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CANONICAL REDUCTION FOR QUADRATIC QUOTIENTS OF THE REES ALGEBRA

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In this paper, we characterize when a quadratic quotient of the Rees algebra, obtained starting with a one-dimensional local ring, has a canonical reduction, generalizing a similar result obtained for Nagata idealization by Rahimi.

Introduction

Let (R,m) be a one-dimensional local ring; we also assume that R is Cohen-Macaulay (briefly C-M). In the last years many authors have studied particular classes of rings in this setting, trying to generalize the notion of Gorenstein and Almost Gorenstein rings. The first concept is so prominent that does not need a presentation; the notion of Almost Gorenstein was introduced by Barucci and Fröberg for one-dimensional analytically unramified rings (see [3]) and it was recently generalized for any C-M one-dimensional local ring possessing a canonical ideal (see [11]). Successively, other classes were introduced and compared to these two. For example, in [5], the authors introduced the notion of 2-almost Gorenstein local rings (2-AGL in short), which are rings where the Sally module of the canonical ideal with respect to one of its reduction has rank

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2. We can find another possible generalization of Gorenstein rings in [13] where the authors give the notion of nearly Gorenstein ring and develope this theory. Comparing these two classes in the one-dimensional case (see [13]), we find that 2-AGL rings are not necessarily nearly Gorenstein and vice versa. Another reasonable generalization of both Gorenstein rings and almost Gorenstein rings is the notion of Generalized Gorenstein local rings (GGL for short), introduced and studied in [10]. Another class was recently introduced and studied in [17], where the author focused on those C-M one-dimensional local rings with the property of canonical reduction, i.e. rings possessing a canonical ideal that is a reduction of the maximal ideal. In the same paper it is studied the behaviour of this class of rings under the idealization, a classical construction introduced by Nagata; in particular it is showed that if R has a canonical reduction, this property is verified for every idealization of R with respect to an ideal satisfying a specific condition; the converse is also true (see Theorem 4.3 of [17]). This fact motivated the problem to understand when quadratic quotients of the Rees algebra have this property. This family of rings is obtained from the Rees algebra $\mathscr{R}_+(I)$ associated to the ring R with respect to an ideal I of R, by quotient with the contraction of the ideal generated by a monic polynomial $t^2+\alpha t+\beta \in R[t]$ and denoted by $R(I)_{\alpha,\beta}$; it was studied in [1] to provide a unified approach to idealization, with respect to an ideal, and amalgamated duplication, another construction introduced by D'Anna and Fontana in [6].

The main result of this paper is a complete answer to the problem stated above (see Theorem 3.6), that generalizes Theorem 4.3 in [17]. To pursue this result, we first approach the problem at numerical semigroup level, studying when the numerical duplication, a construction introduced in [7], has a canonical reduction. We solve this problem in Theorem 2.4 and then we use the ideas developed in this context to obtain similar results in the ring case.

In the first part of this paper, we recall some basic facts on numerical semigroups, in particular we recall some helpful results about numerical duplication of a numerical semigroup S with respect to an ideal E. We show that the canonical ideal of the numerical duplication can be expressed in a "duplication" form and then, we demonstrate that a numerical semigroup S has a canonical reduction if and only if the same property is verified by the numerical duplication of S with respect to a particular ideal E. In the last part, we show that some results obtained in the numerical semigroup case can be generalized in the ring case. In particular we show that Theorem 3.12 of [17] can be generalized for any rings in our setting. Finally, we conclude with a result which allows us to say that Theorem 4.3 of [17] holds in a more general context.

Several computations for this paper were performed by using the GAP system [9] and, in particular, the NumericalSgps package [8].

1. Canonical reduction for numerical semigroups

We start recalling some basic facts on numerical semigroups that can be found in [3]. Let S be a numerical semigroup, i.e. a submonoid of $\mathbb{N}$ such that $|\mathbb{N}\backslash S| < \infty$; we will denote by $f(S) = \max(\mathbb{N}\backslash S)$ its Frobenius number and by $s_1 = \min S\backslash\{0\}$ its multiplicity. If there is no ambiguity about the semigroup S, for simplicity we will write f instead of $f(S)$. The elements of $\mathbb{N}\backslash S$ are called holes of S; in particular we call holes of the first type the set $H(S) = \{f - x \mid x \in S\}$ and holes of the second type the set $L(S) = \{x \in \mathbb{Z} \mid x \notin S$ and $f - x \notin S\}$. Trivially, the elements of $L(S)$ and $H(S)$ are holes and when $\mathbb{N} \neq S$, holds $\mathbb{N} = S \cup H(S) \cup L(S)$. A subset $I \subseteq \mathbb{Z}$ is called relative ideal of S if $i + s \in I$ for all $i \in I$, $s \in S$ and there exists $s \in S$ such that $I + s \subseteq S$. A relative ideal which is contained in S is called integral ideal or simply, ideal of S. For a relative ideal I, we denote by $\mu(I) = \min(I)$ its multiplicity. A relative ideal K' is called canonical ideal if and only if $K' - (K' - I) = I$, for every relative ideal I of S, where in general, with $I - J$ we denote the set $\{x \in \mathbb{Z} \mid x + J \subseteq I\}$. One can prove that the set $K = \{f - x \mid x \in \mathbb{Z}\backslash S\}$ is a canonical ideal and that $K = S \cup L(S)$. We will denote by M the set $\{x \in S \mid x > 0\} = S\backslash\{0\}$; M is an ideal and it is called maximal ideal of S. Finally, we will denote by $T(S)$ the set $(M - M)\backslash S$ and we will call type of S the cardinality of $T(S)$. It is well known that $T(S) \subseteq L(S) \cup \{f\}$; when $L(S) \subseteq T(S)$, the semigroup S is called almost symmetric.
Our first aim is to study the property of canonical reduction for numerical semigroups. In particular, we focus our attention on the existence of a canonical reduction and we will give a characterization, that will allow to easily recover for numerical semigroups the implication almost symmetric implies existence of canonical reduction, proved for rings in [17].

Definition 1.1. Let I, J be relative ideals of S, we will say that $I \sim J$ if and only if there exists $z \in \mathbb{Z}$ such that $I = z + J$.

The following lemma is well known, but we give its proof for convenience of the reader.

Lemma 1.2. *For every relative ideal I of S, there exists I_0 relative ideal of S such that $I \sim I_0$ and $S \subseteq I_0 \subseteq \mathbb{N}$*

Proof. We set $I_0 = I - i_0$, where $i_0 = \mu(I)$. Then for every $x \in I$ we have $x \geq i_0$ and therefore $I_0 \subseteq \mathbb{N}$. Moreover, $I_0 + S = I + S - i_0 \subseteq I - i_0 = I_0$, thus I_0 is a relative ideal of S and by definition, its multiplicity is zero. Finally, for all $s \in S$ we have $s = 0 + s \in I_0$, so $S \subseteq I_0$.

In general, we denote by rI, where $r \in \mathbb{N}^+$, the sum $I + \ldots + I$, r-times.

Definition 1.3. Let I be a relative ideal of S, we denote by $r(I)$ the reduction number of I, i.e. the smallest integer $r \in \mathbb{N}$ such that $(r+1)I = rI + \mu(I)$.

For ideals belonging to the same equivalence class, with respect to $\sim$, we have the following result.

Lemma 1.4. *If I,J are relative ideals of S, such that $I \sim J$, then $r(I) = r(J)$.*

Proof. By the previous lemma, into the class of I, and thus of J, there is a relative ideal I_0 with multiplicity zero such that $S \subseteq I_0 \subseteq \mathbb{N}$; in particular $I_0 = J - \mu(J) = I - \mu(I)$. Therefore, if $r = r(I)$, $(r+1)I = rI + \mu(I)$ is equivalent to $(r+1)I_0 = rI_0$ that in turn is equivalent to $(r+1)J = rJ + \mu(J)$, so we have $r(I) \leq r(J)$ and the other inequality can be proved in the same way.

Definition 1.5. Let I,J be relative ideals of S, with $I \subseteq J$, we will say that I is a reduction of J, if there exists $h \in \mathbb{N}$ such that $(h+1)J = I + hJ$.

Proposition 1.6. *A relative ideal $I \subseteq M$ is a reduction of M if and only if $s_1 = \mu(M) \in I$. In particular $s_1 = \mu(I)$.*

Proof. $\Leftarrow)$ If $h = r(M)$, we have:

$$(h+1)M \supseteq I + hM \supseteq s_1 + hM = (h+1)M$$

therefore I is a reduction of M.
$\Rightarrow)$ If I is a reduction of M, then there exists $h \in \mathbb{N}$, such that $I + hM = (h+1)M$. Since $(h+1)s_1 = \mu((h+1)M) = \mu(I + hM)$, and $hs_1 = \mu(hM)$, we have that $s_1 \in I$ and $s_1 = \mu(I)$.

Definition 1.7. We will say that a numerical semigroup S has a canonical reduction, if there exists a canonical ideal K' that is a reducion of the maximal ideal M.

By the previous result, we can give a characterization of a canonical reduction.

Proposition 1.8. *A canonical ideal $K + z$, with $z \in \mathbb{Z}$, is a canonical reduction of S, if and only if $z = s_1$ and $K + s_1 \subseteq M$.*

Proof. $K+z$ is a canonical reduction if and only if $s_1 = \mu(K+z)$ and $K+z \subseteq M$. Since $0 \in K$ and z is the smallest element of $K+z$, we have that $z = s_1$; the thesis follows immediately.

Now, we can see that a canonical reduction always exists in an almost symmetric semigroup. The following proposition is the numerical counterpart of Proposition 3.1 of [17].

Proposition 1.9. *If S is almost symmetric, it has a canonical reduction.*

Proof. It is well known that, if S is almost symmetric, then $M = K + M$. Therefore, for all $x \in K$, we have $x \in M - M$. Since $s_1 \in M$, it follows that, $x + s_1 \in M$; so $K + s_1 \subseteq M$.

2. Canonical reduction for numerical duplication

Numerical duplication is a construction, defined in [7], that from a numerical semigroup S, a semigroup ideal $E \subseteq S$ and any odd integer b of S, produces a new numerical semigroup. First of all, we define Frobenius number $f(E)$ of an ideal E of S as $\max\{\mathbb{Z} \setminus E\}$. We also observe that, in the case $E \subseteq S$, the inequality $f(E) \geq f$ always holds.

Definition 2.1. With these notations, we define the numerical duplication of S with respect to E and b, as the numerical semigroup

$$S \bowtie^b E := 2 \cdot S \cup (2 \cdot E + b)$$

where with $2 \cdot S$ we indicate the set $\{2s \,|\, s \in S\}$ and with $2 \cdot E$ we indicate the set $\{2t \,|\, t \in E\}$.

To prove that $S \bowtie^b E$ is a numerical semigroup, we observe that $0 = 2 \cdot 0 \in S \bowtie^b E$ and, since $f(E) \geq f$, every integer $n > 2f(E) + b$ is in $S \bowtie^b E$. Moreover, as $b \in S$ and $E \subseteq S$, then $S \bowtie^b E$ is closed with respect to the sum.
Let $E \subseteq S$ be an ideal of S and let $e = f - f(E)$. We indicate with $\tilde{E} = E + e$, the translate of E by e; obviously $f(\tilde{E}) = f$. Let K be the standard canonical ideal of S. Since $S \bowtie^b E$ is a numerical semigroup, the canonical ideal of the numerical duplication is defined as follows:

$$K(S \bowtie^b E) = \left\{ z \in \mathbb{Z} \,|\, f(S \bowtie^b E) - z \notin S \bowtie^b E \right\}.$$

In [7] it is proved that the Frobenius number $f(S \bowtie^b E) = 2f(E) + b$; in particular it holds:

$$z \in K(S \bowtie^b E) \Longleftrightarrow z = 2f(E) + b - a \ \text{ with } \begin{cases} \frac{a}{2} \notin S & a \text{ even} \\ \frac{a-b}{2} \notin E & a \text{ odd} \end{cases}$$

We want to express the canonical ideal of $S \bowtie^b E$ as a "duplication" itself. This way of describing the canonical ideal of $S \bowtie^b E$ will be helpful to understand when $S \bowtie^b E$ has a canonical reduction.

The following lemma could be deduced by Proposition 2.1 of [2], but here we give a direct and simple proof in the semigroup case.

Lemma 2.2. *If E is an ideal of a numerical semigroup S and $b \in S$ is an odd integer, then:*

$$K(S \bowtie^b E) = [(K-E)+f(E)-f] \bowtie^b [K+f(E)-f]$$

Proof. Let $x \in K(S \bowtie^b E)$. We have two cases:

1. If x is even, we know that $x = 2f(E)+b-a$, with $\frac{a-b}{2} \notin E$; then $\frac{x}{2} = f(E)+\frac{b-a}{2} = f(E)+f-\frac{a-b}{2}-f$ and since $\frac{a-b}{2} \notin E$, then $f-\frac{a-b}{2} \in K-E$; so $\frac{x}{2} \in (K-E)+f(E)-f$.
2. If x is odd, we know that $x = 2f(E)+b-a$, with $\frac{a}{2} \notin S$; therefore $\frac{x-b}{2} = f(E)-\frac{a}{2} = f(E)+f-\frac{a}{2}-f$ and since $\frac{a}{2} \notin S$, then $f-\frac{a}{2} \in K$.

In every case, we have proved that $x \in [(K-E)+f(E)-f] \bowtie^b [K+f(E)-f]$.
Conversely:
If $x \in [(K-E)+f(E)-f] \bowtie^b [K+f(E)-f]$, then:

1. if x is even, $x = 2y$ with $y \in (K-E)+f(E)-f$; i.e. $y+f-f(E) \in K-E$, therefore $f-(y+f-f(E)) = f(E)-y \notin E$.
2. If x is odd, $x = 2y+b$ with $y \in K+f(E)-f$; i.e. $y+f-f(E) \in K$, therefore $f-(y+f-f(E)) = f(E)-y \notin S$.

To prove $x \in K(S \bowtie^b E)$, we have to show that $2f(E)+b-x \notin S \bowtie^b E$.

1. If x is even, $2f(E)+b-x = 2f(E)+b-2y$; therefore, if we observe that $\frac{2f(E)+b-2y-b}{2} = f(E)-y \notin E$, we have $2f(E)+b-x \notin S \bowtie^b E$.
2. If x is odd, $2f(E)+b-x = 2f(E)+b-2y-b = 2f(E)-2y = 2(f(E)-y)$; since $f(E)-y \notin S$, we have that $2f(E)+b-x \notin S \bowtie^b E$.

In every case, we have $x \in K(S \bowtie^b E)$.

Proposition 2.3. *If E is an ideal of S such that $K+s_1 \subseteq E$, then $f(E)$ must be equal to f or $f+s_1$.*

Proof. Since $K = \{f-x \mid x \notin S\}$ and $1,\dots,s_1-1 \notin S$, then

$$\{f-1,\dots,f-s_1+1\} \subseteq K;$$

therefore $K+s_1 \supseteq \{f+1,\dots,f+s_1-1\}$. As $K+s_1 \subseteq E \subseteq S$, we have $f \le f(E) \le f+s_1$. Since $E \supseteq \{f+1,\dots,f+s_1-1\}$, then either $f(E) = f$ or $f(E) = f+s_1$.

Theorem 2.4. *For a numerical semigroup S, the following are equivalent.*

1. *S has a canonical reduction $K+s_1$.*

2. *$S \bowtie^b E$ has a canonical reduction for every ideal E such that $K+s_1 \subseteq E \subseteq S$.*

3. *$S \bowtie^b M$ has a canonical reduction.*

Proof. $1) \Rightarrow 2)$ From Proposition 1.2 and Lemma 2.2, it is sufficient to prove that

$$[(K-E)+f(E)-f] \bowtie^b [K+f(E)-f]+2s_1 \subseteq (S \bowtie^b E)\setminus\{0\}.$$

First of all we observe that, under our assumptions, $K-E \subseteq S-s_1$; in fact, since $K+s_1 \subseteq E$, $K-(K+s_1) \supseteq K-E$; but $K-(K+s_1) = (K-K)-s_1 = S-s_1$, so $K-E \subseteq S-s_1$.
Let $x \in [(K-E)+f(E)-f] \bowtie^b [K+f(E)-f]$, we will prove that $x+2s_1 \in S \bowtie^b E$. We have two cases:

(a) If x is even, then $x = 2y$, with $y \in (K-E)+f(E)-f \subseteq S-s_1+f(E)-f$.

(b) If x is odd, then $x = 2y+b$, with $y \in K+f(E)-f$.

To get the thesis, it is sufficient to prove that in the even case, $y+s_1 \in S$ or $S-s_1+f(E)-f+s_1 = S+f(E)-f \subseteq S$, whereas in the odd case we must prove that $y+s_1 \in E$, or rather it is sufficient to prove that $K+f(E)-f+s_1 \subseteq E$. By previous proposition, since $f(E)$ can assume only the value of f or $f+s_1$, these conditions are verified. In effect, if $f(E) = f$, the two conditions become: $S \subseteq S$ and $K+s_1 \subseteq E$. If $f(E) = f+s_1$, they become: $S+s_1 \subseteq S$ e $K+2s_1 \subseteq E$. Therefore, for every $x \in [(K-E)+f(E)-f] \bowtie^b [K+f(E)-f]$ we have that $x+2s_1 \in S \bowtie^b E$ and then the thesis.
$2) \Rightarrow 3)$ Is trivial.
$3) \Rightarrow 1)$ By Proposition 1.2 $K(S \bowtie^b M)+2s_1 \subseteq (S \bowtie^b M)\setminus\{0\}$, i.e. for all $z \in K(S \bowtie^b M)$, $z+2s_1 \in (S \bowtie^b M)\setminus\{0\}$; we must prove that $K+s_1 \subseteq M$.
Let $y \in K$, i.e. $f-y \notin S$; we observe that $2(f-y)+b \notin 2\cdot S$, because is odd, and $2(f-y)+b \notin 2\cdot M+b$, otherwise we have $f-y \in M$. This means that $2(f-y)+b = 2f+b-2y \notin S \bowtie^b M$, thus $2y \in K(S \bowtie^b M)$; by hypothesis $2y+2s_1 \in \left(S \bowtie^b M\right)\setminus\{0\}$, and because it is even, $2y+2s_1 \in (2\cdot S)\setminus\{0\}$; therefore $y+s_1 \in M$. This is verified for all $y \in K$, then we have the thesis.

Remark 2.5. If we find one relative ideal E such that $K+s_1 \subseteq E \subseteq S$ this implies, by Proposition 1.2, that S has a canonical reduction, so the condition 2 of the previous theorem holds for every integral ideal E containing $K+s_1$, in particular for M.

The following example shows that it is possible that $S \bowtie^b E$ has a canonical reduction if $K+s_1 \nsubseteq E$.

Example 2.6. Let

$$S = \langle 7,9,11 \rangle$$

be the numerical semigroup generated by $7,9,11$, by using the software [9], we can compute that $K+s_1 = \langle 7,9 \rangle$, so S has a canonical reduction. Moreover, the integral ideal $E = \langle 11 \rangle$ does not contain $K+s_1$, however $S \bowtie^{23} E$ has a canonical reduction.

3. Canonical reduction for quadratic quotient of the Rees algebra

In this section, we will work with a one-dimensional C-M local ring R with maximal ideal $\underline{m}$, we also assume that the residue field is infinite. We suppose that the integral closure $\overline{R}$ is finitely generated as $R-$module. Thanks to this hypothesis, (see Beweis 4. of Satz 3.6 in [14]) there exists a canonical ideal ω of R such that $R \subseteq \omega \subseteq \overline{R}$.
Let $\mathscr{R}_+(I) = R \oplus It \oplus I^2t^2 \oplus \ldots \subseteq R[t]$ be the Rees algebra associated with the ring R with respect to one of its ideal I. In [1], it has been studied a particular family of rings obtained from the quotient of the Rees algebra with a generic monic polynomial $t^2 + \alpha t + \beta \in R[t]$

$$R_{\alpha,\beta}(I) = \frac{\mathscr{R}_+(I)}{(t^2 + \alpha t + \beta)R[t] \bigcap \mathscr{R}_+(I)} \cong R \oplus It$$

where the last isomorphism is of $R-$modules. In the same paper, the authors showed that, when the monic polymonial is $t^2 \in R[t]$, we have the following isomorphism of rings: $R_{0,0}(I) \cong R \ltimes I$, where $R \ltimes I$ is the Nagata's idealization with respect to the ideal I. We can assume that I is a regular ideal so, by Proposition 2.7 of [1], $R(I)_{\alpha,\beta}$ is also a local C-M ring of dimension one. It is also showed that many relevant properties (as Gorensteinness or type) are independent of the α, β.
In [1] it is proved that if $R = k[[S]] = k[[X^s | s \in S]]$, where S is a numerical semigroup, $I = (X^{e_1}, \ldots, X^{e_h})$ is a monomial ideal of R and $\beta = X^b$, with b an odd integer of S, then $R(I)_{0,\beta} \cong k\left[\left[S \bowtie^b E\right]\right]$, where E is the semigroup ideal generated by $e_1, \ldots, e_h$.
This last fact clarifies why it is natural to try to extend to quadratic quotients of one-dimensional rings the properties obtained for numerical duplication. Hence, as in the second section we characterized the existence of a canonical reduction for numerical duplication expressed in terms of the canonical ideal, we look for a similar characterization in this setting. We start with the following key result.

Proposition 3.1. *R has a canonical reduction if and only if there exists $x \in \underline{m}$, reduction of the maximal ideal, such that $x\omega \subseteq \underline{m}$. If this is the case, every canonical reduction of R is of this form.*

Proof. Let x be a reduction of the maximal ideal $\underline{m}$ such that $x\omega \subseteq \underline{m}$. Since $R \subset \omega$, then $1 \in \omega$, so $(x) \subseteq x\omega \subseteq \underline{m}$. Finally, as x is a reduction of $\underline{m}$, it follows that $x\omega$ is also a reduction of $\underline{m}$, so R has a canonical reduction.
Conversely, let ω' be a canonical ideal of R that is a canonical reduction of the maximal ideal $\underline{m}$. From the characterization of canonical ideals of R, there exists x non zero-divisor of $Q(R)$ such that $\omega' = x\omega$. As 1 is a reduction of ω, by Proposition 3.3(b) of [17], x is a reduction of ω'. From that $(x) \subseteq \omega' \subseteq \underline{m}$, so x is a reduction of $\underline{m}$.

We consider now $(\omega : I)$ that is a fractional ideal of R, so by definition there exists a regular element x of R, such that $x(\omega : I) \subset R$. Let γR be a minimal reduction of $x(\omega : I)$, so we can consider $z = \gamma x^{-1}$ a minimal reduction of $(\omega : I)$.

Remark 3.2. Let $H \subset J \subset I$ be ideals of R, if H is a reduction of I, then H is also a reduction of J. In fact, by Corollary 1.2.5 of [12], H is a reduction of I if and only if $I \subseteq \overline{H}$. Since $J \subset I \subset \overline{H}$, it follows that H is also a reduction of J.

Remark 3.3. Let xR be a minimal reduction of $\underline{m}$, such that $x\omega \subseteq \underline{m}$. For every ideal I of R such that $x\omega \subseteq I$, we have

$$(\omega : I) \subseteq \omega : x\omega = x^{-1}(\omega : \omega) = x^{-1}R$$

so $x(\omega : I) \subseteq R$.
If $1 \in x(\omega : I)$, then $x\omega = I$, so $x(\omega : I) = (x\omega : I) = R$ and a reduction of $(x\omega : I)$ is $\gamma = 1$.
If $1 \notin x(\omega : I)$, then $(x\omega : I) \subseteq \underline{m}$. Therefore:

$$xR \subseteq (x\omega : I) \subseteq \underline{m}.$$

Since xR is a reduction of $\underline{m}$, by Remark 3.2, we obtain that $\gamma = x$ is a reduction of $x(\omega : I)$.

Lemma 3.4. *Let xR be a reduction of the maximal ideal $\underline{m}$. For all α, β and for all the ideals I of R such that $x\omega \subseteq I$, $xR(I)_{\alpha,\beta}$ is a reduction of the maximal ideal $\underline{m} + It$.*

Proof. For all $n \in \mathbb{N}$,

$$(\underline{m} + It)^n = \underline{m}^n + \underline{m}^{n-1}It$$

moreover, since xR is a reduction of $\underline{m}$, there exists $\overline{n} \in \mathbb{N}$ such that, for all $n \geq \overline{n}$, $x\underline{m}^n = \underline{m}^{n+1}$. Therefore, for all $n > \overline{n}$

$$x(\underline{m} + It)^n = x\underline{m}^n + x\underline{m}^{n-1}It = m^{n+1} + \underline{m}^n It = (\underline{m} + It)^{n+1}.$$

In section 2.1 of [2], the authors give an explicit description of a canonical ideal $\omega_{R(I)_{\alpha,\beta}}$ of $R(I)_{\alpha,\beta}$, showing that as R-modul it has the following form:

$$\omega_{R(I)_{\alpha,\beta}} \cong \frac{1}{z}(\omega : I) + \frac{1}{z}\omega$$

where zR is a minimal reduction of $(\omega : I)$.

Theorem 3.5. *If R has a canonical reduction $x\omega$, with xR reduction of $\underline{m}$, $R(I)_{\alpha,\beta}$ has also a canonical reduction for all α, β and for all the ideals I of R such that $x\omega \subseteq I$.*

Proof. By Proposition 3.1, to get the thesis it is sufficient to prove that there exists a reduction h of the maximal ideal of $R(I)_{\alpha,\beta}$ such that:

$$h(\frac{1}{z}(\omega : I) + \frac{1}{z}\omega t) \subseteq \underline{m} + It.$$

By Lemma 3.4, we can choose $xR(I)_{\alpha,\beta}$ as reduction of the maximal ideal $\underline{m} + It$. So the claim is:

$$x(\frac{1}{z}(\omega : I) + \frac{1}{z}\omega t) \subseteq \underline{m} + It.$$

Since $\frac{x}{z}\omega \subseteq I$ if and only if $(\omega : x\omega) \supseteq (\omega : zI)$, if and only if $\frac{x}{z}(\omega : I) \subseteq R$, it is sufficient to prove that $\frac{x}{z}(\omega : I) \subseteq \underline{m}$. By Remark 3.3 that we can choose $z = \gamma x^{-1}$, with γ minimal reduction of $x(\omega : I)$. Moreover, if $1 \in (x\omega : I)$ we can choose $\gamma = 1$, otherwise we can choose $\gamma = x$.
If $\gamma = 1$, then $\frac{x}{z}(\omega : I) = \frac{x^2}{\gamma}(\omega : I) = x^2(\omega : I) \subseteq xR \subseteq \underline{m}$.
If $\gamma = x$, then $\frac{x}{z}(\omega : I) = \frac{x^2}{\gamma}(\omega : I) = x(\omega : I) \subseteq \underline{m}$.

Thanks to the last theorem and Theorem 4.3 of [17], we can prove the following result.

Theorem 3.6. *The following conditions are equivalent:*

1. *R has a canonical reduction J.*

2. *$R(I)_{\alpha,\beta}$ has a canonical reduction for all α, β and for all the ideals I such that $J \subseteq I$.*

Proof. $1) \Rightarrow 2)$ It follows by Theorem 3.1 and by Proposition 3.1.
$2) \Rightarrow 1)$ If $R(I)_{\alpha,\beta}$ has a canonical reduction for all α, β and for all the ideals I such that $J \subseteq I$; as $R \ltimes I$ is a particular case of $R(I)_{\alpha,\beta}$, when $\alpha = \beta = 0$ (see [1]), $R \ltimes I$ has also a canonical reduction for all the ideals I such that $J \subseteq I$. This assumption, by Theorem 4.3 of [17], is equivalent to the fact that R has a canonical reduction J. By Proposition 3.1 $J = x\omega$, where x is minimal reduction of $\underline{m}$.

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