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APPROXIMATE FIXED POINTS VIA COMPLETION

M. AIT MANSOUR - M.A. BAHRAOUI - A. EL BEKKALI

It was proved by S. Tijs, A. Torre and R. Brânzei in [1, Theorem 3.1] that every single-valued contraction from a metric space into itself has an ε-fixed point for every $\varepsilon > 0$. In this paper, we state this result for set-valued mappings and we give a new proof of it by using the concept of completion.

1. Introduction and preliminaries

Let (E,d) be a metric space. We denote by $\mathcal{P}(E)$ the set of nonempty subsets of E. We also denote by $\mathcal{C}(E)$ the set of nonempty and closed subsets of E. The Hausdorff distance between two elements A and B of $\mathcal{P}(E)$ is

$$D(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}.$$

The pair $(\mathcal{C}(E),D)$ is an extended metric[1] space and its completeness is characterized by the following theorem (see for instance [2, Theorem 3.2.4]):

Theorem 1.1. *Let (E,d) be metric space. $(\mathcal{C}(E),D)$ is complete if and only if (E,d) is complete.*

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[1] D is an infinite-valued metric on $\mathcal{C}(E)$.

Let (F,δ) be an another metric space and Δ the Hausdorff distance associed to δ. A set-valued mapping $\Phi : E \rightrightarrows F$ is said to be *Lipschitzian* if there exists a constant $\kappa > 0$ such that $\Delta(\Phi(x),\Phi(x')) \leqslant \kappa d(x,x')$, for all $x,x' \in X$. The constant κ is called *Lipschitz constant* of Φ. If $\kappa < 1$, Φ is said to be a set-valued contraction.

Let $\varepsilon > 0$. A point $x \in E$ is said to be *ε-fixed point* of $\Phi : E \rightrightarrows E$ if and only if $d(x,\Phi(x)) \leqslant \varepsilon$. It was shown by S. Tijs, A. Torre and R. Brânzei in [1], see Theorem 3.1, that every single-valued contraction φ from a metric space into itself has an ε-fixed point for every $\varepsilon > 0$. The proof of this result is obtained by considering the usual sequence defined by $x_{n+1} = \varphi(x_n)$. Exactly in the same immediate way (with obvious variants) one can obtain the result also for set-valued mappings:

Theorem 1.2. *Let (E,d) be a metric space, and let $\Phi : E \rightrightarrows E$ be a set-valued contraction. Then, Φ has an ε-fixed point for every $\varepsilon > 0$.*

The aim of this paper is to give a new alternative proof of Theorem 1.2 based on the notion of completion.

2. Our Proof

Let us first recall the following theorem which will play an important role in our proof of Theorem 1.2, which claims that any metric space E is isometric with a dense subset of a complete metric space called the *completion* of E.

Theorem 2.1 ([3, Theorem 3, p.159])**.** *If (E,d) is a metric space, then there exists a complete metric space (F,δ) and a mapping i from E into F such that:*

1. *i is an isometry, i.e.: for all $x,x' \in E$, $\delta(i(x),i(x')) = d(x,x')$;*

2. *the image of E is dense in F.*

We recall also the following result of extension of contraction mappings.

Theorem 2.2 ([3, Theorem 1, p. 98])**.** *Let (E,d) be a metric space, let (F,δ) be a complete extended metric space, and let A be a dense subset of E. Then, for every Lipschitzian mapping $f : A \to F$, there exists a Lipschitzian mapping $g : E \to F$, with the same Lipschitz constant as f, such that $g_{|A} = f$.*

Another ingredient of our proof of Theorem 1.2 is a very classical result by Nadler which says that every set-valued contraction from a complete metric space into itself with closed values has a fixed point.

Theorem 2.3 ([4, Theorem 5]). *Let (E,d) be a complete metric space, and let $\Phi : E \rightrightarrows E$ be a set-valued contraction with closed values. Then Φ has a fixed point, i.e., there exists a point $x \in E$ such that $x \in \Phi(x)$.*

We are now in a position to give our proof of Theorem 1.2.

Proof of Theorem 1.2. Fix some $\varepsilon > 0$ and denote by κ the Lipschitz constant of Φ. From Theorem 2.1, there exists a complete metric space (F,δ) and an isometry $i : E \to F$ such that $i(E)$ is dense in F.

Since i is an isometry, it is injective. Then, $i : E \to i(E)$ is a bijective mapping. Furthermore, $i^{-1} : i(E) \to E$ is an isometry. Indeed, for all $y,y' \in i(E)$, one has

$$d(i^{-1}(y),i^{-1}(y')) = \delta(i(i^{-1}(y)),i(i^{-1}(y'))) = \delta(y,y').$$

Now, we consider the mapping $\tilde{i} : \mathcal{P}(E) \to \mathcal{C}(F)$ defined, for all $A \in \mathcal{P}(E)$ by

$$\tilde{i}(A) = \overline{i(A)} = \overline{\{i(x) : x \in A\}},$$

where $\overline{i(A)}$ is the closure of $i(A)$ in F. So, for all $A,B \in \mathcal{P}(E)$, we have

$$\begin{aligned}\Delta(\tilde{i}(A),\tilde{i}(B)) &= \Delta(\overline{i(A)},\overline{i(B)}) = \Delta(i(A),i(B)) \\ &= \max\left\{\sup_{a\in A}\inf_{b\in B}\delta(i(a),i(b)),\sup_{b\in B}\inf_{a\in A}\delta(i(a),i(b))\right\} \\ &= \max\left\{\sup_{a\in A}\inf_{b\in B}d(a,b),\sup_{b\in B}\inf_{a\in A}d(a,b)\right\} \\ &= D(A,B).\end{aligned}$$

Hence, $\tilde{i}$ is an isometry. Now, consider the composition map $\Psi_1 := \tilde{i}\circ\Phi\circ i^{-1}$ with $i^{-1} : i(E) \to E$, $\Phi : E \to \mathcal{P}(E)$ and $\tilde{i} : \mathcal{P}(E) \to \mathcal{C}(F)$. Since $\tilde{i}$ and i^{-1} are isometries and Φ is a contraction with constant κ, for all $x,x' \in E$, it follows that

$$\begin{aligned}\Delta(\Psi_1(i(x)),\Psi_1(i(x'))) &= \Delta(\tilde{i}(\Phi(x)),\tilde{i}(\Phi(x'))) \\ &= D(\Phi(x),\Phi(x')) \\ &\leqslant \kappa d(x,x') \\ &\leqslant \kappa\delta(i(x),i(x')).\end{aligned}$$

Thus, $\Psi_1 : i(E) \to \mathcal{C}(F)$ is a contraction with constant κ and closed values. Since (F,δ) is complete, from Theorem 1.1 we infer that $(\mathcal{C}(F),\Delta)$ is also complete. According to Theorem 2.2, there exists a contraction mapping $\Psi : F \to \mathcal{C}(F)$ extending Ψ_1 to F. Accordingly, since F is complete then thanks to Theorem 2.3, Ψ has a fixed point, i.e., there exists $y \in F$ such that $y \in \Psi(y)$. On the other hand, since $i(E)$ is dense in F, there exists $x \in E$ such that $\delta(i(x),y) \leqslant \frac{\varepsilon}{2}$. Thus,

$$\begin{aligned} d(x,\Phi(x)) &= \delta(i(x),\tilde{i}(\Phi(x))) = \delta(i(x),\Psi_1(i(x))) = \delta(i(x),\Psi(i(x))) \\ &\leqslant \delta(i(x),y) + \delta(y,\Psi(i(x))) \\ &\leqslant \delta(i(x),y) + \Delta(\Psi(y),\Psi(i(x))) \\ &\leqslant \delta(i(x),y) + \kappa\delta(y,i(x)) = (\kappa+1)\delta(y,i(x)) \\ &\leqslant \varepsilon. \end{aligned}$$

Consequently, x is a ε-fixed point of Φ. This completes the proof. □

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M. AIT MANSOUR
Département de Physiques, Faculté Poly-disciplinaire, Safi, Université Cadi Ayyad, Morocco
e-mail: `ait.mansour.mohamed@gmail.com`

M.A. BAHRAOUI
Département de Mathématiques, Faculté des Sciences et techniques, Université Abdelmalek Essaadi, Morocco.
e-mail: mbahraoui@hotmail.com

A. EL BEKKALI
Département de Mathématiques, Faculté des Sciences et techniques, Université Abdelmalek Essaadi, Morocco.
e-mail: adhamcpge@gmail.com