial^2 h_0^{'\alpha}}{\partial\lambda^2}\,(\delta\lambda)^2 + 2\,\frac{\partial^2 h_0^{'\alpha}}{\partial\lambda\,\partial\lambda_\theta}\,\delta\lambda\,\delta\lambda_\theta + 2\,\frac{\partial\,h_1^{'\alpha\mu\nu}}{\partial\lambda}\,\delta\lambda\,\delta\Sigma_{\mu\nu} + \right.$$

$$\left. + \frac{\partial^2 h_0^{'\alpha}}{\partial\lambda_\beta\,\partial\lambda_\theta}\,\delta\lambda_\beta\,\delta\lambda_\theta + 2\,\frac{\partial h_1^{'\alpha\mu\nu}}{\partial\lambda_\beta}\,\delta\lambda_\beta\,\delta\Sigma_{\mu\nu} + h_2^{'\alpha\beta\gamma\mu\nu}\,\delta\Sigma_{\beta\gamma}\,\delta\Sigma_{\mu\nu}\right].$$

Here it was not necessary to take the traceless parts of $h_1^{'\alpha\mu\nu}$ and of $h_2^{'\alpha\beta\gamma\mu\nu}$ because they are contracted with traceless tensors.
We now calculate this expression of Q in the reference frame where $l^\alpha \equiv (c\,,0\,,0\,,0)$, $n^\alpha \equiv (0\,,1\,,0\,,0)$ and use the above expressions of $h_0^{'\alpha}$, $h_1^{'\alpha\mu\nu}$, $h_2^{'\alpha\beta\gamma\mu\nu}$. Moreover, we take as independent variables $\delta\Sigma_{23}$, $\delta D = \delta\Sigma_{22} - \delta\Sigma_{33}$, $\delta\mu_6 = \delta\Sigma_{22} + \delta\Sigma_{33} - 2\,\delta\Sigma_{11}$, $\delta\lambda_2$, $\delta\Sigma_{02}$, $\delta\Sigma_{12}$, $\delta\lambda_3$, $\delta\Sigma_{03}$, $\delta\Sigma_{13}$, $\delta\mu_1 = \delta\lambda$, $\delta\mu_2 = \delta\lambda_0$, $\delta\mu_3 = \delta\Sigma_{00}$, $\delta\mu_4 = \delta\lambda_1$, $\delta\mu_5 = \delta\Sigma_{01}$; since $\delta\Sigma_{00} = \delta\Sigma_{11} + \delta\Sigma_{22} + \delta\Sigma_{33}$, it follows that

$$\delta\Sigma_{11} = \frac{1}{3}\left(\delta\Sigma_{00} - \delta\mu_6\right),\ \delta\Sigma_{22} = \frac{1}{6}\left(2\,\delta\Sigma_{00} + \delta\mu_6 + 3\,\delta D\right)$$
$$\delta\Sigma_{33} = \frac{1}{6}\left(2\,\delta\Sigma_{00} + \delta\mu_6 - 3\,\delta D\right).$$

By using these properties, we obtain the following results:

- The derivatives of Q with respect to $\delta\Sigma_{23}$, δD lead to the equations

$$\tilde{h}_5\,\mu\,\delta\Sigma_{23} = 0 \quad , \quad \tilde{h}_5\,\mu\,\delta D = 0\,,$$

  which has the eigenvalue $\mu = 0$ with multiplicity 2.

- The derivatives of Q with respect to $\delta\lambda_2$, $\delta\Sigma_{02}$, $\delta\Sigma_{12}$, lead to the equations

$$\mu\left[\frac{1}{\gamma^2}\left(\tilde{h}_0 - \gamma\frac{\partial\tilde{h}_0}{\partial\gamma}\right)(c\,\delta\lambda_2) - 2\,\frac{\partial\,\tilde{h}_2}{\partial\gamma}\,(c^2\,\delta\Sigma_{02})\right] + \frac{2}{\gamma}\,\tilde{h}_2\,(c^2\,\delta\Sigma_{12}) = 0\,,$$

$$\mu\left[-2\,\frac{\partial\,\tilde{h}_2}{\partial\gamma}\,(c\,\delta\lambda_2) - \frac{4}{15}\left(2\tilde{h}_5 + \gamma\,\frac{\partial\,\tilde{h}_5}{\partial\gamma}\right)(c^2\,\delta\Sigma_{02})\right] + \frac{4}{15}\tilde{h}_5\,(c^2\,\delta\Sigma_{12}) = 0\,, \qquad (22)$$

$$\frac{2}{\gamma}\tilde{h}_2\,(c\,\delta\lambda_2) + \frac{4}{15}\,\tilde{h}_5\,(c^2\,\delta\Sigma_{02}) + \mu\,\frac{4}{15}\,\tilde{h}_5\,(c^2\,\delta\Sigma_{12}) = 0\,.$$

We note that this system has the symmetric form; its eigenvalues are $\mu = 0$ and two others of opposite sign which are the solutions of the equations

$$\begin{vmatrix} \frac{1}{\gamma^2}\left(\tilde{h}_0 - \gamma \frac{\partial \tilde{h}_0}{\partial \gamma}\right) & -2\,\frac{\partial \tilde{h}_2}{\partial \gamma} & \frac{2}{\gamma}\,\tilde{h}_2 \\ -2\,\frac{\partial \tilde{h}_2}{\partial \gamma} & -\frac{4}{15}\left(2\tilde{h}_5 + \gamma \frac{\partial \tilde{h}_5}{\partial \gamma}\right) & \frac{4}{15}\tilde{h}_5 \\ \frac{2}{\gamma}\,\tilde{h}_2 & \frac{4}{15}\tilde{h}_5 & \mu^2\,\frac{4}{15}\,\tilde{h}_5 \end{vmatrix} = 0\,. \tag{23}$$

- The derivatives of Q with respect to $\delta\lambda_3$, $\delta\Sigma_{03}$, $\delta\Sigma_{13}$, lead again to the previous equations but with $\delta\lambda_2$, $\delta\Sigma_{02}$, $\delta\Sigma_{12}$ replaced by the new variables. It follows that all the 3 eigenvalues of this system have multiplicity 2.

- The derivatives of Q with respect to $\delta\mu_j$ lead to the equations $a_{ij}\,\delta\mu_j = 0$ with $a_{ij} = a_{ji}$ and

$$a_{11} = \mu\,\tilde{h}_0 = \mu\,a^*_{11}\,,\, a_{12} = -\mu\,\frac{\partial \tilde{h}_0}{\partial \gamma} = \mu\,a^*_{12}\,,\, a_{13} = -\mu\left(\tilde{h}_2 + \gamma \frac{\partial \tilde{h}_2}{\partial \gamma}\right) = \mu\,a^*_{13}\,,$$

$$a_{14} = \frac{1}{\gamma}\tilde{h}_0\,,\, a_{15} = 2\tilde{h}_2 \quad , \quad a_{16} = 0 \quad , \quad a_{22} = \mu\,\frac{\partial^2 \tilde{h}_0}{\partial \gamma^2} = \mu\,a^*_{22}\,,$$

$$a_{23} = \mu\left(2\,\frac{\partial \tilde{h}_0}{\partial \gamma} + \gamma \frac{\partial^2 \tilde{h}_0}{\partial \gamma^2}\right) = \mu\,a^*_{23}\,,\, a_{24} = \frac{1}{\gamma^2}\left(\tilde{h}_0 - \gamma \frac{\partial \tilde{h}_0}{\partial \gamma}\right),\, a_{25} = -2\,\frac{\partial \tilde{h}_2}{\partial \gamma}\,,\, a_{26} = 0\,,$$

$$a_{33} = \frac{1}{45}\,\mu\left(17\tilde{h}_5 + 15\gamma \frac{\partial \tilde{h}_5}{\partial \gamma} + 3\gamma^2\,\frac{\partial^2 \tilde{h}_5}{\partial \gamma^2}\right) = \mu\,a^*_{33}\,,\, a_{34} = \frac{1}{3\gamma}\left(5\tilde{h}_2 - 3\gamma \frac{\partial \tilde{h}_2}{\partial \gamma}\right),$$

$$a_{35} = -\frac{2}{45}\left(\tilde{h}_5 + 3\gamma \frac{\partial \tilde{h}_5}{\partial \gamma}\right) \quad , \quad a_{36} = 0 \quad , \quad a_{44} = \mu\,\frac{1}{\gamma^2}\left(\tilde{h}_0 - \gamma \frac{\partial \tilde{h}_0}{\partial \gamma}\right) = \mu\,a^*_{44}$$

$$a_{45} = -2\mu\,\frac{\partial \tilde{h}_2}{\partial \gamma} = \mu\,a^*_{45}\,,\, a_{46} = -\frac{2}{3}\,\frac{1}{\gamma}\,\tilde{h}_2\,,\, a_{55} = -\frac{4}{15}\,\mu\left(2\tilde{h}_5 + \gamma \frac{\partial \tilde{h}_5}{\partial \gamma}\right) = \mu\,a^*_{55}\,,$$

$$a_{56} = -\frac{4}{45}\,\tilde{h}_5 \quad , \quad a_{66} = \frac{1}{45}\,\mu\,\tilde{h}_5 = \mu\,a^*_{66}\,.$$

We appreciate that also this system has the symmetric form. Moreover, we see that it gives again the eigenvalue $\mu = 0$ with multiplicity two and a bi-quadratic equation for the determination of the other 4 eigenvalues.

In fact, the determinant of the coefficients can be written in the form

$$\begin{vmatrix} \mu a^*_{11} & \mu a^*_{12} & \mu a^*_{13} & a_{14} & a_{15} & 0 \\ \mu a^*_{12} & \mu a^*_{22} & \mu a^*_{23} & a_{24} & a_{25} & 0 \\ \mu a^*_{13} & \mu a^*_{23} & \mu a^*_{33} & a_{34} & a_{35} & 0 \\ a_{14} & a_{24} & a_{34} & \mu a^*_{44} & \mu a^*_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & \mu a^*_{45} & \mu a^*_{55} & a_{56} \\ 0 & 0 & 0 & a_{46} & a_{56} & \mu a^*_{66} \end{vmatrix}$$

It follows that, in correspondence to $\mu = 0$ we find $\delta\mu_4 = 0$, $\delta\mu_5 = 0$, while the other 4 unknowns are linked by only 2 equations, that is

$$a_{14}\,\delta\mu_1 + a_{24}\,\delta\mu_2 + a_{34}\,\delta\mu_3 + a_{46}\,\delta\mu_6 = 0,$$
$$a_{15}\,\delta\mu_1 + a_{25}\,\delta\mu_2 + a_{35}\,\delta\mu_3 + a_{56}\,\delta\mu_6 = 0\,.$$

Finally, the other eigenvalues are the solutions of the equation

$$\begin{vmatrix} a^*_{11} & a^*_{12} & a^*_{13} & a_{14} & a_{15} & 0 \\ a^*_{12} & a^*_{22} & a^*_{23} & a_{24} & a_{25} & 0 \\ a^*_{13} & a^*_{23} & a^*_{33} & a_{34} & a_{35} & 0 \\ a_{14} & a_{24} & a_{34} & \mu^2 a^*_{44} & \mu^2 a^*_{45} & a_{46} \\ a_{15} & a_{25} & a_{35} & \mu^2 a^*_{45} & \mu^2 a^*_{55} & a_{56} \\ 0 & 0 & 0 & a_{46} & a_{56} & a^*_{66} \end{vmatrix}. \qquad (24)$$

So we have found the eigenvalue $\mu = 0$ with multiplicity 6, the other 4 eigenvalues of eq. (24) each with multiplicity 1 and the other 2 corresponding to eq. (23) with multiplicity 2.
Let us now study the solutions of these equations for the different possible values of a.

3.5. The case $a < 2$.

With the above expressions of $\tilde{h}_i$, we have that eq. (23) can be written as

$$\begin{vmatrix} 2\Gamma(2-a) & 5\Gamma(3-a)(a+5) & 5\Gamma(3-a)(a+5) \\ 5\Gamma(3-a)(a+5) & 6\Gamma(4-a)(a+4)(a+11) & 6\Gamma(4-a)(a+4)(a+11) \\ 5\Gamma(3-a)(a+5) & 6\Gamma(4-a)(a+4)(a+11) & 30\mu^2\Gamma(4-a)(a+4)(a+11) \end{vmatrix} = 0\,.$$

If we subtract the second column from the third one, we find

$$30\left(\mu^2-\frac{1}{5}\right)\Gamma(4-a)\Gamma(2-a)\Gamma(3-a)\,(a+4)(a+11)\,f(a)=0\,,$$

with $f(a)=13a^3+56a^2+137a+334$. We note now that $f'(a)>0\,\forall\, a$ so that $f(a)=0$ has only one real solution. But $\lim_{a\to-\infty} f(a)=-\infty$ and $f(-1)=240$ so that the only real root of $f(a)$ is less than -1 and for $a>-1$ we have $f(a)>0$. Consequently, our equation gives $\mu^2=\frac{1}{5}$.
Instead of this, eq. (24) with the above expressions of $\tilde{h}_i$ becomes

$$\begin{vmatrix}
1 & 3 & \frac{4}{3}(a+5) & 1 & \frac{2}{3}(a+5) & 0 \\
3 & 12 & \frac{20}{3}(a+5) & 4 & \frac{10}{3}(a+5) & 0 \\
\frac{4}{3}(a+5) & \frac{20}{3}(a+5) & \frac{16}{9}\frac{(a+4)(a+11)(3-a)}{2-a} & \frac{20}{9}(a+5) & \frac{8}{9}\frac{(a+4)(a+11)(3-a)}{2-a} & 0 \\
1 & 4 & \frac{20}{9}(a+5) & 4\mu^2 & \frac{10}{3}(a+5)\mu^2 & -\frac{2}{9}(a+5) \\
\frac{2}{3}(a+5) & \frac{10}{3}(a+5) & \frac{8}{9}\frac{(a+4)(a+11)(3-a)}{2-a} & \frac{10}{3}(a+5)\mu^2 & \frac{4}{3}\frac{(a+4)(a+11)(3-a)}{2-a}\mu^2 & -\frac{4}{45}\frac{(a+4)(a+11)(3-a)}{2-a} \\
0 & 0 & 0 & -\frac{2}{9}(a+5) & -\frac{4}{45}\frac{(a+4)(a+11)(3-a)}{2-a} & \frac{1}{45}\frac{(a+4)(a+11)(3-a)}{2-a}
\end{vmatrix}=0\,.$$

This equation has solutions $\mu^2=\frac{1}{3}$ and $\mu^2=\frac{3}{5}$. (We have to take into account that $2a^3+10a^2+18a+23>0$ for $a\geq-1$ and that $13a^3+56a^2+137a+334>0$ for $a\geq-1$).

3.6. The case $a=2$.

We note that $\tilde{h}_1=3\tilde{h}_2$ and $\tilde{h}_4=\frac{10}{3}\tilde{h}_5$ so that we can add to the third column of eq. (23) the second one multiplied by $-\frac{1}{5}$ and we find $\mu^2-\frac{1}{5}$ multiplied by a term proportional (with positive coefficient) to

$$\begin{vmatrix}
-4\ln\gamma+1 & \frac{70}{3} \\
\frac{70}{3} & 56
\end{vmatrix}$$

and the limit of this determinant is 56. So we have found also in this case the eigenvalue $\mu^2 = \frac{1}{5}$.
Let us consider now eq. (24); with the above expressions of $\tilde{h}_i$, and after some easy simplifications, it becomes

$$\begin{vmatrix}
-\ln\gamma & -3\ln\gamma+1 & \frac{4}{3}(a+5)\Gamma(3-a) & -\ln\gamma & \frac{2}{3}(a+5)\Gamma(3-a) & 0 \\
-3\ln\gamma+1 & -12\ln\gamma+7 & \frac{20}{3}(a+5)\Gamma(3-a) & -4\ln\gamma+1 & \frac{10}{3}(a+5)\Gamma(3-a) & 0 \\
\frac{4}{3}(a+5)\Gamma(3-a) & \frac{20}{3}(a+5)\Gamma(3-a) & \frac{16}{9}(a+4)(a+11)\Gamma(4-a) & \frac{20}{9}(a+5)\Gamma(3-a) & \frac{8}{9}(a+4)(a+11)\Gamma(4-a) & 0 \\
-\ln\gamma & -4\ln\gamma+1 & \frac{20}{9}(a+5)\Gamma(3-a) & (-4\ln\gamma+1)\mu^2 & \frac{10}{3}(a+5)\Gamma(3-a)\mu^2 & -\frac{2}{9}(a+5)\Gamma(3-a) \\
\frac{2}{3}(a+5)\Gamma(3-a) & \frac{10}{3}(a+5)\Gamma(3-a) & \frac{8}{9}(a+4)(a+11)\Gamma(4-a) & \frac{10}{3}(a+5)\Gamma(3-a)\mu^2 & \frac{4}{3}(a+4)(a+11)\Gamma(4-a)\mu^2 & -\frac{4}{45}(a+4)(a+11)\Gamma(4-a) \\
0 & 0 & 0 & -\frac{2}{9}(a+5)\Gamma(3-a) & -\frac{4}{45}(a+4)(a+11)\Gamma(4-a) & \frac{1}{45}(a+4)(a+11)\Gamma(4-a)
\end{vmatrix} = 0.$$

We divide now its columns 1, 2, 4 by $\ln\gamma$ and see that the eigenvalues μ are functions of $\frac{1}{\ln\gamma}$; to find the limiting terms of these functions we take the limit for $\gamma \to 0$, and obtain

$$\begin{vmatrix}
-1 & -3 & -1 \\
-3 & -12 & -4 \\
-1 & -4 & -4\mu^2
\end{vmatrix} = 0 \quad \text{and} \quad
\begin{vmatrix}
\frac{32}{3}13 & \frac{16}{3}13 & 0 \\
\frac{16}{3}13 & 8\cdot 13\mu^2 & -\frac{8}{15}13 \\
0 & -\frac{8}{15}13 & \frac{2}{15}13
\end{vmatrix} = 0;$$

the first one of these determinants is the minor of the previous one where the lines 1, 2, 4 intersect the column 1, 2, 4 and, similarly, the second one of these determinants is the minor of the previous one where the lines 3, 5, 6 intersect the column 3, 5, 6. So we find the eigenvalues $\mu^2 = \frac{1}{3}$ and $\mu^2 = \frac{3}{5}$, as in the previous case.

3.7. The case $2 < a < 3$.

We can add to the third column of eq. (23) the second one multiplied by $-\frac{1}{5}$ and we find $\mu^2 - \frac{1}{5}$ multiplied by a term proportional to

$$\begin{vmatrix} 2(a+2)R_{-a}\gamma^3 & \frac{20}{3}\Gamma(3-a)(a+5) \\ \frac{20}{3}\Gamma(3-a)(a+5) & \frac{8}{3}\Gamma(4-a)(a+4)(a+11) \end{vmatrix}$$

and this determinant is different from zero in the limit for $\gamma \to 0$. So we have found also in this case the eigenvalue $\mu^2 = \frac{1}{5}$.
Let us consider now eq. (24), with the above expressions of $\tilde{h}_i$; we multiply its first line by γ^{a+1}, its lines 2 and 4 by γ^{a+2}, its remaining lines by γ^5; after that, we multiply its columns 2 and 4 by γ, and its lines 3, 5, 6 by γ^2. So it becomes

$$\begin{vmatrix} R_{-a} & (a+1)R_{-a} & \frac{4}{3}\Gamma(3-a)(a+5)\gamma^{a-2} & R_{-a} & \frac{2}{3}\Gamma(3-a)(a+5)\gamma^{a-2} & 0 \\ (a+1)R_{-a} & (a+1)(a+2)R_{-a} & \frac{20}{3}\Gamma(3-a)(a+5)\gamma^{a-2} & (a+2)R_{-a} & -\frac{10}{3}\Gamma(3-a)(a+5)\gamma^{a-2} & 0 \\ \frac{4}{3}\Gamma(3-a)(a+5) & \frac{20}{3}\Gamma(3-a)(a+5) & \frac{16}{9}\Gamma(4-a)(a+4)(a+11) & \frac{20}{3}\Gamma(3-a)(a+5) & \frac{8}{9}\Gamma(4-a)(a+4)(a+11) & 0 \\ R_{-a} & (a+2)R_{-a} & \frac{20}{3}\Gamma(3-a)(a+5)\gamma^{a-2} & (a+2)R_{-a}\mu^2 & -\frac{10}{3}\Gamma(3-a)(a+5)\gamma^{a-2}\mu^2 & -\frac{2}{9}\Gamma(3-a)(a+5)\gamma^{a-2} \\ \frac{2}{3}\Gamma(3-a)(a+5) & -\frac{10}{3}\Gamma(3-a)(a+5) & \frac{8}{9}\Gamma(4-a)(a+4)(a+11) & -\frac{10}{3}\Gamma(3-a)(a+5)\mu^2 & \frac{4}{3}\Gamma(4-a)(a+4)(a+11)\mu^2 & -\frac{4}{45}\Gamma(4-a)(a+4)(a+11) \\ 0 & 0 & 0 & -\frac{2}{9}\Gamma(3-a)(a+5) & -\frac{4}{45}\Gamma(4-a)(a+4)(a+11) & \frac{1}{45}\Gamma(4-a)(a+4)(a+11) \end{vmatrix} = 0.$$

This equation shows that the eigenvalues are functions of γ^{a-2}; their leading terms can be obtained by taking the limit of the above determinant for $\gamma \to 0$. In this way it splits into the following two equations

$$\begin{vmatrix} 1 & a+1 & 1 \\ a+1 & (a+1)(a+2) & a+2 \\ 1 & a+2 & (a+2)\mu^2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} \frac{16}{9} & \frac{8}{9} & 0 \\ \frac{8}{9} & \frac{4}{3}\mu^2 & -\frac{4}{45} \\ 0 & -\frac{4}{45} & \frac{1}{45} \end{vmatrix} = 0; \qquad (25)$$

the first one of these determinants is the minor of the previous one where the lines 1, 2, 4 intersect the column 1, 2, 4 and, similarly, the second one of these determinants is the minor of the previous one where the lines 3, 5, 6 intersect the column 3, 5, 6. The first one of these equations has the solution $\mu^2 = \frac{1}{a+1}$ and we note that its limit for $a \to 2$ gives the solution $\mu^2 = \frac{1}{3}$ of the previous cases; the second equation has solutions $\mu^2 = \frac{3}{5}$, like in the previous cases.

3.8. The case $a = 3$.

We have that eq. (23) becomes

$$\begin{vmatrix} 10R_{-3}\,\gamma^{-6} & \frac{32}{3}\gamma^{-6}(1-5\ln\gamma) & -\frac{32}{3}\gamma^{-6}\ln\gamma \\ \frac{32}{3}\gamma^{-6}(1-5\ln\gamma) & \frac{16}{3}\cdot 49\,\gamma^{-7} & \frac{16}{15}\cdot 49\,\gamma^{-7} \\ -\frac{32}{3}\gamma^{-6}\ln\gamma & \frac{16}{15}\cdot 49\,\gamma^{-7} & \frac{16}{15}\cdot 49\,\gamma^{-7}\mu^2 \end{vmatrix} = 0\,,$$

from which

$$\mu^2 = \frac{1}{5} + \frac{32}{5}\,\frac{\gamma}{5\cdot 147 R_{-3} - 32\,\gamma(1-5\ln\gamma)^2}$$

whose limit for $\gamma \to 0$ is again $\mu^2 = \frac{1}{5}$ because $\lim_{\gamma\to 0}\gamma(1-5\ln\gamma)^2 = 0$.
Let us consider now eq. (24), with the above expressions of $\tilde{h}_i$; we multiply its first column by $\gamma^4\ln\gamma$, its second and 4^{th} column by $\gamma^5\ln\gamma$, the other 3 columns by γ^5; after that, we multiply its second and 4^{th} line by γ, and the lines 3, 5, 6 by γ^2; after that, we divide its lines 1, 2, 4 by $\ln\gamma$. So it becomes

$$\left|\begin{matrix} R_{-3} & 4R_{-3} & \frac{8}{3}\left(-4+\frac{1}{\ln\gamma}\right) \\ 4R_{-3} & 20R_{-3} & \frac{8}{3}\left(-20+\frac{9}{\ln\gamma}\right) \\ 8\gamma\ln\gamma(-4\ln\gamma+1) & \frac{8}{3}\ln\gamma(-20\gamma\ln\gamma+9\gamma) & \frac{16}{9}\cdot 7\cdot 14 \\ R_{-3} & 5R_{-3} & -\frac{8}{9}\left(20-\frac{3}{\ln\gamma}\right) \\ -\frac{16}{3}\gamma(\ln\gamma)^2 & \frac{16}{3}(-5\gamma\ln\gamma+\gamma)\ln\gamma & \frac{8}{9}\cdot 7\cdot 14 \\ 0 & 0 & 0 \end{matrix}\right.$$

$$\begin{vmatrix} R_{-3} & -\frac{16}{3} & 0 \\ 5R_{-3} & \frac{16}{5}\left(-5+\frac{1}{\ln\gamma}\right) & 0 \\ -\frac{8}{9}\ln\gamma(20\gamma\ln\gamma-3\gamma) & \frac{8}{9}\cdot 7\cdot 14 & 0 \\ 5R_{-3}\mu^2 & \frac{16}{3}\left(-5+\frac{1}{\ln\gamma}\right) & \frac{16}{9} \\ \frac{16}{3}\ln\gamma(-5\gamma\ln\gamma+\gamma)\mu^2 & \frac{4}{3}\cdot 7\cdot 14\mu^2 & -\frac{4}{45}\cdot 7\cdot 14 \\ \frac{16}{9}\gamma(\ln\gamma)^2 & -\frac{4}{45}\cdot 7\cdot 14 & \frac{1}{45}\cdot 7\cdot 14 \end{vmatrix} = 0\,.$$

Now, we have $\lim_{\gamma\to 0}\gamma^{\frac{1}{2}}\ln\gamma = 0$, from which $\lim_{\gamma\to 0}\gamma(\ln\gamma)^2 = 0$; it follows that at the limit for $\gamma\to 0$ our equation splits into $(25)_1$ calculated in $a=3$ and in $(25)_2$. So we obtain again the solutions $\mu^2=\frac{1}{4}$ and $\mu^2=\frac{3}{5}$, like in the previous cases.

3.9. The case $3<a<4$.

We have that eq. (23) gives

$$\mu^2 = \frac{1}{5} + \frac{12}{5}\,\frac{(a+1)^2(3-a)^2(R_{1-a})^2\gamma^{4-a}}{R_{-a}\Gamma(4-a)(a+2)(a+4)(a+11)a^2 - 12(a+1)^2(a+2)^2(R_{1-a})^2\gamma^{4-a}}$$

whose limit is $\mu^2=\frac{1}{5}$.
Let us consider now eq. (24), with the above expressions of $\tilde{h}_i$; we multiply its first column by γ^4 and the other columns by γ^5; after that, we multiply its first line by γ^{a-3}, its second and fourth lines by γ^{a-2}, and the other lines by γ^2. So it becomes

$$\begin{vmatrix} R_{-a} & (a+1)R_{-a} & 2\frac{(a+1)^2}{a-3}R_{-1-a} \\ (a+1)R_{-a} & (a+1)(a+2)R_{-a} & 2\frac{(a+1)^2(a+2)}{a-3}R_{-1-a} \\ 2\frac{(a+1)^2}{a-3}R_{-1-a}\gamma^{4-a} & 2\frac{(a+1)^2(a+2)}{a-3}R_{-1-a}\gamma^{4-a} & \frac{16}{9}(a+4)(a+11)\Gamma(4-a) \\ R_{-a} & (a+2)R_{-a} & \frac{2}{3}\frac{(a+1)(3a+11)}{a-3}R_{-1-a} \\ 4\frac{a+1}{a-3}R_{-1-a}\gamma^{4-a} & 4\frac{(a+1)(a+2)}{a-3}R_{-1-a}\gamma^{4-a} & \frac{8}{9}(a+4)(a+11)\Gamma(4-a) \\ 0 & 0 & 0 \end{vmatrix}$$

$$
\left.\begin{array}{ccc}
R_{-a} & 4\,\frac{a+1}{a-3}\,R_{-1-a} & 0 \\
(a+2)\,R_{-a} & 4\,\frac{(a+1)(a+2)}{a-3}\,R_{-1-a} & 0 \\
\frac{2}{3}\,\frac{(a+1)(3a+11)}{a-3}\,R_{-1-a}\,\gamma^{4-a} & \frac{8}{9}\,(a+4)(a+11)\,\Gamma(4-a) & 0 \\
(a+2)\,R_{-a}\,\mu^2 & 4\,\frac{(a+1)(a+2)}{a-3}\,R_{-1-a}\,\mu^2 & -\frac{4}{3}\,\frac{a+1}{a-3}\,R_{-1-a} \\
4\,\frac{(a+1)(a+2)}{a-3}\,R_{-1-a}\,\mu^2\,\gamma^{4-a} & \frac{4}{3}\,(a+4)(a+11)\,\Gamma(4-a)\,\mu^2 & -\frac{4}{45}\,(a+4)(a+11)\,\Gamma(4-a) \\
-\frac{4}{3}\,\frac{a+1}{a-3}\,R_{-1-a}\,\gamma^{4-a} & -\frac{4}{45}\,(a+4)(a+11)\,\Gamma(4-a) & \frac{1}{45}\,(a+4)(a+11)\,\Gamma(4-a)
\end{array}\right| = 0\,.
$$

This equation shows that the eigenvalues are functions of γ^{4-a}; their leading terms can be obtained by taking the limit of the above determinant for $\gamma \to 0$. It follows that at the limit for $\gamma \to 0$ our equation splits into (25). So we obtain again the solutions $\mu^2 = \frac{1}{a+1}$ and $\mu^2 = \frac{3}{5}$, like in the previous cases.

3.10. The case $a = 4$.

We have that eq. (23) gives

$$
\mu^2 = \frac{1}{5} + \frac{1}{5}\,\frac{125\,(R_{-3})^2 - 192 R_{-4} + 25 (R_{-3})^2 (\ln\gamma)^{-1}}{48 R_{-4}(-5\ln\gamma + 1) - 900 (R_{-3})^2}
$$

whose limit is $\mu^2 = \frac{1}{5}$.
Let us consider now eq. (24), with the above expressions of $\tilde{h}_i$; we multiply its first column by γ^5 and the other columns by γ^6; after that, we multiply the second and fourth line by γ and the lines 3, 5, 6 by $\frac{\gamma}{\ln\gamma}$. So it becomes

$$
\left|\begin{array}{ccc}
R_{-4} & 5R_{-4} & 50R_{-5} \\
5R_{-4} & 30R_{-4} & 300R_{-5} \\
50R_{-5}\,\frac{1}{\ln\gamma} & 300R_{-5}\,\frac{1}{\ln\gamma} & -\frac{40}{3}\left(16 - \frac{6}{\ln\gamma}\right) \\
R_{-4} & 6R_{-4} & \frac{230}{3} \\
20R_{-5}\,\frac{1}{\ln\gamma} & 120R_{-5}\,\frac{1}{\ln\gamma} & -\frac{40}{3}\left(8 - \frac{6}{5\ln\gamma}\right) \\
0 & 0 & 0
\end{array}\right.
$$

$$\begin{array}{ccc|}
R_{-4} & 20R_{-5} & 0 \\
6R_{-4} & 120R_{-5} & 0 \\
\frac{230}{3}\,\frac{1}{\ln\gamma} & -\frac{40}{3}\left(8-\frac{6}{5\ln\gamma}\right) & 0 \\
6R_{-4}\,\mu^2 & 120R_{-5}\,\mu^2 & -\frac{20}{3}R_{-5} \\
120R_{-5}\,\mu^2\,\frac{1}{\ln\gamma} & 32\left(-5+\frac{1}{\ln\gamma}\right)\mu^2 & \frac{32}{3} \\
-\frac{20}{3}R_{-5}\,\frac{1}{\ln\gamma} & \frac{32}{3} & -\frac{8}{3}
\end{array} = 0 \,.$$

This equation shows that the eigenvalues are functions of $\frac{1}{\ln\gamma}$; their leading terms can be obtained by taking the limit of the above determinant for $\gamma \to 0$. It follows that at the limit for $\gamma \to 0$ our equation splits into (25) calculated in $a = 4$. So we obtain again the solutions $\mu^2 = \frac{1}{a+1}$ and $\mu^2 = \frac{3}{5}$, like in the previous cases. But in this case we note that $\mu^2 = \frac{1}{a+1}$ becomes $\mu^2 = \frac{1}{5}$, the same value which we obtained from eq. (23). So in the present case we have the eigenvalues $\mu = 0$ with multiplicity 6, and $\mu^2 = \frac{1}{5}$ with multiplicity 3 and $\mu^2 = \frac{3}{5}$ with multiplicity 1.

3.11. The case $a > 4$.

We have that eq. (23) gives the above reported eq. $(19)_2$. Through numerical calculus we obtain the graphics in picture 1 which shows that $0 < \mu^2 \leq 1$.
Let us consider now eq. (24), with the above expressions of $\tilde{h}_i$; we divide its first line by $\gamma^{-a-1} R_{-a}$ and the other lines by $\gamma^{-a-2} R_{-a}$; after that, we multiply all the columns, except the first one, by γ; finally, we use the identity $R_{2-a} = \frac{a-1}{a-4} R_{-a}$ and exchange the third with the fourth line and the third with the fourth column. So it becomes the above reported eq. (20).
Through numerical calculus we obtain that its solutions are real and do not exceed the speed of light.

A. The R_k numbers

Since the R_k numbers are heavily present in our limits, it is important to see that their expression is

$$R_k = \frac{\sqrt{\pi}}{4} \quad \frac{\Gamma\left(-1-\frac{k}{2}\right)}{\Gamma\left(\frac{1-k}{2}\right)} \,. \tag{26}$$

To prove this relation, we consider firstly the function $f(\gamma, y) = e^{-\gamma y} - 1 - \gamma y$; since its derivative with respect to γ is negative $\forall\, y \geq 1$ and, moreover, $f(0, y) =$

0 we obtain that $e^{-\gamma y} \leq 1 + \gamma y$.
Let us multiply now the relation $1 < e^{-\gamma y} \leq 1 + \gamma y$ times $\sqrt{y^2-1}\, y^k$ and integrate in dy:

$$\int_1^{+\infty} \sqrt{y^2-1}\, y^k\, dy < \int_1^{+\infty} e^{-\gamma y} \sqrt{y^2-1}\, y^k\, dy < \int_1^{+\infty} \sqrt{y^2-1}\, y^k\, dy + \\ + \gamma \int_1^{+\infty} \sqrt{y^2-1}\, y^{k+1}\, dy .$$

If $k < -3$ all integrals in the left hand side and right hand side of this relation are finite numbers and we can take the limit of this relation for $\gamma \to 0$; we see that the left hand side and the right hand side have the same limit, so that

$$R_k = \int_1^{+\infty} \sqrt{y^2-1}\, y^k\, dy . \tag{27}$$

Since this integral is known in literature, we can use its expression and find (26). But, for the sake of completeness, we include here the proof of how (27) leads to the expression (26). To this end, let us use in eq. (27) the change of variables $y = \cosh x$ and find

$$R_k = \int_0^{+\infty} \sinh^2 x \cosh^k x\, dx . \tag{28}$$

Now we use the definition of the Beta function

$$B(t,s) = \int_0^1 (1-x)^{t-1} x^{s-1}\, dx ,$$

and use for it the change of variables $x = \cosh^{-2} q$ so that it becomes

$$B(t,s) = -2 \int_{+\infty}^0 \left(\frac{\cosh^2 q - 1}{\cosh^2 q} \right)^{t-1} (\cosh q)^{-2s-1} \sinh q\, dq = \\ = 2 \int_0^{+\infty} (\cosh q)^{-2t-2s+1} (\sinh q)^{2t-1}\, dq .$$

By comparing this result with (28), we find

$$R_k = \frac{1}{2} B\left(\frac{3}{2}, -\frac{k}{2} - 1 \right) .$$

But the Beta function satisfies the condition

$$B(t,s) = \frac{\Gamma(t)\,\Gamma(s)}{\Gamma(t+s)} .$$

so that

$$R_k = \frac{1}{2}\frac{\Gamma\left(-\frac{k}{2}-1\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(-\frac{k}{2}+\frac{1}{2}\right)} = \frac{1}{4}\frac{\Gamma\left(-\frac{k}{2}-1\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(-\frac{k}{2}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{4}\frac{\Gamma\left(\frac{-k-2}{2}\right)}{\Gamma\left(\frac{-k+1}{2}\right)},$$

where we have used the known result $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. In this way we have proved eq. (26) in the case $k<-3$.
There remains to prove it also in the case $-3\leq k<-2$.
To this end, let us consider the above eq. $(4)_2$ and integrate it by parts as follows

$$\begin{aligned}\bar{R}_k &= \int_1^{+\infty} e^{-\gamma y}\sqrt{y^2-1}\,y^k\,dy = \frac{1}{3}\int_1^{+\infty} y^{k-1}e^{-\gamma y}\left[\left(y^2-1\right)^{\frac{3}{2}}\right]' dy = \\ &= \frac{1}{3}\left|y^{k-1}e^{-\gamma y}\left(y^2-1\right)^{\frac{3}{2}}\right|_1^{+\infty} - \frac{1}{3}\int_1^{+\infty}\left(y^2-1\right)^{\frac{3}{2}}\left[(k-1)y^{k-2}e^{-\gamma y}-\gamma y^{k-1}e^{-\gamma y}\right]dy = \\ &= -\frac{1}{3}\int_1^{+\infty}\left(y^2-1\right)^{\frac{1}{2}}e^{-\gamma y}\left[(k-1)y^k-\gamma y^{k+1}-(k-1)y^{k-2}+\gamma y^{k-1}\right]dy = \\ &= -\frac{k-1}{3}\bar{R}_k+\frac{k-1}{3}\bar{R}_{k-2}+\frac{\gamma}{3}\bar{R}_{k+1}-\frac{\gamma}{3}\bar{R}_{k-1},\end{aligned} \tag{29}$$

from which we deduce that

$$(k+2)\bar{R}_k = (k-1)\bar{R}_{k-2}+\gamma\bar{R}_{k+1}-\gamma\bar{R}_{k-1}. \tag{30}$$

Now, when $k<-2$, all the terms $\bar{R}_k$, $\bar{R}_{k-2}$, $\bar{R}_{k-1}$ are convergent, as we have seen above, while $\gamma\bar{R}_{k+1}$ has limit zero (we omit the proof for the sake of brevity); so we can take the limit of the above relation for $\gamma\to 0$ and obtain

$$(k+2)R_k = (k-1)R_{k-2} \quad \text{or} \quad R_k = \frac{k-1}{k+2}R_{k-2} \quad \text{for} \quad k<-2. \tag{31}$$

Since we have to consider only the case $-3\leq k<-2$, it follows that $k-2<-4$. So we can use (26) for R_{k-2}; in this way, $(31)_2$ gives

$$R_k = \frac{k-1}{k+2}\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(-\frac{k}{2}\right)}{\Gamma\left(\frac{3-k}{2}\right)} = \frac{k-1}{k+2}\frac{\sqrt{\pi}}{4}\frac{\Gamma\left(-\frac{k}{2}-1\right)}{\Gamma\left(\frac{1-k}{2}\right)}\frac{-\frac{k}{2}-1}{\frac{1-k}{2}}.$$

So we find again (26) but $\forall k<-2$.
If k is an integer number, (26) assumes simpler expressions

$$R_{-2n} = \frac{(2n-2)!!}{2n-2}\frac{2n+1}{(2n+1)!!} \quad , \quad R_{-(2n+1)} = \frac{(2n-1)!!}{2n-1}\frac{1}{(2n)!!}\frac{\pi}{2}, \tag{32}$$

thanks to the known result

$$\Gamma\left(n+\frac{1}{2}\right) = \left(\frac{1}{2}\right)^n\frac{(2n+1)!!}{2n+1}\sqrt{\pi}.$$

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