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SINGULAR CUBIC FOURFOLDS CONTAINING A PLANE

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In this paper we consider cubic 4-folds containing a plane whose discriminant curve is a reduced nodal plane sextic. In particular, we describe the singular points of such cubic 4-folds and we give conditions on the geometry of the plane sextics so that all the associated cubic 4-folds are singular. Moreover, we construct a family of smooth rational cubic 4-folds whose discriminant curve is reduced but reducible.

1. Introduction.

In this paper, a cubic fourfold is a hypersurface of degree 3 in $\mathbb{P}^5$. Beauville and Voisin (see [1], [2] and [16]) proved that if C is a smooth plane sextic and θ is an odd theta-characteristic such that $h^0(C, \theta) = 1$, then the pair (C, θ) determines a cubic 4-fold containing a plane. In [8], Friedman and Casalaina-Martin showed that when C is an irreducible nodal plane quintic and θ is the push-forward of a theta-characteristic on the complete normalization $\tilde{C}$ of C, then the corresponding cubic 3-fold is smooth.

The aim of this paper is to study the relation between cubic fourfolds containing a plane and reduced nodal plane sextics endowed with odd theta-characteristics. Our main result is Theorem 3.1 which gives an explicit description of the singular points of a cubic 4-fold X containing a plane P which is associated to a reduced nodal plane sextic C and to a generalized theta-characteristic θ on C with $h^0(C, \theta) = 1$. In particular, this result analyzes

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the connections with the geometry of θ and C, it counts the number of singular points and it gives an estimate of the rank of the free abelian group $\mathrm{NS}_2(X)$ generated by the equivalence classes of the algebraic cycles of codimension 2 in X. It is worth noticing that, due to the work of Hassett ([11] and [12]), a better understanding of the lattice-theoretic properties of this group can give interesting information about the geometry of the cubic 4-folds (for example, their rationality).

Proposition 4.2 gives some sufficient conditions on the geometry of a plane sextic which imply that all the associated cubic 4-folds are singular. Proposition 4.7 shows the existence of a family of smooth rational cubic 4-folds whose discriminant curve is a reduced nodal plane sextic.

In Section 2 we prove some general facts about cubic 4-folds and theta-characteristics for nodal reduced (but possibly reducible) curves. We will work over the complex numbers.

2. Cubic 4-folds and theta-characteristics.

Given a smooth curve C, a *theta-characteristic* is a line bundle θ such that

$$\theta^{\otimes 2} = \omega_C,$$

where ω_C is the canonical sheaf of C. We say that a theta-characteristic is *even* if $h^0(C,\theta) \equiv 0 \pmod 2$ and *odd* if $h^0(C,\theta) \equiv 1 \pmod 2$. In particular, by the adjunction formula, if C is a smooth plane sextic, a theta-characteristic on C is a line bundle such that

$$\theta^{\otimes 2} = \mathcal{O}_{\mathbb{P}^2}(3)|_C.$$

It is a classical result (see for example [9]) that the number of non-isomorphic theta-characteristics is equal to 2^{2g}, where g is the genus of C. More precisely, there are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd theta-characteristics.

When C is a singular curve, the general picture becomes slightly different. We recall the following definition:

Definition 2.1. A *stable spin curve* is a pair (Y, L), where Y is a reduced connected curve with the following properties:

(a.1) it has only ordinary double points;
(b.1) each smooth rational component contains at least 2 nodes;
(c.1) two smooth rational components never meet each other

(we will often call *exceptional component* a smooth rational component of C satisfying property (b.1)). Moreover L is a line bundle such that, if $Z =$

$\overline{Y - (\cup_{i \in I} E_i)}$, where $\{E_i : i \in I\}$ is the set of the exceptional components of Y, then

(a.2) $(L|_Z)^{\otimes 2} \cong \omega_Z$;
(b.2) $L|_{E_i} \cong \mathcal{O}_{E_i}(1)$, for all $i \in I$.

A *stable model* of a stable spin curve (Y, L) is a curve C which is obtained by contracting all the exceptional components of Y. The map $\nu : Y \to C$ contracting the exceptional components of Y is the *contraction map*.

Cornalba showed (see [9], page 566) that the isomorphism classes of stable spin curves whose stable model is a nodal curve C are the natural analogues of the theta-characteristics for C. In particular, one can prove (see [5]) that any nodal curve is the stable model of 2^{2g} isomorphism classes of spin curves (counted with multiplicity), where g is the arithmetic genus of the curve C.

Definition 2.2. A *theta-characteristic* on a reduced nodal curve C is a sheaf θ on C, such that there exists a stable spin curve (Y, L) whose stable model is C and satisfying the following condition:

$$\theta = \nu_* L,$$

where $\nu : Y \to C$ is the contraction map.

Also in this case, we say that a theta-characteristic θ is even (resp. odd) if $h^0(C, \theta) \equiv 0 \bmod 2$ (resp. $h^0(C, \theta) \equiv 1 \bmod 2$). In particular, one can prove that there are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd theta-characteristics, where these sheaves are counted with multiplicity and g is the arithmetic genus of C.

Let C be a reduced nodal curve and θ a theta-characteristic on C. We can associate to θ a set of points $S_\theta \subset \mathrm{Sing}\,(C)$ defined in the following way:

$$S_\theta := \{p \in \mathrm{Sing}(C) : \theta_p \ncong \mathcal{O}_{C,p}\}.$$

The following lemma gives a different characterization of the set S_θ.

Lemma 2.3. *Let C be a reduced nodal curve and θ a theta-characteristic such that $\theta = \nu_* L$, where (Y, L) is a stable spin curve whose stable model is C and ν is the contraction map. Then*

$$S_\theta = \{p \in \mathrm{Sing}(C) : \nu^{-1}(p)\ is\ \ an\ \ exceptional\ component\}.$$

Proof. By Definition 2.2, if p is such that $\nu^{-1}(p)$ is an exceptional component then θ is not invertible in p and thus $p \in S_\theta$. Let $p \in S_\theta$. Then we have only two possibilities: (1) $\nu^{-1}(p)$ is one point or (2) $\nu^{-1}(p)$ is an exceptional component. It is easy to see that (1) cannot occur. Indeed, in this case, by Definitions 2.1 and 2.2, θ would be locally invertible in p.

If p is as in case (2) then $\theta_p/m_p\theta_p \cong \mathbb{C}^2$, where m_p is the maximal ideal of θ_p. Indeed, let U be an open neighborhood of p. Then we have the following restriction map:

$$\rho : \theta(U) \to L(U_1) \oplus L(U_2) \oplus L(E),$$

where U_1 and U_2 are open neighborhoods of the two intersection points p_1 and p_2 of the strict transform of C with the exceptional component E. Since $\theta(U) = (\nu_* L)(U) = L(\nu^{-1}(U))$ and $(U_1) \oplus L(U_2) \oplus L(E) = L(U_1) \oplus L(U_2) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, let

$$\rho : s \longmapsto (s_1, s_2, v) \in L(U_1) \oplus L(U_2) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)).$$

We want to show that v is uniquely determined by s_1 and s_2 and that every v occurs in the image of ρ (i.e. the values of $s_1(p)$ and $s_2(p)$ are arbitrary). This would imply $\theta_p/m_p\theta_p \cong \mathbb{C}^2$.

The fact that v is determined by s_1 and s_2 follows from easy calculations (recalling that v must be properly glued with s_1 and s_2 in p_1 and p_2). Consider the exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1)(-p_1 - p_2) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathcal{O}_{p_1} \oplus \mathcal{O}_{p_2} \longrightarrow 0,$$

where $\mathcal{O}_{p_i}$ is the skyscraper sheaf in p_i. The first sheaf is isomorphic to the sheaf $\mathcal{O}_{\mathbb{P}^1}(-1)$. Thus, considering the long exact sequence in cohomology, we get

$$0 \longrightarrow 0 \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \longrightarrow \mathbb{C} \oplus \mathbb{C} \longrightarrow 0.$$

The isomorphism $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{C} \oplus \mathbb{C}$ proves that v is uniquely determined by s_1 and s_2. $\square$

Corollary 2.4. *Let C be a reduced nodal curve and θ a theta-characteristic such that $\theta = \nu_* L$, where (Y, L) is a stable spin curve whose stable model is C and ν is the contraction map. Let $Z := \overline{Y - (\cup_{i \in I} E_i)}$, where $\{E_i : i \in I\}$ is the set of the irreducible components of C and $\nu_Z := \nu|_Z$. Then $\theta = (\nu_Z)_*(L|_Z)$.*

Proof. This easily follows form the previous lemma and remarks. $\square$

Corollary 2.4. *Let C be a reduced nodal curve and θ a theta-characteristic. Let us suppose that there are two stable spin curves (Y_1, L_1) and (Y_2, L_2) such that $\theta = (\nu_1)_* L_1 = (\nu_2)_* L_2$, where ν_1 and ν_2 are the corresponding contraction maps. Then $Y_1 = Y_2$.*

Proof. Let $E := \{E_1, \dots, E_m\}$ and $E' := \{E'_1, \dots, E'_n\}$, be the sets of the irreducible components of Y_1 and Y_2 respectively. Let S_1 and S_2 be the subsets of $\mathrm{Sing}(C)$ defined in the following way:

$$S_1 := \nu_1(E) \qquad S_2 := \nu_2(E').$$

Let us suppose that $S_1 \not\subseteq S_2$. If $p \in S_1 - S_2$, then, $\theta = (\nu_1)_* L_1$ is not invertible in p. We have also $\theta = (\nu_2)_* L_2$ and, due to Lemma 2.3, θ must be invertible in p. This gives a contradiction. The same holds if $S_2 \not\subseteq S_1$. Hence $S_1 = S_2$. $\square$

We prove the following generalization of Corollary 4.2 in [1] and Theorem 4.1 in [8]. The techniques are very similar to the ones used by Friedman and Casalaina-Martin.

Proposition 2.6. *Let C be a reduced nodal sextic contained in $\mathbb{P}^2$ and let θ be a theta-characteristic on C such that $h^0(C, \theta) = 1$. Chosen homogeneous coordinates x_1, x_2, x_3 in $\mathbb{P}^2$, there exists a matrix*

$$M = \begin{pmatrix} l_{11} & l_{12} & l_{13} & q_1 \\ l_{21} & l_{22} & l_{23} & q_2 \\ l_{31} & l_{32} & l_{33} & q_3 \\ q_1 & q_2 & q_3 & f \end{pmatrix}, \tag{1}$$

where l_{ij}, q_k and f are polynomials respectively of degree 1, 2 and 3 in x_1, x_2 and x_3 and $l_{ij} = l_{ji}$, such that the following sequence is exact:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^3 \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^3 \oplus \mathcal{O}_{\mathbb{P}^2} \longrightarrow \theta \longrightarrow 0. \tag{2}$$

In particular, $\det M$ *is the equation which defines C.*

Conversely, if C is a reduced nodal plane sextic and M is a matrix of type (1) fitting in a short exact sequence as (2), then θ is a theta-characteristic and $h^0(\theta) = 1$.

Proof. Let $\theta = \nu_* L$, where (Y, L) is a stable spin curve whose stable model is C and let $\nu : Y \to C$ be the contraction map. Let $Z := \overline{Y - (\cup_{i \in I} E_i)}$ and $\tilde{L} := L|_Z$, where $\{E_i : i \in I\}$ is the set of the irreducible components of Y. By Definition 2.1, we have $(\tilde{L})^{\otimes 2} = \omega_Z$.

Now we get the following isomorphisms:

$$\begin{aligned}\mathcal{H}om(\theta,\omega_C) &\cong \mathcal{H}om(\nu_*\tilde{L},\omega_C)\\ &\cong \nu_*(\tilde{L}^{-1}\otimes\mathcal{O}(-\nu^{-1}(S_\theta))\otimes\nu^*\omega_C)\\ &\cong \nu_*(\tilde{L}^{-1}\otimes\omega_Z)\\ &\cong \nu_*\tilde{L}\cong\theta,\end{aligned}$$

where the second isomorphism follows from the same calculation as in Theorem 4.1 in [8].

Adapting the proof of Theorem B in [1] and using the same calculations of the proof of Corollary 4.2 in [1] we get the desired result (the previous isomorphism shows that the matrix M is symmetric). The converse follows using the results in [1]. □

For a reduced nodal plane sextic C in $\mathbb{P}^5$, let $\Pi(C)\subset\mathbb{P}^5$ be the plane containing C. Let X be a cubic 4-fold in $\mathbb{P}^5$ containing a plane P and let C be a reduced nodal plane sextic in $\mathbb{P}^5$ such that $P\cap\Pi(C)=\emptyset$. Assume that θ is a theta-characteristic on C so that $h^0(C,\theta)=1$.

Definition 2.7. We say that (X,P) is *associated to* (C,θ) or that (C,θ) is a *discriminant curve* of (X,P) if there exist homogeneous coordinates x_1,x_2,x_3,u_1,u_2,u_3 in $\mathbb{P}^5$ such that:

(a) the equations of P are $x_1=x_2=x_3=0$, the equations of $\Pi(C)$ are $u_1=u_2=u_3=0$ while the equation of X is

$$F:=\sum_{i,j=1,2,3} l_{ij}u_iu_j+2\sum_{k=1,2,3} q_ku_k+f,$$

where l_{ij}, q_k and f are polynomials in x_1, x_2 and x_3 of degree 1, 2 and 3 respectively;

(b) if $M(X,P,C)$ is the matrix

$$M(X,P,C)=\begin{pmatrix} l_{11} & l_{12} & l_{13} & q_1\\ l_{21} & l_{22} & l_{23} & q_2\\ l_{31} & l_{32} & l_{33} & q_3\\ q_1 & q_2 & q_3 & f\end{pmatrix},$$

whose coefficients are the polynomials l_{ij}, q_k and f as in (a), then θ and $M(X,P,C)$ fit in a short exact sequence

$$0\longrightarrow\mathcal{O}_{\Pi(C)}(-2)^3\oplus\mathcal{O}_{\Pi(C)}(-3)\xrightarrow{M(X,P,C)}\mathcal{O}_{\Pi(C)}(-1)^3\oplus\mathcal{O}_{\Pi(C)}\longrightarrow\theta\longrightarrow 0$$

of sheaves on $\Pi(C)$ of type (2);

(c) the equations of C in $\mathbb{P}^5$ are $\det M(X,P,C)=u_1=u_2=u_3=0$.

We will often say that a cubic 4-fold X containing a plane P is *associated* to a plane reduced nodal sextic C in $\mathbb{P}^5$ if there exists a theta-characteristic θ on C so that $h^0(C, \theta) = 1$ and (X, P) is associated to (C, θ). Furthermore, a reduced nodal plane sextic $C \subset \Pi(C) \subset \mathbb{P}^5$ is a *discriminant curve* of a cubic 4-fold X containing a plane P is there is a theta-characteristic θ on C such that $h^0(C, \theta) = 1$ and (C, θ) is a discriminant curve of (X, P).

Let $X \subset \mathbb{P}^5$ be a cubic 4-fold containing a plane P and let C be a reduced nodal plane sextic in $\mathbb{P}^5$. Assume that θ is a theta-characteristic on C such that $h^0(C, \theta) = 1$ and that (C, θ) is a discriminant curve of (X, P). We define the rational map

$$\pi_{P,C} : X - - \to \Pi(C)$$

as the projection from the plane P onto $\Pi(C)$. Given a point $p \in \Pi(C)$ and the 3-dimensional projective space $\mathbb{P}^3_p$ containing p and P, we define $F_p := \mathbb{P}^3_p \cap X$. We have $F_p = Q_p \cup P$, where Q_p is a quadric surface which is called the *fiber* of the projection $\pi_{P,C}$ over p.

Remark 2.8. We can give a more geometric interpretation of a discriminant curve C of a cubic 4-fold X containing a plane P. Indeed, it is a classical result (see [16]) that if X and C are smooth, the curve C can be thought as the discriminant curve of the quadric bundle $\pi : \mathrm{Bl}_P(X) \to \Pi(C)$ obtained from $\pi_{P,C} : X - - \to \Pi(C)$ by blowing up P inside X. Proposition 3.4 and Lemmas 3.3 and 3.5 show that the same holds true when C is a nodal reduced plane sextic.

We write $W(X, P, C)$ for the net of conics given by

$$C_p := Q_p \cap P,$$

when p varies in $\Pi(C)$. $B(X, P, C)$ is the base points locus of $W(X, P, C)$. To the pair (C, θ) it is naturally associated a cubic plane curve $D \subset \Pi(C)$. If M is the matrix with polynomial coefficients defined as in Equations (1) and (2), the equation of D in $\Pi(C)$ is $\det G = 0$, where

$$G := \begin{pmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

is 3×3 minor corresponding to the linear part of M. We introduce the sets

$$I_C := D \cap C \qquad\qquad S_C := \mathrm{Sing}(C) - I_C$$

and

$$\tilde{S}_\theta := \{p \in \mathrm{Sing}(C) : p \in \mathrm{supp}(\theta / s\mathcal{O}_C)\},$$

where s is such that $\langle s \rangle = H^0(C, \theta)$.

Remark 2.9. Let (C, θ) be a discriminant curve of (X, P), where $X \subset \mathbb{P}^5$ is a cubic 4-fold, $P \subset X$ is a plane, C is a reduced nodal plane sextic in $\mathbb{P}^5$ and θ is a theta-characteristic on C such that $h^0(C, \theta) = 1$. Assume that $P_1, P_2 \subset X$ are two planes so that $P_1 \cap P_2 = \emptyset$. Consider $\varphi \in \mathrm{PGL}(6)$ such that $\varphi(P_1) = P$ and $\varphi(P_2) = \Pi(C)$. By Definition 2.7, $(\varphi^{-1}(C), \varphi^*(\theta))$ is a discriminant curve of (X, P_1). Moreover, any two discriminant curves (C_1, θ_1) and (C_2, θ_2) of (X, P) are isomorphic.

3. Singular points of cubic 4-folds containing a plane.

This section is devoted to prove the following theorem which describes the singular points of a cubic 4-fold containing a plane which is associated to a reduced plane sextic with at most nodal singularities. This results also gives an estimate of the rank of the free abelian group $\mathrm{NS}_2(X)$ generated by the equivalence classes of the algebraic cycles of codimension 2 in X. We use the notations introduced in Section 2.

Theorem 3.1. *Let X be a cubic 4-fold containing a plane P with discriminant curve (C, θ), where C is a nodal reduced plane sextic and θ is a theta-characteristic on C such that $h^0(C, \theta) = 1$. Then all the singular points of X are double points, the set* $\mathrm{Sing}(X)$ *is zero dimensional and*

$$\begin{aligned}\mathrm{Sing}(X) &= \left(\bigcup\nolimits_{p \in (\mathrm{Sing}(C) - \tilde{S}_\theta)} \mathrm{Sing}(Q_p)\right) \coprod B(X, P, C).\\ &= \left(\bigcup\nolimits_{p \in S_C} \mathrm{Sing}(Q_p)\right) \coprod B(X, P, C),\end{aligned}$$

where Q_p is the fiber over $p \in \Pi(C)$ of the projection $\pi_{P,C}$. In particular, X is irreducible and

$$\#S_C \leq \#\mathrm{Sing}(X) \leq \#S_C + 3.$$

Moreover, the free abelian group $\mathrm{NS}_2(X)$ *generated by the equivalence classes of the algebraic cycles of codimension 2 in X contains* $2(\#S_\theta)+1$ *distinct classes represented by planes and*

$$\mathrm{rkNS}_2(X) \geq \#S_\theta + 2.$$

The proof of this theorem requires some preliminary results. We start with the following lemmas and propositions.

Lemma 3.2. *Let X be a cubic 4-fold containing a plane P associated to a reduced nodal plane sextic C with a theta-characteristic θ such that $h^0(C,\theta)=1$. Let $x \in \mathrm{Sing}(X) - P$. If $x_1, x_2, x_3, u_1, u_2, u_3$ are homogeneous coordinates in $\mathbb{P}^5$ as in Definition 2.7 and if $x = (a_1 : a_2 : a_3 : b_1 : b_2 : b_3)$, then*

$$p := \pi_{P,C}(x) = (a_1 : a_2 : a_3) \in \mathrm{Sing}(C).$$

Furthermore, if f, q_1, q_2 and q_3 are the polynomials appearing in the equation of X as in Definition 2.7, then

$$f(p) = q_1(p) = q_2(p) = q_3(p) = 0$$

and all the first partial derivatives of f evaluated in p are zero.

Proof. For $a_j \neq 0$, the map $g \in \mathrm{PGL}(6)$ defined by

$$(x_1 : x_2 : x_3 : u_1 : u_2 : u_3) \mapsto (x_1 : x_2 : x_3 : u_1 - (b_1/a_j)x_j : \\ u_2 - (b_2/a_j)x_j : u_3 - (b_3/a_j)x_j)$$

is such that $g(x) = (a_1 : a_2 : a_3 : 0 : 0 : 0)$. Obviously, $g|_P = \mathrm{Id}$ and $\pi_{P,C}(x) = \pi_{P,C}(g(x)) = p$ while $P' := g(\Pi(C))$ can be different from $\Pi(C)$. Thus, up to passing to the discriminant curve $(g(C), g_*(\theta))$ (see Remark 2.9), we can suppose without loss of generality that $x = (a_1 : a_2 : a_3 : 0 : 0 : 0)$.

Evaluating the equation of X in x and calculating its partial derivatives in x in these new coordinates, we get the following equalities:

$$f(a_1, a_2, a_3) = q_1(a_1, a_2, a_3) = q_2(a_1, a_2, a_3) = q_3(a_1, a_2, a_3) = 0,$$

where f, q_1, q_2 and q_3 are the polynomials appearing in the equation of X as in Definition 2.7. Moreover, the point $(a_1 : a_2 : a_3 : 0 : 0 : 0)$ is a singular point for the cubic $f = 0$ in $\Pi(C)$. Thus all the first partial derivatives of f in p are zero.

If h is the equation of C in the local ring of $\Pi(C)$ at p, we get $h = gf$ modulo m_p^2, where m_p is the maximal ideal and g is a polynomial of degree 3. The cubic $f = 0$ is singular in p and thus $h = 0$ modulo m_p^2.
This implies that p is a singular point of C (similar calculations are done in [8] for cubic 3-folds). □

Lemma 3.3. *Let X be a cubic 4-fold containing a plane P associated to a reduced nodal plane sextic C with a theta-characteristic θ such that $h^0(C,\theta)=1$.*

(i) If $p \in \tilde{S}_\theta - S_\theta$, then Q_p is a cone (i.e $\mathrm{Sing}(Q_p)$ consists of one point) and $\mathrm{Sing}(Q_p) \subset P$.
(ii) If $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$, then Q_p is a cone and $Q_p \cap P$ is a smooth conic.

Proof. Let $x_1, x_2, x_3, u_1, u_2, u_3$ be homogeneous coordinates in $\mathbb{P}^5$ satisfying (a)–(c) in Definition 2.7. By Definition 2.2, $\theta = \nu_* L$, where L is a line bundle defined on a stable spin curve Y whose stable model is C and ν is the contraction map. Moreover, C is obtained contracting all the exceptional components of Y and S_θ is the image in C of all these components. If $p \notin S_\theta$, θ is locally invertible in p (see Lemma 2.3). This implies that if $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$ or $p \in \mathrm{Sing}(C) - S_\theta$, the rank of $M(X, P, C)(p)$ must be 3 (where $M(X, P, C)(p)$ means the matrix $M(X, P, C)$ evaluated in p).

For any $p = (a : b : c : 0 : 0 : 0) \in \Pi(C)$, let $\mathbb{P}^3_p$ be the 3-dimensional projective space generated by p and P. Chosen in $\mathbb{P}^3_p$ the homogeneous coordinates u_1, u_2, u_3, t, the equation of $F_p = \mathbb{P}^3_p \cap X$ in $\mathbb{P}^3_p$ is

$$(3)\quad L := \sum_{i,j=1,2,3} l_{ij}(a, b, c) t u_i u_j + 2 \sum_{k=1,2,3} q_k(a, b, c) t^2 u_k + f(a, b, c) t^3 = 0.$$

When $p \in \mathrm{Sing}(C) - S_\theta$, the matrix $M(X, P, C)(p)$ has rank 3. Hence, by Equation (3), the quadric $Q_p \subset F_p$ is a cone and it has only one singular point. This proves the first part of (i) and (ii).

Let us consider the cubic plane curve $D \subset \Pi(C)$ defined in the last part of Section 2. Its equation in $\Pi(C)$ is the determinant of the matrix G which is the linear part of $M(X, P, C)$. In [8] it is proved (in the case of a cubic 3-fold, but their proof can be easily modified for our purposes) that $p \in D \cap C$ if and only if p is in the support of $\theta / \mathcal{O}_C$ (the quotient is given by the inclusion $\mathcal{O}_X \hookrightarrow \theta$, defined by a section s of θ such that $H^0(C, \theta) = \langle s \rangle$). This means that if $G(p) \equiv 0$, then θ is not locally isomorphic to $\mathcal{O}_X$ in p. By definition, this would imply $p \in \tilde{S}_\theta$. Thus, if $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$, then $G(p) \not\equiv 0$ and, more precisely, $\mathrm{rk} G(p) = 3$. On the other hand, if $p \in \tilde{S}_\theta - S_\theta$, by definition, det $G(p) = 0$ and thus the rank of $G(p)$ is less than three.

The equation of the conic $Q_p \cap P$ is

$$\sum_{i,j} l_{ij}(a, b, c) u_i u_j = 0.$$

Hence, if $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$, then the curve $Q_p \cap P$ is smooth since the rank of $M(X, P, C)(p)$ and of $G(p)$ is 3. If $p \in \tilde{S}_\theta - S_\theta$, then $\mathrm{Sing}(Q_p) \subset P$, because the rank of $G(p)$ is smaller than 3. $\square$

Proposition 3.4. *Let X be a cubic 4-fold containing a plane P associated to a reduced nodal plane sextic C with a theta-characteristic θ such that $h^0(C, \theta) = 1$. Let $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$. Then*

$$\mathrm{Sing}(X) \supseteq \mathrm{Sing}(Q_p).$$

In particular, if $\mathrm{Sing}(C) \neq \tilde{S}_\theta$, *then Sing*$(X) \neq \emptyset$.

Proof. As usual, assume that $x_1, x_2, x_3, u_1, u_2, u_3$ are homogeneous coordinates in $\mathbb{P}^5$ satisfying (a)–(c) in Definition 2.7. We prove this result in three steps.

STEP 1. If $p \in C - \mathrm{Sing}(C)$ then $\mathrm{rk} M(X, P, C)(p) = 3$, where $M(X, P, C)(p)$ means the matrix $M(X, P, C)$ evaluated at p. Moreover, Sing (Q_p) consists of one point and Q_p is a cone.
Proof. This follows from the trivial remark that if $\mathrm{rk} M(X, P, C)(p) < 3$, then the determinants of all the 3×3 minors of $M(X, P, C)$ are zero in p. Moreover, the partial derivatives of the equation of C is $\sum_i d_i m_i$, where d_i is the derivative of one of the polynomials entries of $M(X, P, C)$ and m_i is the determinant of a 3×3 minor of $M(X, P, C)$. $\square$

STEP 2. If $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$ then there is an open neighborhood U of p in C such that the map $\varphi : U \to X$ defined by

$$\varphi : q' \longmapsto \mathrm{Sing}(Q_{q'})$$

is an isomorphism onto its image and $\pi_{P,C}|_{\varphi(U)} : \varphi(U) \to U$ is an isomorphism.
Proof. We start considering the simplest case of a smooth plane sextic C. It is possible to introduce the embedding $\psi_{|\theta(1)|} : C \longrightarrow \mathbb{P}^5$, defined by the linear system $|\theta(1)|$. It can be geometrically described by taking the singular points of the singular quadrics in the fibration $\pi_{P,C} : X - - \to \Pi(C)$ (see, for example, [16] and [2]). In particular, the image in $\mathbb{P}^5$ of a smooth sextic is smooth itself.

We can produce an analogous construction in the reduced nodal case. Indeed, let U be a suitable open neighborhood of p. By Step 1, if $p \in U - \mathrm{Sing}(C)$, then Q_p has only one singular point. By Lemma 3.3(ii) the same is true when $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$. Thus, at least, φ is well-defined.

The injectivity follows easily. Indeed, let $q_1, q_2 \in U$ be such that

$$\{x\} = \mathrm{Sing}(Q_{q_1}) = \mathrm{Sing}(Q_{q_2}),$$

where $Q_{p_1} \not\equiv Q_{p_2}$. Then there is a plane P_1 meeting P along a line and such that

$$C' := Q_{q_1} \cap P_1 \qquad \text{and} \qquad C'' := Q_{q_2} \cap P_1$$

are irreducible plane conics. This means that F_{q_1} would contain C', C'' and x. Thus $F_{q_1} \supset Q_{q_1} \cup Q_{q_2}$, where $Q_{q_1} \not\equiv Q_{q_2}$. This is absurd, since $\overline{F_{q_1} - P}$ is of degree 2.

Now we can show that φ, restricted to a suitable open neighborhood U is an isomorphism onto its image and that $\pi_{P,C}|_{\varphi(U)}$ maps $\varphi(U)$ isomorphically onto U. By Lemma 3.3, when $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$, there is an open neighborhood V of p such that $\mathrm{Sing}(Q_q) \not\subseteq P$, when $q \in V$. Now we recall that the equation of Q_q inside the projective space $\mathbb{P}^3_q$ containing q and P is obtained by dividing up Equation (3) by t. Thus, by looking at the partial derivatives of (3), we see that there is an open neighborhood W of $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$ such that the map φ is given by

$$(a, b, c) \longmapsto (a, b, c, \psi_1(a, b, c), \psi_2(a, b, c), \psi_3(a, b, c)),$$

where ψ_1, ψ_2 and ψ_3 are rational functions defined in W. Restricting W to C we get the thesis. $\square$

STEP 3. If $q \in \mathrm{Sing}(C) - \tilde{S}_\theta$ then $\mathrm{Sing}(Q_q) \subseteq \mathrm{Sing}(X)$.
Proof. Let us consider the differential

$$(\mathrm{d}\pi_{P,C})_{q'} : T_{q'}\varphi(U) \longrightarrow T_q\mathbb{P}^2,$$

where $q' := \varphi(q)$. It has rank 2 in q'. Moreover, by definition, q' is the singular point of the quadric Q_q which is the fiber of the projection $\pi_{P,C}$ over the node q. This means that $\dim T_{q'}Q_q = 3$. Since

$$\dim T_{q'}X = \dim T_{q'}Q_q + \dim(\mathrm{Im}(d\pi_{q'})) = 3 + 2 = 5,$$

q' is singular in X and this proves Step 3. $\square$

This concludes the proof of the lemma. $\square$

Lemma 3.5. *Assume that X is a cubic 4-fold containing a plane P associated to the pair (C, θ), where C is a reduced nodal plane sextic, θ is a theta-characteristic such that $h^0(C, \theta) = 1$ and $p \in S_\theta$. Then*
(i) $\mathrm{Sing}(X) \cap Q_p \subset P$*;*
(ii) $Q_p = P_1 \cup P_2$, where P_1 and P_2 are distinct planes.
Moreover, $S_\theta \subseteq \tilde{S}_\theta$.

Proof. Looking at the short exact sequence of sheaves in $\Pi(C) \cong \mathbb{P}^2$

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\Pi(C)}(-2)^3 \oplus \mathcal{O}_{\Pi(C)}(-3) &\xrightarrow{M(X,P,C)} \\ \mathcal{O}_{\Pi(C)}(-1)^3 \oplus \mathcal{O}_{\Pi(C)} &\longrightarrow \theta \longrightarrow 0, \end{aligned} \tag{4}$$

and at Definition 2.2, we see that if $p \in S_\theta$, then $\mathrm{rk} M(X, P, C)(p) = 2$. In particular, Q_p is the union of two distinct planes as stated in (ii).

Suppose that $x_1, x_2, x_3, u_1, u_2, u_3$ are homogeneous coordinates in $\mathbb{P}^5$ satisfying (a)–(c) in Definition 2.7. Let $x \in \mathrm{Sing}(X)$ be such that $p := \pi_{P,C}(x) \in S_\theta$. By Lemma 3.2, if $x \in \mathrm{Sing}(X)$ then $p = \pi_{P,C}(x) \in \mathrm{Sing}(C)$ and the polynomials f, q_i (for $i = 1, 2, 3$) and all the first partial derivatives of f are zero in p (see Definition 2.7 and Lemma 3.2 for the definitions of the polynomials f and q_i). If $x \notin P$ but $p \in S_\theta$, the previous remarks implies that, up to a change of variables, we can suppose that the matrix $M(X, P, C)(p)$ (which has rank 2) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, the equation of the curve C, in the local ring R of $\Pi(C)$ at p, would be congruent to $-l_{12}^2 \bmod m_p^3$, where m_p is the maximal ideal of R. This gives the desired contradiction since C has no cusp of multiplicity 2 (similar calculations are done in [5] for cubic 3-folds in a slightly different context). Hence $x \in P$ and (i) holds.

The last statement in the lemma is a consequence of the fact that, since θ is the push-forward of a line bundle on a stable spin curve (see Definition 2.1), the sheaf θ is not locally isomorphic to $\mathcal{O}_C$ in p (by Lemma 2.3), for every $p \in S_\theta$. Thus p is in the support of $\theta/s\mathcal{O}_C$, where s is such that $\langle s \rangle = H^0(C, \theta)$. This implies that p is in $\tilde{S}_\theta$. $\square$

Remark 3.6. We can find examples which show that S_θ can be different from $\tilde{S}_\theta$. One can consider the case of an irreducible plane sextic C with a node p and a theta-characteristic θ on C such that $H^0(C, \theta) = \langle s \rangle$, the section s is a cubic intersecting C in p and θ does not come from a theta-characteristic on the total normalization of C. It is easy to see that, in this case, $S_\theta = \emptyset$ and $\tilde{S}_\theta = \{p\}$.

We can describe some more interesting examples. Indeed, let $x_1, x_2, x_3, u_1, u_2, u_3$ be homogeneous coordinates in $\mathbb{P}^5$. Assume that C is the union of the nodal plane cubic C_1 whose equations are $u_1 = u_2 = u_3 = x_2^2 x_3 - x_1^3 + x_1^2 x_3 = 0$ and of a smooth plane cubic C_2 whose equations are $u_1 = u_2 = u_3 = f = 0$. Moreover, suppose that $C_1 \cap C_2$ consists of nine points distinct from the nodes of C_1. There exists a theta-characteristic θ_1 on C_1 such that $h^0(C_1, \theta_1) = 0$. In particular, we can write the equation of C_1 in $\Pi(C)$ as the determinant of the matrix

$$M_1 := \begin{pmatrix} 0 & x_1 & x_2 \\ x_1 & -x_3 & 0 \\ x_2 & 0 & x_1 + x_3 \end{pmatrix}.$$

The equation of C in $\Pi(C)$ is then the determinant of the matrix

$$M_2 := \begin{pmatrix} M_1 & 0 \\ 0 & f \end{pmatrix}.$$

As we have seen in the proof of Lemma 3.2, $p \in \mathrm{Sing}(C)$ is in $\tilde{S}_\theta$ if and only if $p \in C \cap D$. This means that the point $p := (0 : 0 : 1 : 0 : 0 : 0)$ is in $\tilde{S}_\theta$ but not in S_θ. This is clear since the rank of M_2 in p is 3 (simply because C_2 does not intersect C_1 in its node) but the intersection of the quadric Q_p with the plane P is the union of two lines (because the rank of M_1 in p is 2).

Definition 3.7. A *couple of planes* is given by two planes P_1 and P_2 such that

$$P_1 \cap P_2 = r,$$

where r is a line. We write (P_1, P_2) for such a couple.

We can prove the following proposition.

Proposition 3.8. *Let X be a cubic 4-fold containing a plane P associated to a discriminant curve (C, θ). If $q_1, q_2 \in S_\theta$ and $Q_{q_1} = P_1 \cup P_2$, $Q_{q_2} = S_1 \cup S_2$, where P_i and S_j are planes, then*

$$P_i \cap S_j = 1 \textit{ point},$$

for any $i, j \in \{1, 2\}$. Furthermore, given a couple of planes (P_1, P_2) in X, there exists a plane $P \subset X$ such that $P_1 \cup P_2$ is a fiber of the projection $\pi_{P,C}$.

Proof. Observe that, due to Lemma 3.5, if $p \in S_\theta$ then $Q_p = P_1 \cup P_2$, where P_1 and P_2 are planes. Moreover $P_1 \neq P$ and $P_2 \neq P$. Indeed, assume that $x_1, x_2, x_3, u_1, u_2, u_3$ are homogeneous coordinates in $\mathbb{P}^5$ as in Definition 2.7. Let $\mathbb{P}^3_p$ be the projective space of dimension 3 containing $p = (a : b : c : 0 : 0 : 0)$ and P. As we noticed in the proof of Lemma 3.3, the equation of F_p in $\mathbb{P}^3_p$ is the polynomial L defined by Equation (3) in the homogeneous coordinates u_1, u_2, u_3, t. The choice of these coordinates in Lemma 3.3 was such that the equation of P in $\mathbb{P}^3_p$ is $t = 0$. Thus $P_i \equiv P$ if and only if, in $\mathbb{P}^3_p$, $L = t^2 M$, where M is a polynomial of degree 1 in u_1, u_2, u_3 and t. This happens if and only if the matrix G (see the end of Section 2 and Lemma 3.3 for its definition) is the zero matrix in p (i.e. $G(p) \equiv 0$). This would imply $\mathrm{rk}M(X, P, C)(p) = 1$ but this is a contradiction since $\mathrm{rk}M(X, P, C)(p) = 2$, when $p \in S_\theta$.

Let (P_1, P_2) and (S_1, S_2) be two couples of planes such that $Q_{q_1} = P_1 \cup P_2$ and $Q_{q_2} = S_1 \cup S_2$, for $q_1, q_2 \in S_\theta$. By Lemma 3.5 the intersection of $P_1 \cup P_2$

(and $S_1 \cup S_2$) with the plane of projection P is the union of two lines (not necessarily distinct). Hence S_i and P_j meet each other at least in one point $P_i \cap S_j \cap P$. Suppose that there exist $i, j \in \{1, 2\}$ such that $S_i \cap P_j$ is a line l_1 and let l_2 be the line $P_1 \cap P_2$.

If $l_1, l_2 \subset P$, then $l_1 \equiv l_2$ because, otherwise, $P_j \equiv P$ (recall that $l_1, l_2 \subset P_j$). Hence, we would have a plane R meeting P, S_i, P_1 and P_2 along 4 lines r_1, r_2, r_3 and r_4. Thus $F_{q_1} \supset P_1 \cup P_2$ would contain l_1, r_1, r_2, r_3, r_4 and so it would also contain $S_i (\not\equiv P)$. If $l_1 \subset P$ but $l_2 \not\subset P$, then $S_i \cup P_1 \cup P_2 \subset F_{q_1}$ because $S_i \cup P_j \subset \mathbb{P}^3_{q_1}$. On the other hand, if $l_1 \not\subset P$, then $(S_i \cap P) \cup l_1 \subset F_{q_1}$ and $S_i \subset F_{q_1}$. In all these cases we get a contradiction since $\overline{F_{q_1} - P}$ is a quadric. Hence we can conclude that, for any $i, j \in \{1, 2\}$, $S_i \cap P_j$ consists of one point.

Suppose now that S_1 and S_2 are a couple of planes. Let $P' \subset \mathbb{P}^5$ be a plane such that $P' \cap S_1 = \emptyset$. Consider the projection $\pi_{S_1, P'} : X - - \to P'$ from S_1. Remark 2.9 shows that there is a nodal plane sextic C' with a theta-characteristic θ' such that (C', θ') is a discriminant curve of (X, S_1), $P' \equiv \Pi(C')$ and $\pi_{S_1, P'}$ coincides with the projection $\pi_{S_1, C'} : X - - \to \Pi(C')$. The plane S_2 is contained in a fiber of $\pi_{S_1, P'}$. Indeed, since S_1 and S_2 meet along a line, there exists a $\mathbb{P}^3$ containing both planes and meeting P' in one point. This implies that S_2 is a component of a fiber of $\pi_{S_1, P'} = \pi_{S_1, C'}$. Applying Lemma 3.5 to the pairs (X, S_1) and (C', θ'), we see that such a fiber of $\pi_{S_1, C'}$ must contain two distinct planes. Hence there exists a plane S_3 meeting S_1 and S_2 along two lines.

This concludes the proof. □

Remark 3.9. Looking more carefully at the proof of Proposition 3.8 and using Remark 2.9, we can say that each couple of planes (P_1, P_2) contained in a cubic 4-fold can actually be described as a triple of planes (P_1, P_2, P_3) such that $P_i \cap P_j$ is a line for $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Lemma 3.10. *Let X be a smooth cubic 4-fold containing a plane P with discriminant curve (C, θ), where C is a nodal reduced plane sextic and θ is a theta-characteristic on C such that $h^0(C, \theta) = 1$. Then*

$$\mathrm{rkNS}_2(X) \geq \#S_\theta + 2.$$

Moreover, $\mathrm{NS}_2(X)$ *contains* $2(\#S_\theta) + 1$ *distinct classes represented by planes.*

Proof. By Proposition 3.8, we know that X contains $M := \#S_\theta$ couples of planes

$$(P_{1,1}, P_{1,2}), \ldots, (P_{M,1}, P_{M,2}).$$

Let h^2, P and Q be the classes in $\mathrm{NS}_2(X)$ corresponding, respectively, to the hyperplane section, the plane P and the general quadric in the fiber of $\pi_{P,C}$. We recall that, in $\mathrm{NS}_2(X)$, $h^2 = Q + P$.

If "$\cdot$" is the cup product, we have $Q \cdot P = Q \cdot (h^2 - Q) = 2 - 4 = -2$ and

$$P_{i,j} \cdot P = \frac{1}{2}(Q^2 \cdot P) = -1,$$

for $i \in \{1, \dots, 12\}$ and $j \in \{1, 2\}$. Hence, since $P \cdot P = 3$, the class of $P_{i,j}$ is distinct from the one of P, for any $i \in \{1, \dots, M\}$ and $j \in \{1, 2\}$. Moreover,

$$P_{i,1} \cdot P_{i,2} = \frac{1}{2}\left(Q^2 - P_{i,1}^2 - P_{i,2}^2\right) = -1$$

and the classes of $P_{i,1}$ and $P_{i,2}$ are distinct in $\mathrm{NS}_2(X)$, because $P_{i,j} \cdot P_{i,j} = 3$. On the other hand, by Proposition 3.8, $P_{i,k} \cdot P_{j,h} = 1$, for i distinct from j and $k, h \in \{1, 2\}$. Thus the classes of $P_{i,k}$ and of $P_{j,h}$ are not the same in $\mathrm{NS}_2(X)$ when $k \neq h$ and $i, j \in \{1, \dots, M\}$. Summarizing, we have proved that P, $P_{1,1}, \dots, P_{M,1}$ and $P_{1,2}, \dots, P_{M,2}$ are $2(\#S_\theta) + 1$ distinct classes in $\mathrm{NS}_2(X)$.

Let us consider the sublattice $N \subseteq \mathrm{NS}_2(X)$ defined by:

$$N := \langle P, P_{1,2}, P_{i,1} : i \in \{1, \dots, M\}\rangle.$$

Clearly $\mathrm{rkNS}_2(X) \geq \mathrm{rk}N$. Consider the matrix $A = (a_{ij})$ representing the intersection form in N. Using the previous calculations, we see that the coefficients of A are defined as follows (keeping the planes ordered as in the definition of N):

$$a_{ij} = \begin{cases} 3 & \text{if } i = j; \\ -1 & \text{if } i = 1, j > 1 \text{ or } j = 1, i > 1 \text{ or } i = 2, j = 3 \text{ or } i = 3, j = 2; \\ 1 & \text{otherwise.} \end{cases}$$

By an easy calculation we see that the determinant of A is different from 0. Hence A has maximal rank and the $\#S_\theta + 2$ planes generating N are linearly independent. □

Remark 3.11. The estimate in Lemma 3.10 does not depend on the choice of the discriminant curve (C, θ). Indeed, by Remark 2.9, two discriminant curves (C, θ) and (C', θ') of (X, P) are isomorphic.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. For simplicity, we assume from the beginning homogeneous coordinates $x_1, x_2, x_3, u_1, u_2, u_3$ in $\mathbb{P}^5$ satisfying (a)–(c) in Definition 2.7.

Let $q := (a : b : c : 0 : 0 : 0) \in \Pi(C)$ and let $\mathbb{P}^3_q$ be the projective space of dimension 3 containing q and the plane P. Depending on the fact that $a \neq 0$ or $a = 0$, such a 3-dimensional space can be described by the equations

$$bx_1 - ax_2 = cx_1 - ax_3 = 0 \qquad \text{or} \qquad x_1 = cx_2 - bx_3 = 0.$$

In these two cases, the Jacobian matrix L_q of the equations describing $F_q = Q_p \cup P$ (see Equation (3)) is either

$$\begin{pmatrix} b & -a & 0 & 0 & 0 & 0 \\ c & 0 & -a & 0 & 0 & 0 \\ F_{x_1} & F_{x_2} & F_{x_3} & F_{u_1} & F_{u_2} & F_{u_3} \end{pmatrix} \quad \text{or} \tag{5}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & -b & 0 & 0 & 0 \\ F_{x_1} & F_{x_2} & F_{x_3} & F_{u_1} & F_{u_2} & F_{u_3} \end{pmatrix},$$

where F is the equation of X as in item (a) of Definition 2.7 while F_{x_i} and F_{u_j} are the partial derivatives of F with respect to the variables x_i and u_j ($i, j \in \{1, 2, 3\}$).

From (5) it easily follows that the singular points of X are contained in the union of the singular points of F_q when q varies in $\Pi(C)$. Since $\mathrm{Sing}(F_q) = \mathrm{Sing}(Q_q) \cup (P \cap Q_q)$,

$$\mathrm{Sing}(X) \subseteq \left(\bigcup_{p \in C} \mathrm{Sing}(Q_p) \right) \cup \left(\bigcup_{p \in \Pi(C)} (Q_p \cap P) \right).$$

By Lemma 3.2, if $x \in \mathrm{Sing}(X) - P$ then $\pi_{P,C}(x) \in \mathrm{Sing}(C)$. Moreover, by Lemma 3.3(i) and Lemma 3.5(i), if $x \in \mathrm{Sing}(X) - P$ then $\pi_{P,C}(x) \in \mathrm{Sing}(C) - \tilde{S}_\theta$. Furthermore, Proposition 3.4 shows that if $p \in \mathrm{Sing}(C) - \tilde{S}_\theta$ then $\mathrm{Sing}(Q_p) \subseteq \mathrm{Sing}(X)$. Hence

$$\mathrm{Sing}(X) \subseteq \left(\bigcup_{p \in (\mathrm{Sing}(C) - \tilde{S}_\theta)} \mathrm{Sing}(Q_p) \right) \cup \left(\bigcup_{p \in \Pi(C)} (Q_p \cap P) \right).$$

Let $x \in \mathrm{Sing}(X) \cap P$. Then the matrix $L_q(x)$ has rank 2, for every $q \in \Pi(C)$. This means that x is contained in $P \cap Q_q$, for every $q \in \Pi(C)$. Thus

$x \in B(X, P, C)$. The converse is also true. Indeed, let $y \in B(X, P, C)$. It is easy to see that if $e_1 = (1 : 0 : 0)$, $e_2 = (0 : 1 : 0)$ and $e_3 = (0 : 0 : 1)$ in P, then $F_{x_i}(y) = G_i(y)$, for $i \in \{1, 2, 3\}$, where G_i is the equation of the conic $C_i = Q_{e_i} \cap P$. Since $y \in B(X, P, C) \subset P$, $F_{x_i}(y) = F_{u_j}(y) = 0$, for $i, j \in \{1, 2, 3\}$. Hence $y \in \mathrm{Sing}(X)$ and

$$\mathrm{Sing}(X) = \left(\bigcup_{p \in (\mathrm{Sing}(C) - \tilde{S}_\theta)} \mathrm{Sing}(Q_p) \right) \cup B(X, P, C).$$

We showed in the proof of Lemma 3.3 that $S_C = \mathrm{Sing}(C) - \tilde{S}_\theta$ while Lemma 3.3 proves that $\mathrm{Sing}(Q_q) \not\subseteq P$, when $q \in \mathrm{Sing}(C) - \tilde{S}_\theta$. This implies that

$$\mathrm{Sing}(X) = \left(\bigcup_{p \in S_C} \mathrm{Sing}(Q_p) \right) \coprod B(X, P, C).$$

If $o \in X$ is a singular point with multiplicity ≥ 3, then $o \in P$. Indeed, if $o \notin P$, then we can suppose, without loss of generality, that $o = (a : b : c : 0 : 0 : 0)$ (as in the proof of Lemma 3.2). By simple calculations, for $i, j \in \{1, 2, 3\}$, $F_{u_i u_j}(x) = l_{ij}(x) = 0$ where $F_{u_i u_j}(x)$ is the second partial derivative of F with respect to u_i and u_j and $l_{ij}(x)$ is the (i, j)-linear coefficient in $M(X, P, C)$ evaluated in x. Moreover, if $p := \pi_{P,C}(o)$, then $f(p) = q_1(p) = q_2(p) = q_3(p) = 0$ (Lemma 3.2) and $M(X, P, C)(p)$ would have rank 0. This is absurd. If $o \in P$, then, for any $p \in \Pi(C)$, Q_p is singular in o because F_p has multiplicity greater or equal to 3 in o. In particular, we would get the absurd conclusion that $\det M_X \equiv 0$. Therefore, each singular point of X has multiplicity at most 2.

In Proposition 3.8 we proved that $P \not\subset Q_p$, for all $p \in \Pi(C)$. Moreover, the conics $C_q := Q_q \cap P$ in $W(X, P, C)$ can not coincide when q varies in $\Pi(C)$ because, otherwise, $B(X, P, C) = C_q$ and $P - C_q \subseteq \mathrm{Sing}(X)$. This would contradict the previous description of the singular locus of X. Hence $B(X, P, C)$ is not a conic.

If $B(X, P, C)$ contains a line l then all the conics in $W(X, P, C)$ are reducible. On the other hand, the smooth part of C contains at least one point q such that $C_q = Q_q \cap P$ is a smooth conic (this easily follows, as in the smooth case, since the rank of the matrix $M(X, P, C)$ calculated in this point is 3. Lemma 3.3 gives directly this result for the points in $\tilde{S}_\theta$). Hence $B(X, P, C)$ contains a finite number of points. Obviously, such a number is at most 4.

Let us suppose

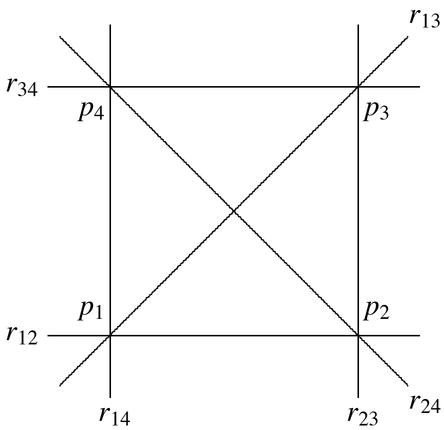
$$B(X, P, C) = \{p_1, p_2, p_3, p_4\}.$$

Given a triple $\{p_{i_1}, p_{i_2}, p_{i_3}\}$, they are in general position. Indeed, if $l = \langle p_{i_1}, p_{i_2}, p_{i_3}\rangle$ is a line, then each conic in $W(X, P, C)$ would pass through p_{i_1}, p_{i_2} and p_{i_3} and it would contain l. Then $l \subseteq B(X, P, C)$ which contradicts the fact that $B(X, P, C)$ is zero-dimensional.

Let us define the following six lines in P:

$$r_{ij} := \langle p_i, p_j\rangle,$$

where $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. The geometric picture is summarized by the following diagram:



Let $D \subset \Pi(C)$ be the plane cubic defined in Section 2. The multiplicity of intersection of the points in the set $D \cap C$ can be at most 2 (see [8]). Hence $D \cap C$ contains at least nine points and $C_p := Q_p \cap P$ is reducible for $p \in D \cap C$.

If $q_1, q_2 \in D \cap S_\theta$, Proposition 3.8(i) shows that C_{q_1} and C_{q_2} are distinct because $Q_{q_1} \cap Q_{q_2}$ consists of a finite number of points. If $q_1, q_2 \in D \cap C$ but $q_1, q_2 \notin S_\theta$, then, by the same argument as in the proof of Step 2 of Proposition 3.4 (see the proof of the injectivity of the map φ),

$$\mathrm{Sing}(Q_{q_1}) \neq \mathrm{Sing}(Q_{q_2}).$$

Since $\mathrm{Sing}(Q_{q_i}) = \mathrm{Sing}(C_{q_i})$, C_{q_1} and C_{q_2} are distinct. Thus $W(X, P, C)$ must contain at least five reducible distinct conics. This gives a contradiction because

$W(X, P, C)$ contains only three reducible conics, given by:

$$\begin{aligned} V_1 &:= r_{13} \cup r_{24}; \\ V_2 &:= r_{12} \cup r_{34}; \\ V_3 &:= r_{14} \cup r_{23}. \end{aligned}$$

Hence $B(X, P, C)$ contains at most 3 points and the inequalities about the number of points in $\operatorname{Sing}(X)$ stated in Theorem 3.1 hold true.

Furthermore, since $B(X, P, C)$ is finite, $\operatorname{Sing}(X)$ is zero-dimensional and X is irreducible because the set $\operatorname{Sing}(X)$ is finite. The last part of Theorem 3.1 is exactly Lemma 3.10. $\square$

We will discuss later (see Examples 4.5 (i) and 4.5 (ii)) the cases of cubic 4-folds whose singularities coincide with the singular points of Q_q for $q \in \operatorname{Sing}(C) - \tilde{S}_\theta$ or with extra points in P.

4. Smoothness and rationality.

It is clear that, given a nodal plane sextic there are many associated cubic 4-folds. The first result in this paragraph describes some sufficient conditions on the geometry of a plane sextic C such that all the associated cubic 4-folds are singular. We also give some explicit examples which clarify the numerical bounds given in Theorem 3.1.

In the second part of this section, we consider the case of smooth cubic 4-folds and we introduce a family of smooth rational cubic 4-folds whose discriminant curve is reduced but reducible.

Roughly speaking, Theorem 3.1 proves that the number of singular points of a cubic 4-folds X containing a plane P depends on the number of nodes of a discriminant curve (C, θ) of (X, P) which are not in S_θ. Thus a cubic 4-folds X containing a plane P such that each node of a discriminant curve C gives rise to a singular point of X realizes the maximal number of singular points compatible with the geometry of the curve C. This is summarized by the following definition:

Definition 4.1. Let X be an irreducible cubic 4-fold containing a plane P. We say that X *realizes the maximal number of singular points* if any discriminant curve (C, θ) satisfies the following conditions:

(i) C is a reduced nodal plane sextic and $\#\operatorname{Sing}(C) > 0$;
(ii) $h^0(C, \theta) = 1$ and $\#S_\theta = 0$.

We say that a reducible plane curve C *has general irreducible components* if, for each irreducible component C' and for each theta-characteristic θ over C', if θ is odd (resp. even) then $h^0(C,\theta) = 1$ (resp. $h^0(C,\theta) = 0$).

Proposition 4.2. *(i) Let C be a reduced nodal plane sextic in $\mathbb{P}^5$ with general irreducible components and such that either*

(a) *C is irreducible with 10 nodes or*
(b) *C is reducible and it does not contain neither a smooth cubic nor a quartic with n nodes nor a quintic with m nodes, where $0 \leq n \leq 2$ and $0 \leq m \leq 5$.*

Then there exists at least one singular cubic 4-fold X containing a plane P whose discriminant curve is C. Moreover, all the cubic 4-folds containing a plane associated to C are singular.

(ii) Let X be a cubic 4-fold containing a plane P realizing the maximal number of singular points. Let C be any discriminant curve of (X, P) with general irreducible components. Then either C is nodal and irreducible or C is the union of three smooth conics or of a smooth conic and a quartic (possibly nodal).

Proof. Let θ be a theta-characteristic on C such that $h^0(C,\theta) = 1$ and let X be a cubic 4-fold containing a plane P associated to (C,θ). If $p \in \tilde{S}_\theta - S_\theta$, using Lemma 3.3(i) and the same techniques as in the proof of Steps 2 and 3 of Proposition 3.4, one can prove that $\mathrm{Sing}(Q_p) \subseteq \mathrm{Sing}(X)$. In particular, by Theorem 3.1, $\mathrm{Sing}(Q_p) \subset B(X, P, C)$. Analogously, if $x \in \mathrm{Sing}(X) \cap B(X, P, X)$, then $\pi_{P,C}(x) \in \tilde{S}_\theta - S_\theta$. Hence all the cubic 4-folds containing a plane associated to a curve C satisfying the hypotheses of item (i) are singular if C does not have a theta-characteristic θ such that

$$\#\mathrm{Sing}(C) = \#S_\theta \quad \text{and} \quad h^0(C,\theta) = 1. \tag{6}$$

Consider the following fact (see [9] or [5] for the proof and [5] for the definitions and the techniques involving the dual graphs):

<ins>FACT.</ins> Let $\tilde{C}$ be a stable spin curve whose stable model is C. Let $Z_{\tilde{C}} := \tilde{C} - (\cup_{i\in I} E_i)$, where $\{E_i : i \in I\}$ is the set of the irreducible components in $\tilde{C}$, and let $\Gamma_{Z_{\tilde{C}}}$ be the dual graph of the curve $Z_{\tilde{C}}$ (the vertices of $\Gamma_{Z_{\tilde{C}}}$ are the irreducible components of $Z_{\tilde{C}}$ and its edges are the nodes of $\tilde{C}$). If $\Gamma_{Z_{\tilde{C}}}$ is even (i.e. to each vertex converges an even number of edges) then the number M of distinct theta-characteristics over C is such that $M \geq 2^{b_1(\Gamma_{Z_{\tilde{C}}})}$, where $b_1(\Gamma_{Z_{\tilde{C}}})$ is the first Betti number of $\Gamma_{Z_{\tilde{C}}}$.

From this and from the easy remark that the push-forward via the contraction map preserves the parity of the theta-characteristic, we get two very easy consequences (see Section 4 in [9] for a brief discussion):

(a.1) if $2^{b_1(\Gamma_{Z_{\tilde{C}}})} > 1$ then C has an odd theta-characteristic coming from a line bundle on $\tilde{C}$ as in Definition 2.2;

(b.1) if $\Gamma_{Z_{\tilde{C}}}$ has m disjoint components $Y_1, \dots, Y_m$, an odd theta-characteristic for C coming from $\tilde{C}$ is given by the choice of an odd theta-characteristic for an odd number of curves C_i corresponding to the graphs Y_i.

If θ satisfies (6), then, by Definitions 2.1 and 2.2, there exists a stable spin curve $(\tilde{C}, L)$ whose stable model is C and such that $\mathrm{Sing}(C)$ is the image via the contraction map $\nu : \tilde{C} \to C$ of all the exceptional components of $\tilde{C}$ and $\theta = \nu_* L$. In this case, by (b.1), there would be an irreducible component C_1 of $\tilde{C}$ such that $h^0(C_1, L|_{C_1}) = 1$. This is impossible, since when C satisfies the hypotheses of Proposition 4.2(i), the irreducible components of such a curve $\tilde{C}$ have genus 0.

By the remarks at the beginning of this proof and by (b.1), a plane sextic C as in item (i) is the discriminant curve of at least one (singular) cubic 4-fold X containing a plane P if there are a set $S \subset \mathrm{Sing}(C)$ with $S \neq \mathrm{Sing}(C)$ and a stable spin curve $(\tilde{C}, L)$ such that:

(a.2) $\nu(\tilde{C}) = C$;

(b.2) S is the image via ν of all the exceptional components of $\tilde{C}$;

(c.2) the curve $Z_{\tilde{C}} := \overline{\tilde{C} - (\cup_{i \in I} E_i)}$ has an irreducible component of arithmetic genus one, where $\{E_i : i \in I\}$ is the set of the irreducible components in $\tilde{C}$.

Furthermore, a curve C as in item (i) can only be:

(a.3) the union of two singular cubics C_1 and C_2;

(b.3) the union of a line and of a quintic with 6 nodes;

(c.3) the union of two lines and of a quartic with 3 nodes;

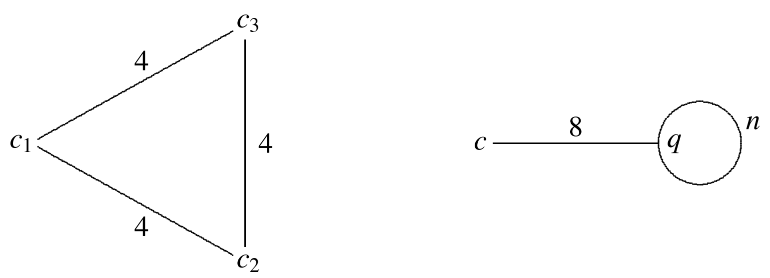
(d.3) an irreducible sextic with 10 nodes.

In the first case we put $S = \mathrm{Sing}(C_1) \cup (C_1 \cap C_2)$. In case (b.3) we define S as the union of 5 of the 6 nodes of the quintic and of the 5 intersection points of the line and the quintic. In the third case, S is the set of all the intersection points and of 2 of the 3 nodes. In case (d.3), we observe that the curve C blown up in 9 of the 10 nodes has genus 1. In all of these cases, due to (a.1)–(b.1), a curve $\tilde{C}$ satisfying (a.2)–(c.2) always exists.

For item (ii), observe that if (X, P) realizes that maximal number of singular points then there exists a discriminant curve (C, θ) such that $\theta^2 = \omega_C$. Consider the following cases:

CASE 1: C is irreducible with m nodes ($m < 11$);

CASE 2: C is reducible and it is either the union of 3 smooth conics or the union of a smooth conic and of a quartic with n nodes. In these cases the dual graphs correspond respectively to the diagrams:



where c_1, c_2, c_3, c are conics, q is a quartic and the bold number above the edges stands for the number of edges connecting two vertices. The circle in the second graph means the possible existence in q of n nodes.

In both cases, to each vertex of the dual graph of C converges an even number of edges. Moreover, the dual graph of C has first Betti number greater than zero. (a.1) (with $\tilde{C} = C$) implies that there exists an odd theta-characteristic θ on C such that $\theta^2 = \omega_C$. If C is reducible but it is not as in Cases 1 and 2, then the dual graph of C is not even. Indeed, C would contain at least a line and such a line would intersect the union of the other irreducible components of C in an odd number of points. By the results in [9] and [5], there are no sheaves L on C such that (C, L) is a stable spin curve with stable model C. $\square$

The following corollary is a trivial consequence of the techniques described in the previous proof.

Corollary 4.3. *Let X be a cubic 4-fold containing a plane P associated to the pair (C, θ), where C is a reduced nodal plane sextic with general irreducible components and θ is an odd theta-characteristic with $h^0(C, \theta) = 1$. Then X is smooth if and only if not all the irreducible components of the total normalization $\tilde{C}$ of C are rational.*

Remark 4.4. Assume that a cubic 4-fold X containing a plane P is associated to a reduced nodal plane sextic C. Let X contain ten couples of planes

$$(P_{1,1}, P_{1,2}), \dots, (P_{10,1}, P_{10,2})$$

which corresponds to ten different fibers of $\pi_{P,C}$. By Proposition 3.8 and Theorem 3.1, $P_{i,j} \cap P_{k,h}$ consists of one point (for $i \neq k$). Then C has one of the following configurations:

(a) 6 lines;
(b) 3 conics;
(c) 2 conics and 2 lines;
(d) 1 conic and 4 lines;
(e) 3 lines and 1 cubic with 1 node;
(f) 1 line and a quintic with 5 or 6 nodes;
(g) 1 smooth cubic and 1 cubic with 1 node;
(h) 1 quartic with 1 or 2 nodes and 2 lines.

Moreover, if X is smooth then its discriminant curve is of type (f) (and it has 5 nodes), (g) or (h) (where the quartic has just 1 node). Indeed, due to Proposition 3.8, Theorem 3.1 and Proposition 4.2, we just need to prove that (a)–(h) are the only curves which admit a theta-characteristic θ such that $h^0(\theta) = 1$ and $\#S_\theta = 10$. Since X contains ten couples of planes, C cannot be

(a.1) an irreducible sextic with m nodes ($m < 10$);
(b.1) two smooth cubics;
(c.1) a line and a quintic with m nodes ($m < 5$);
(d.1) a smooth quadric and a conic,

because C must have at least ten nodes. We can exclude the case of an irreducible sextic with ten nodes since the total normalization of such a curve is a smooth rational curve.

Looking at the dual graph of the remaining possible configurations for nodal plane sextics, the only ones which are even (i.e. each vertex has an even number of edges), once we take away ten of their edges, are those that correspond to sextics as in (a)–(h).

In particular, we get examples of cubic 4-folds containing 10 planes meeting each other in one point and whose discriminant curve is nodal (see also Example 4.5(ii)). Theorem 3.1 implies that some of them can be smooth. Observe that cubic 4-folds of this type are closely related to the problem of describing the moduli space of Enriques surfaces (for a more precise discussion see [10]).

Now we would like to analyze some explicit examples of singular cubic 4-folds containing a plane. These will show that the bounds for the number of points in $\mathrm{Sing}(X)$ given by Theorem 3.1 are optimal.

Example 4.5. Let C be a reducible nodal plane sextic in $\mathbb{P}^5$ and let $x_1, x_2, x_3, u_1, u_2, u_3$ be homogeneous coordinates in $\mathbb{P}^5$ such that the equations of $\Pi(C)$ are $u_1 = u_2 = u_3 = 0$.

(i) Assume moreover that the equation of C in $\Pi(C)$ is the determinant of the following matrix:

$$M_1 := \begin{pmatrix} 0 & x_1 & x_2 & 0 \\ x_1 & 0 & x_3 & 0 \\ x_2 & x_3 & 0 & 0 \\ 0 & 0 & 0 & f \end{pmatrix},$$

where the polynomial f defines a degree 3 plane curve C_1 meeting the cubic C_2, whose equation is $x_1x_2x_3 = 0$, in nine distinct points $p_1, \dots, p_9$ not coinciding with the nodes of C_2. In this case $\tilde{S}_\theta = \{p_1, \dots, p_9\}$.

Let X the cubic 4-fold whose equation is $2x_1u_1u_2 + 2x_2u_1u_3 + 2x_3u_2u_3 + f = 0$. X contains the plane P with equations $x_1 = x_2 = x_3 = 0$ and C is a discriminant curve of (X, P) (by Definition 2.7). It is very easy to verify that

$$\begin{aligned} B(X, P, C) = \{&(0:0:0:1:0:0), (0:0:0:0:1:0), \\ &(0:0:0:0:0:1)\} \subseteq P. \end{aligned}$$

Moreover, Proposition 4.2 and Theorem 3.1 imply that $\#\mathrm{Sing}(X) = \#B(X, P, C) + \#\mathrm{Sing}(C_1) = 3 + \#\mathrm{Sing}(C_1) = \#S_C + 3$.

(ii) Consider the case when C is a plane curve which is the union of six lines $l_1, \dots, l_6$ in general position in $\Pi(C) \subset \mathbb{P}^5$. Assume that the equation of C in $\Pi(C)$ is the determinant of the matrix

$$M_2 := \begin{pmatrix} l_1 & 0 & 0 & 0 \\ 0 & l_2 & 0 & 0 \\ 0 & 0 & l_3 & 0 \\ 0 & 0 & 0 & l_4l_5l_6 \end{pmatrix}.$$

Once more, consider the cubic 4-fold X whose equation is $l_1u_1^2 + l_2u_2^2 + l_3u_3^2 + l_4l_5l_6 = 0$ and containing the plane P with equations $x_1 = x_2 = x_3 = 0$. As in the previous case, (X, P) is associated to C while the base locus $B(X, P, C)$ is empty. Let θ be the odd theta-characteristic on C given by the matrix M_2. The set $\tilde{S}_\theta$ contains 12 of the 15 nodes of C. By Theorem 3.1,

$$\mathrm{Sing}(X) = \mathrm{Sing}(Q_{p_1}) \cup \mathrm{Sing}(Q_{p_2}) \cup \mathrm{Sing}(Q_{p_3}),$$

where $\mathrm{Sing}(C) = \tilde{S}_\theta \cup \{p_1, p_2, p_3\}$ and $p_1 = l_4 \cap l_5$, $p_2 = l_4 \cap l_6$, $p_3 = l_5 \cap l_6$. Hence $\#\mathrm{Sing}(X) = \#S_C$.

Example 4.6. Let C be a plane sextic in $\mathbb{P}^5$ which is the union of a general quartic C_1 and of two lines in general position or of a general quintic C_2 and a line in general position. We choose homogeneous coordinates $x_1, x_2, x_3, u_1, u_2, u_3$ in $\mathbb{P}^5$ such that the equations of $\Pi(C)$ are $u_1 = u_2 = u_3 = 0$. Since the curves C_1 and C_2 are supposed to be general, there are two theta-characteristics θ_1 and θ_2 such that

$$h^0(C_1, \theta_1) = 1 \qquad \text{and} \qquad h^0(C_2, \theta_2) = 1.$$

By Corollary 4.2 in [1], we get the following two matrices whose determinants are the equations of C_1 and C_2:

$$M_1 := \begin{pmatrix} l_{11} & q_1 \\ q_1 & f \end{pmatrix} \qquad M_2 := \begin{pmatrix} l_{11} & l_{12} & q_1 \\ l_{12} & l_{22} & q_2 \\ q_1 & q_2 & f \end{pmatrix}.$$

In these two cases, the sextic C has equation in $\Pi(C)$ described by the determinant of the following two matrices:

$$N_1 := \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & M_1 \end{pmatrix} \qquad N_2 := \begin{pmatrix} l_1 & 0 \\ 0 & M_2 \end{pmatrix},$$

where l_1 and l_2 are the equations of the lines. The corresponding cubic 4-folds are singular if C_1 and C_2 have at least one node.

More explicitly, the matrix corresponding to a sextic which is union of the Fermat's quartic $x_1^4 + x_2^4 + x_3^4 = 0$ and of the two lines $x_1 = 0$ and $x_1 + x_2 = 0$ is given by:

$$M := \begin{pmatrix} -x_1 & 0 & 0 & 0 \\ 0 & x_1 + x_2 & 0 & 0 \\ 0 & 0 & -(x_1 - \omega x_2) & x_3^2 \\ 0 & 0 & x_3^2 & (x_1 + \omega x_2)(x_1^2 + i x_2^2) \end{pmatrix},$$

where $i, \omega \in \mathbb{C}$ are such that $i^2 = -1$ and $\omega^2 = -i$.

The following theorem proves, in particular, that there is a smooth rational cubic 4-fold X containing a plane P associated to a reducible nodal plane sextic. We write Mat(3) for the algebra of 3×3 matrices with complex coefficients.

Proposition 4.7. *Given 3 lines in $\mathbb{P}^5$ meeting in three distinct points, there exists a family of smooth rational cubic 4-folds X containing a plane such that a discriminant curve of the pair (X, P) is a reduced nodal plane sextic containing these three lines. This family is parametrized by points in a non-empty open subset of* Mat(3). *Moreover, the group* $\mathrm{NS}_2(X)$ *contains 25 distinct classes corresponding to planes in X and* $\mathrm{rkNS}_2(X) \geq 14$.

In particular, there exists a smooth rational cubic 4-fold containing a plane associated to reduced nodal plane sextic.

Proof. Let $l_1, l_2, l_3 \subset \mathbb{P}^5$ be three lines meeting each other in three distinct points. Fix homogeneous coordinates $x_1, x_2, x_3, u_1, u_2, u_3$ in $\mathbb{P}^5$ such that the equations of l_1, l_2 and l_3 are

$$l_1 : u_1 = u_2 = u_3 = x_1 = 0\,;\quad l_2 : u_1 = u_2 = u_3 = x_2 = 0\,;$$
$$l_3 : u_1 = u_2 = u_3 = x_3 = 0\,.$$

The lines l_1, l_2 and l_3 are contained in the plane Π whose equations are $u_1 = u_2 = u_3 = 0$.

If P is the plane described by the equations $x_1 = x_2 = x_3 = 0$, given a matrix $A := (a_{ij}) \in \mathrm{Mat}(3)$, the equations of a plane P_A such that $P \cap P_A = \emptyset$ can be obtained as the zero-locus of

$$\begin{aligned} u_1 - a_{11}x_1 - a_{12}x_2 - a_{13}x_3 = 0\\ u_1 - a_{21}x_1 - a_{22}x_2 - a_{23}x_3 = 0\\ u_1 - a_{31}x_1 - a_{32}x_2 - a_{33}x_3 = 0\,. \end{aligned}$$

Let C'_A be the plane cubic in Π defined by the polynomial

$$f_A := \sum_{i=1}^{3}(a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3)^2 x_i\,.$$

If we require that C'_A is smooth and that $C'_A \cap (l_1 \cup l_2 \cup l_3) \neq \{(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)\}$, we impose that A belongs to an open subset $U \subseteq \mathrm{Mat}(3)$. Given $A \in U$, we get a nodal reduced plane sextic $C_A := C'_A \cup l_1 \cup l_2 \cup l_3$. By Proposition 2.6, the matrix

$$M_A := \begin{pmatrix} x_1 & 0 & 0 & 0\\ 0 & x_2 & 0 & 0\\ 0 & 0 & x_3 & 0\\ 0 & 0 & 0 & -f_A \end{pmatrix}$$

determines a theta-characteristic θ_A on C_A such that $h^0(C_A, \theta_A) = 1$. Consider the cubic 4-fold X_A whose equation is

$$F_A := x_1u_1^2 + x_2u_2^2 + x_3u_3^2 - f_A\,.$$

Obviously $P, P_A \subset X_A$ and (X_A, P) is associated to (C_A, θ_A). As it was shown in [14] and [15], the planes P_A gives a section for the projection $\pi_{P,C_A} : X_A - - \to \Pi(C_A)$ and X_A is rational. It is very easy to see that $\tilde{S}_{\theta_A} = \mathrm{Sing}(C_A)$ and that $B(X_A, P, C_A) = \emptyset$. Hence Theorem 3.1 implies that X_A is smooth.

To show that the open subset U is non-empty, let C' be the Fermat plane cubic in Π whose equation in Π is $f = x_1^3 + x_2^3 + x_3^3$. Clearly, the three lines l_1, l_2 and l_3 meet the cubic in nine points distinct from the three intersection points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$. By simple calculations we see that the plane P' described by the equations

$$u_1 - x_1 = u_2 - x_2 = u_3 - x_3 = 0$$

is contained in the cubic 4-fold X whose equation in $\mathbb{P}^5$ is $x_1u_1^2 + x_2u_2^2 + x_3u_3^2 - f = 0$. Moreover, $P \cap P' = \emptyset$. Hence X is rational. This implies that the matrix $\mathrm{Id} \in \mathrm{Mat}(3)$ is in U.

Given $A \in U$, we have the equalities $S_{\theta_A} = \mathrm{Sing}(C_A)$ and $\#S_{\theta_A} = 12$. Theorem 3.1 gives the desired estimate about the number of distinct planes in $\mathrm{NS}_2(X_A)$ and the rank of $\mathrm{NS}_2(X_A)$. $\square$

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