LE MATEMATICHE
Vol. LVII (2002) – Fasc. I, pp. 3–19

# ON STABILITY OF SHOCK WAVES IN A COMPRESSIBLE VISCOUS GAS

ALEXANDER M. BLOKHIN

On the example of the Navier-Stokes model, this paper discusses the approach in which the surface of a strong discontinuity in a a compressible viscous gas is considered as a shock wave. It is proved that this approach contains essential lacks. This conclusion follows from existence of special exponentially increasing solutions to the problem on shock wave stability.

## 1. Introduction.

In a moving continuum there often appear transitional zones where parameters which characterize such a continuum (density, pressure, temperature, velocity, etc.) vary rapidly (have large gradients) with respect to spacial variables. If dissipative mechanisms are neglected in a mathematical model of the continuum motion, then these thin zones are usually considered as surfaces of strong discontinuities. In this case the flow parameters change step-wise with jumps, generally speaking, on a propagating surface of some strong discontinuity (e.g., on a shock wave). We note that motions of perfect continua (say, in gas dynamics, magnetohydrodynamics, etc.) are described, as a rule, by hyperbolic systems of conservation laws for which the mathematical theory of strong discontinuities has been well developed not only for one-dimensional (see, e.g., [14]) but for multidimensional flows too (see, especially, [1]–[5], [12]).

Entrato in redazione il 19 luglio 2001.

As far as motions of continua with dissipation (i.e., with viscosity, heat conductivity, etc.) are concerned, we have observed above that there appear zones of large gradients; consequently, the necessity in mathematical simulation of such phenomena arises. In this paper we consider the motion of a viscous gas within the framework of the known Navier-Stokes model of a compressible liquid [9]. As known, exactly the Navier-Stokes equations are used for solution of the problem on the shock wave front in a viscous heat-conducting gas (see [14] for the description of the classical approach to solution of this problem). In the problem on the structure we consider a thin transitional zone (the viscous profile) where the gas flow parameters vary continuously instead of the surface of the strong shock.

They say that the strong shock in a perfect medium have the structure if this discontinuous flow is the limiting flow in the medium with dissipation as dissipation coefficients tends to zero. We however note that problems on the structure of shock waves for different model of continuum mechanics have been studied on the 1-D level as yet. By this reason, the approach connected with structures can not be considered in full measure as the alternative to the discontinuous approach for multi-D shock fronts in perfect media. Moreover, only the structural approach has a satisfactory theoretical justification for continua with dissipations (although on the 1-D level too, see, e.g., [11]).

The following facts indirectly confirm the validity of this structural approach for "viscous" conservation laws. Hyperbolic conservation laws which simulate motions of perfect media have the property that their solutions stay single-valued and continuous only for a short time even if the initial data are smooth. Then the so-called gradient catastrophe is observed (see, e.g., [14]), and one has to consider strong discontinuities. It seems that solutions of "viscous" conservation laws do not possess such a property. Numerous results in articles [6]–[8], [16] on the global existence theorems for the Navier-Stokes equations indirectly prove this assumption. In this connection, of special interest is the article [6] where the theorem on global existence and uniqueness of the generalized solution to the initial problem for the 1-D system of the Navier-Stokes equations written in the Lagrangian coordinates with discontinuous initial data. It has been shown that under certain restrictions on initial data discontinuities of the shock wave type do not appear in solutions to the Navier-Stokes equations.

At the same time, it should be noted that the discontinuous approach has been used in a large number of works on shock waves in a viscous gas. For example, in the article [19], while studying the stability of planar shock waves in order to estimate the influence of small viscosity on perturbations of planar gas dynamic shock waves, it has been assumed that the width of the transitional zone is negligibly small. By this reason, the problem on perturbations propagation is reduced, as well as for an inviscid gas, to a linear initial-boundary value problem with boundary conditions on the shock wave. Another typical example is the article [18] where the 2-D steady viscous flow around immovable blunt bodies has been studied numerically. Here the bow compression shock has been treated as a strong discontinuity surface on which the corresponding jump conditions (modified Rankine-Hugoniot conditions) have been fulfilled. Clearly, the bow compression shock has been introduced in order to bound essentially the calculation domain where solutions to the Navier-Stokes equations have been sought. Steady flow regimes have been found by the stabilization method, i.e. steady-state solutions to the Navier-Stokes equations have been found as a limit as $t \to \infty$ (see [3] on the stabilization method in gas dynamics).

In the present article, on the example of the Navier-Stokes model for a compressible liquid, we will show inadmissibility of the discontinuous approach. We can make this conclusion even at the linear level. We begin with studying the initial boundary value problem (IBVP) obtained by linearization of the Navier-Stokes equations and jump conditions with respect to a piecewise constant solution. This piecewise constant solution describes the following flow regime for a viscous gas: the supersonic steady viscous flow (at $x > 0$) is separated from the subsonic flow by a planar shock wave (with the equation $x = 0$). We show that the shock is unstable independently on the character of linearized boundary conditions on $x = 0$. This directly follows from the fact that the number of independent parameters which determine an arbitrary small perturbation of the shock front is greater than the number of the linearized boundary conditions. So, the shock wave in a viscous gas which is treated as a surface of the strong discontinuity is similar to nonevolutionary (undercompressive) discontinuities in perfect media ([8], [10]).

In order to prove the linear instability we construct exponentially increasing in time particular solutions which, from the mathematical point of view, are the Hadamard examples (see, e.g., [3, 17]) and prove ill-posedness of the linear IBVP. The discovered instability indirectly justifies that steady-state flows around blunt bodies in a viscous gas with a bow compression shock can not be calculated with the stabilization method. From the physical point of view, this means that the above described stationary regime of a viscous gas with a shock wave does not practically realize.

## 2. Compressible Navier-Stokes equations and modified Rankine-Hugoniot conditions.

Since the mathematical model of a viscous gas is widely known we describe it schematically. Following [19], we suppose that the heat coefficient equals zero. The mentioned model is the Navier-Stokes equations for a compressible liquid:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{2.1}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\rho u_i u_k + P_{ik}) = 0, (i = 1, 2, 3),$$

$$\frac{\partial}{\partial t}\Big(\rho(e_0 + \frac{1}{2}|\mathbf{u}|^2)\Big) + \operatorname{div}\Big(\rho(e_0 + \frac{1}{2}|\mathbf{u}|^2 + pV)\mathbf{u} - \boldsymbol{\xi}\Big) = 0.$$

Here $\rho$ denotes the density; $\mathbf{u} = (u_1, u_2, u_3)$, the velocity of the gas; $P_{ik} = p\delta_{ik} - \sigma_{ik}$ are the components of the stress tensor; $p$ is the pressure; $\delta_{ik}$, the Kronecker delta; $\sigma_{ik} = \eta\Big(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik} \operatorname{div} \mathbf{u}\Big) + \zeta\delta_{ik} \operatorname{div} \mathbf{u}$ are the components of the viscous stress tensor; $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, $\xi_i = \sum\limits_{k=1}^{3} \sigma_{ik} u_k$, $(i = 1, 2, 3)$; $e_0$ is the internal energy, $V = 1/\rho$; $\eta$ and $\zeta$ are the first and second viscosity coefficients (they are usually assumed to be functions of $\rho$ and $s$), $s$ is the internal entropy. To make the system (1.1) closed we complete it with the Gibbs relation

$$T ds = de_0 + p dV,$$

where $T$ is the temperature, and the state equation

$$e_0 = e_0(\rho, s).$$

Then the thermodynamical parameters $T$ and $p$ are defined as follows

$$T = \frac{\partial e_0}{\partial s},\ p = \rho^2 \frac{\partial e_0}{\partial \rho},$$

and we can regard (1.1) as a system for the components of the vector $(p, s, \mathbf{u})$.

Reasoning in the usual way (see, e.g., [13], [15]), we derive the following jump conditions (the modified Rankine-Hugoniot conditions) from the system (1.1) of viscous conservation laws

$$[\rho(u_{\mathbf{n}} - D_{\mathbf{n}})] = [j] = 0, \tag{2.2}$$

$$[u_{\mathbf{n}}]j + [\mathcal{P}] = 0,$$

$$[u_{\mathbf{k},\mathbf{l}}]j = \Big[\sum_{i,k=1}^{3} \sigma_{ik}\tau_i n_k\Big],$$

$$[e_0 + \frac{1}{2}|\mathbf{u}|^2]j + [p\, u_{\mathbf{n}} - \sum_{i,k=1}^{3} \sigma_{ik} n_i u_k] = 0.$$

Here the equation $f(t, x_2, x_3) - x_1 = 0$ represents the surface of propagating strong discontinuity, $[g] = g - g_\infty$ denotes the jump of values of a discontinuous function $g$, $g_\infty = g|_{f(t,x_2,x_3)-x_1 \to +0}$ is the value on left side of the shock; $u_{\mathbf{n}} = (\mathbf{u}, \mathbf{n})$, $u_{\mathbf{k}} = (\mathbf{u}, \mathbf{k})$, $u_{\mathbf{l}} = (\mathbf{u}, \mathbf{l})$, $\mathbf{n} = (n_1, n_2, n_3) = \frac{1}{\sqrt{1+f_{x_2}^2+f_{x_3}^2}}(-1, f_{x_2}, f_{x_3})$ is the unit normal to the discontinuity front, $D_{\mathbf{n}} = -\frac{f_t}{\sqrt{1+f_{x_2}^2+f_{x_3}^2}}$ is the projection of the strong discontinuity velocity onto $\mathbf{n}$; $\mathbf{k} = (\tau_1, \tau_2, \tau_3) = (f_{x_2}, 1, 0)$ and $\mathbf{l} = (\tau_1, \tau_2, \tau_3) = (f_{x_3}, 0, 1)$ are the vectors, orthogonal to the vector $\mathbf{n}$; $j = \rho(u_{\mathbf{n}} - D_{\mathbf{n}})$ is the mass transfer flux across the discontinuity surface, $\mathcal{P} = p - \sum_{i,k=1}^{3} \sigma_{ik} n_i n_k$.

In the case of the shock wave ($[\rho] \neq 0$, $j \neq 0$), (2.2) easily reduces to a system which is similar to the Rankine-Hugoniot relations in gas dynamics (see [13]):

$$[j] = 0, \tag{2.2$'$}$$

$$[u_{\mathbf{n}}]^2 + [\mathcal{P}][V] = 0,$$

$$[u_{\mathbf{k},\mathbf{l}}] = \frac{1}{j}\Big[\sum_{i,k=1}^{3} \sigma_{ik}\tau_i n_k\Big],$$

$$[e_0] + \frac{\mathcal{P} + \mathcal{P}_\infty}{2}[V] =$$

$$= \frac{1}{2j^2}\Big[\sum_{i=1}^{3}\Big(\sum_{k=1}^{3}\sigma_{ik} n_k\Big)^2 - \Big(\sum_{i,k=1}^{3}\sigma_{ik} n_i n_k\Big)^2\Big].$$

**3. Formulation of the linear IBVP.**

Now we formulate the linear IBVP mentioned in Section 1. With this purpose we consider solutions to the system (2.1) which describe steady-state flows in a viscous gas with a strong shock. Obviously, we can take, for example, the following piecewise constant solution which, on the other hand, is the approximate solution to the system (2.1) with small viscosity and heat conductivity:

$$\mathbf{u} = (\hat{u}_1, 0, 0),\ \rho = \hat{\rho},\ s = \hat{s} \text{ for } x_1 > 0, \tag{3.1}$$

$$\mathbf{u} = (\hat{u}_{1\infty}, 0, 0),\ \rho = \hat{\rho}_\infty,\ s = \hat{s}_\infty \text{ for } x_1 < 0,$$

where the constants $\hat{u}_1$, $\hat{\rho}$, $\hat{s}$, $\hat{u}_{1\infty}$, $\hat{\rho}_\infty$, and $\hat{s}_\infty$ satisfy the jump conditions (2.2′) on the plane $x_1 = 0$:

$$\widehat{j} = \hat{\rho}\hat{u}_1 = \hat{\rho}_\infty \hat{u}_{1\infty}, \tag{3.2}$$

$$(\hat{u}_1 - \hat{u}_{1\infty})^2 + (\hat{p} - \hat{p}_\infty)(\widehat{V} - \widehat{V}_\infty) = 0,$$

$$(\hat{e}_0 - \hat{e}_{0\infty}) + \frac{(\hat{p} + \hat{p}_\infty)}{2}(\widehat{V} - \widehat{V}_\infty) = 0.$$

The constants $\hat{u}_{1\infty}$, $\hat{\rho}_\infty$, and $\hat{s}_\infty$ are parameters of the coming flow of a viscous and heat conducting gas, moreover,

$$\hat{u}_{1\infty} > \hat{c}_\infty > 0,\ \ \hat{\rho}_\infty > 0, \tag{3.3}$$

$$\hat{p}_\infty = \hat{\rho}_\infty^2 \frac{\partial e_0}{\partial \rho}(\hat{\rho}_\infty, \hat{s}_\infty),\ \widehat{V}_\infty = 1/\hat{\rho}_\infty,\ \hat{e}_{0\infty} = e_0(\hat{\rho}_\infty, \hat{s}_\infty);\ \hat{c}_\infty =$$

$$= \sqrt{\frac{\partial}{\partial \rho}\Big(\rho^2 \frac{\partial e_0}{\partial \rho}\Big)(\hat{\rho}_\infty, \hat{s}_\infty)}$$

is the sound speed in the coming flow (see [13]); the constants $\hat{u}_1$, $\hat{\rho}$, and $\hat{s}$ are parameters behind the shock wave,

$$0 < \hat{u}_1 < \hat{c},\ \ \hat{\rho} > 0, \tag{3.4}$$

$$\hat{p} = \hat{\rho}^2 \frac{\partial e_0}{\partial \rho}(\hat{\rho}, \hat{s}),\ \widehat{V} = 1/\hat{\rho},\ \hat{e}_0 = e_0(\hat{\rho}, \hat{s}),\ \hat{c} = \sqrt{\frac{\partial}{\partial \rho}\Big(\rho^2 \frac{\partial e_0}{\partial \rho}\Big)(\hat{\rho}, \hat{s})}$$

is the sound speed behind the shock wave. Additionally, we suppose that the state equation $e_0 = e_0(\rho, s)$ satisfies the requirements for the so-called normal gas (see, e.g., [13]). As known, in this case (see, e.g., [13, 14]) inequalities (3.3), (3.4), and the following compressibility conditions

$$\hat{p} > \hat{p}_\infty,\ \hat{\rho} > \hat{\rho}_\infty,\ \hat{u}_{1\infty} > \hat{u}_1,\ \hat{s} > \hat{s}_\infty \tag{3.5}$$

are valid.

Thus, from the physical point of view, existence of such a piecewise constant solution means that we have a planar shock wave which separates the supersonic coming steady viscous flow (the Mach number ahead of the shock wave, $M_\infty = \hat{u}_{1\infty}/\hat{c}_\infty > 1$; see (3.3)) and the subsonic flow behind the shock wave (the Mach number behind the shock wave, $M = \hat{u}_1/\hat{c} < 1$; see (3.4)). It is natural to raise the issue if such a flow regime realizes physically. For an inviscid gas this issue has been studied in detail in [3]. Below we show that in a viscous heat conducting gas such a regime is unstable with respect to small perturbations.

After linearization of the system (2.1) and the jump conditions (2.2) with respect to the piecewise constant solution (3.1), (3.2) we obtain the linear IBVP in the dimensionless form. Its one-dimensional variant looks as follows

$$\begin{gathered} Lp + u_x = 0, \\ Lu + \frac{1}{M^2}\, p_x = r\ \ u_{xx}, \\ Ls = 0 \\ \text{for}\ \ x > 0; \end{gathered} \tag{3.6}$$

$$\begin{gathered} L_\infty p_\infty + (u_\infty)_x = 0, \\ L_\infty u_\infty + \frac{1}{M_\infty^2}\,(p_\infty)_x = r_\infty (u_\infty)_{xx} \\ \text{for}\ x < 0; \end{gathered} \tag{3.7}$$

$$\begin{gathered} u + dp - \hat{d} r u_x = \hat{u}\Big\{u_\infty + d_\infty p_\infty - \hat{d}_\infty r_\infty (u_\infty)_x\Big\}, \\ \nu p + \widehat{N} s - \hat{\nu} r u_x = \hat{u}\Big\{\nu_\infty p_\infty + \hat{\nu} r_\infty (u_\infty)_x\Big\}, \\ F_t = \mu\{u + p - u_\infty - p_\infty - \widehat{N} s\} \end{gathered} \tag{3.8}$$

for $x = 0$.

Here $p$, $u$, $s$ are small perturbations of pressure, velocity and entropy at $x > 0$; $p_\infty$, $u_\infty$ are small perturbations of pressure and velocity at $x < 0$ (without the loss of generality we can suppose that the perturbations of the entropy $s_\infty$ equal zero at $x < 0$); $p$, $u$, $s$, $p_\infty$, $u_\infty$ are related to the following characteristic parameters: $\hat\rho\hat c^2$ is the pressure; $\hat u_1$, the velocity; $\hat s$, the entropy; $\hat\rho_\infty\hat c_\infty^2$, $\hat u_{1\infty}$; $L = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$, $L_\infty = \frac{1}{\hat u}\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$; the time $t$ and the coordinate $x$ are related to the following characteristic values: $\hat l/\hat u_1$ is the time, $\hat l$ the length; $r = \frac{4}{3R_1} + \frac{1}{R_2}$, $r_\infty = \frac{4}{3R_{1\infty}} + \frac{1}{R_{2\infty}}$; $R_{1,2,1\infty,2\infty}$ are the Reynolds numbers, where: $R_1 = \frac{\hat\rho\,\hat u_1\hat l}{\widehat\eta}$, $R_2 = \frac{\hat\rho\,\hat u_1\hat l}{\widehat\zeta}$ and so on; $\widehat\eta = \eta(\hat\rho, \hat s)$, $\widehat\zeta = \zeta(\hat\rho, \hat s)$; $\hat u = \hat u_{1\infty}/\hat u_1$, where $\hat u > 1$ (see (3.5)); $d = \frac{1+M^2}{2M^2} + \frac{\beta^2}{2M^2}\widehat L$, $\beta^2 = 1 - M^2$, $\hat d = \frac{1+\widehat L}{2}$, $d_\infty = \frac{M_\infty^2+1}{2M_\infty^2} + \frac{\beta_\infty^2}{2M_\infty^2}\widehat L$, $\beta_\infty^2 = M_\infty^2 - 1$, $\hat d_\infty = \frac{1-\widehat L}{2}$, $\nu = \frac{\beta^2}{M^2}\widehat L$, $\hat\nu = \widehat L$, $\nu_\infty = \frac{\beta_\infty^2}{M_\infty^2}\widehat L$, $\widehat N = -\frac{\hat s\,(e_0)_{Vs}(\hat\rho,\hat s)}{\widehat V\,(e_0)_{VV}(\hat\rho,\hat s)}$, $\mu = \frac{\hat u}{\hat u - 1}$, $\widehat L = \frac{1}{1-\widehat D}$, $\widehat D = \frac{2\widehat T\hat s}{\hat u_1^2(\hat u-1)\widehat N}$; $F(t)$ is a small perturbation of the shock wave front. We note that the problem (3.6)–(3.8) does not contain the characteristic length $\widehat l$. However, we will see later that the obtained result does not depend on the concrete value $\widehat l$.

Following [19] and taking zero $p_\infty$, $u_\infty$, instead of (3.6)–(3.8), we consider a simpler variant:

$$Lp + u_x = 0, \tag{3.9}$$

$$Lu + \frac{1}{M^2}\,p_x = r\,u_{xx}$$

$$\text{for } x > 0;$$

$$u + dp = \hat d\,r\,u_x \ \text{ for } x = 0. \tag{3.10}$$

The function $s$ is found from the problem

$$Ls = 0 \ \text{ at } x > 0,$$

$$\widehat N\,s = \hat\nu\,r\,u_x - \nu\,p \ \text{ at } x = 0;$$

and the function $F(t)$ is determined from the relation

$$F_t = \mu\,\{u + p - \widehat N\,s\} \ \text{ at } x = 0.$$

The problems (3.6)–(3.8) and (3.9)–(3.10) reduce to simpler problems. Indeed, rewriting the second equation in (3.6)

$$\tau u + \xi\Big(\frac{1}{M^2}p + u - r\,\xi u\Big) = 0,\ \tau = \frac{\partial}{\partial t},\ \xi = \frac{\partial}{\partial x},$$

and introducing the potential $\varphi = \varphi(t, x)$

$$u = \xi\varphi,\ p = M^2 r\,\xi^2\varphi - M^2 L\varphi, \tag{3.11}$$

we obtain from (3.11) and the first equation in (3.6) that

$$\Big\{M^2L^2 - \xi^2 - M^2 r\,L\xi^2\Big\}\,\varphi = 0 \ \text{ at } \ x > 0. \tag{3.12}$$

In a similar way we introduce the potential $\psi = \psi(t, x)$ for the system (3.7):

$$u_\infty = \xi\psi,\ \ p_\infty = r_\infty\, M_\infty^2\xi^2\psi - M_\infty^2 L_\infty\psi \tag{3.13}$$

and

$$\Big\{M_\infty^2 L_\infty^2 - \xi^2 - M_\infty^2\, r_\infty\, L_\infty\xi^2\Big\}\,\psi = 0 \ \text{ at } \ x < 0. \tag{3.14}$$

Using (3.11), (3.13), we rewrite the first boundary condition in (3.8) as follows:

$$\Big\{(1 - \widehat{L})\,(M^2 r\,\xi^2 + \beta^2\,\xi) - 2M^2 d\,\tau\Big\}\,\varphi = \tag{3.15}$$

$$= \Big\{\hat{u}(1 + \widehat{L})\,(M_\infty^2 r_\infty\,\xi^2 - \beta_\infty^2\,\xi) - 2M_\infty^2 d_\infty\tau\Big\}\,\psi \ \text{ at } \ x = 0.$$

Consequently, instead of the mixed problem (3.6)–(3.8) we obtain the problem (3.12), (3.14), (3.15) (the function $s$ is found from the equation $Ls = 0$ at $x > 0$ and the second boundary condition in (3.8)). The function $F(t)$ is determined from the third relation in (3.8).

The problem (3.9), (3.10) reduces to the following:

$$\Big\{M^2L^2 - \xi^2 - M^2 r\,L\xi^2\Big\}\,\varphi = 0 \ \text{ at } \ x > 0, \tag{3.12}$$

$$\tau\varphi = \gamma\,\{M^2 r\,\xi^2 + \beta^2\,\xi\}\,\varphi \ \text{ at } \ x = 0. \tag{3.16}$$

Here $\gamma = \frac{1-\widehat{L}}{2M^2 d}$.

Concluding the paragraph, we place the two-dimensional variant of the linear IBVP on stability of the shock wave in a viscous gas:

$$\begin{gathered} Lp + u_x + v_y = 0, \\ Lu + \frac{1}{M^2} p_x = r\, u_{xx} + \frac{1}{R_1} u_{yy} + r_1 v_{xy}, \\ Lv + \frac{1}{M^2} p_y = \frac{1}{R_1} v_{xx} + r\, v_{yy} + r_1 u_{xy}, \\ Ls = 0 \ \text{ at } \ x > 0; \end{gathered} \tag{3.17}$$

$$\begin{gathered} L_\infty p_\infty + (u_\infty)_x + (v_\infty)_y = 0, \\ L_\infty u_\infty + \frac{1}{M_\infty^2}(p_\infty)_x = r_\infty (u_\infty)_{xx} + \frac{1}{R_{1\infty}}(u_\infty)_{yy} + r_{1\infty}(v_\infty)_{xy}, \\ L_\infty v_\infty + \frac{1}{M_\infty^2}(p_\infty)_y = \frac{1}{R_{1\infty}}(v_\infty)_{xx} + r_\infty (v_\infty)_{yy} + r_{1\infty}(u_\infty)_{xy} \\ \text{at } \ x < 0; \end{gathered} \tag{3.18}$$

$$\begin{gathered} u + dp - \hat{d}\, r\, u_x - \hat{d}\, r_2\, v_y = \\ = \hat{u}\Big\{ u_\infty + d_\infty p_\infty - \hat{d}_\infty r_\infty (u_\infty)_x - \hat{d}_\infty r_{2\infty}(v_\infty)_y \Big\}, \\ F_y = \frac{1}{\hat{u}-1}\Big\{ v - \frac{1}{R_1}(u_y + v_x) - \hat{u}\, v_\infty + \frac{\hat{u}}{R_{1\infty}}\,((u_\infty)_y + (v_\infty)_x)\Big\}, \\ F_t = \mu\,(u + p - u_\infty - p_\infty - \widehat{N}\, s), \\ \nu p + \widehat{N} s - \hat{\nu}\, r\, u_x - \hat{\nu}\, r_2\, v_y = \hat{u}\{\nu_\infty p_\infty + \hat{\nu}\, r_\infty (u_\infty)_x + \hat{\nu}\, r_{2\infty}(v_\infty)_y\} \\ \text{at } \ x = 0. \end{gathered} \tag{3.19}$$

Here $r_1 = \frac{1}{3R_1} + \frac{1}{R_2}$, $r_2 = \frac{1}{R_2} - \frac{2}{3R_1}$, $r_{1\infty} = \frac{1}{3R_{1\infty}} + \frac{1}{R_{2\infty}}$, $r_{2\infty} = \frac{1}{R_{2\infty}} - \frac{2}{3R_{1\infty}}$. The rest definitions have been given above.

We seek a special solution to (3.17)–(3.19). Now we obtain some relations which simplify the seeking procedure. With this end we rewrite the first three equations in the form:

$$Lp + \bar{\lambda} = r\,\Delta p, \tag{3.20}$$

$$L\bar{u} + \frac{1}{M^2}\,\xi p = \frac{1}{R_1}\,\eta\Omega,$$

$$L\bar{v} + \frac{1}{M^2}\,\eta p = -\frac{1}{R_1}\,\xi\Omega.$$

Here $\bar{u} = u + r\,\xi p$, $\bar{v} = v + r\,\eta p$, $\eta = \frac{\partial}{\partial y}$, $\Delta = \xi^2 + \eta^2$, $\Omega = \eta u - \xi v = \eta\bar{u} - \xi\bar{v}$, $\bar{\lambda} = \xi\bar{u} + \eta\bar{v}$. It follows from (3.20) that

$$L\Omega = \frac{1}{R_1}\,\Delta\Omega, \tag{3.21}$$

$$L\bar{\lambda} + \frac{1}{M^2}\,\Delta p = 0.$$

We analyze (3.20), (3.21) and conclude that if the functions $\varphi = \varphi(t, x, y)$ and $\Phi = \Phi(t, x, y)$ satisfy the following equations

$$\{M^2 L^2 - \Delta - M^2 r\, L\, \Delta\}\,\varphi = 0, \tag{3.22}$$

$$\left\{L - \frac{1}{R_1}\,\Delta\right\}\Phi = 0$$

$$\text{at } x > 0,$$

the functions $p$, $u$, and $v$,

$$p = -M^2 L\varphi, \tag{3.23}$$

$$u = \xi\varphi + \eta\Phi + M^2 r\, L\, \xi\varphi,$$

$$v = \eta\varphi - \xi\Phi + M^2 r\, L\, \eta\varphi,$$

are the solution to (3.17); the function $s$ can be found from the equation

$$Ls = 0 \text{ at } x > 0$$

and the last boundary condition in (3.19). Similarly, let the functions $\psi(t, x, y)$ and $\Psi(t, x, y)$ satisfy the equations

$$\{M_\infty^2 L_\infty^2 - \Delta - M_\infty^2 r_\infty\, L_\infty\, \Delta\}\,\psi = 0, \tag{3.24}$$

$$\left\{L_\infty - \frac{1}{R_{1\infty}}\,\Delta\right\}\Psi = 0$$

$$\text{at } x < 0.$$

Then the functions

$$p_\infty = -M_\infty^2 \, L_\infty \psi, \tag{3.25}$$

$$u_\infty = \xi\psi + \eta\Psi + M_\infty^2 \, r_\infty \, L_\infty \, \xi\psi,$$

$$v_\infty = \eta\psi - \xi\Psi + M_\infty^2 \, r_\infty \, L_\infty \, \eta\psi$$

are the solution to (3.18). Boundary conditions for the systems (3.22), (3.24) are obtained by substituting the presentation (3.23), (3.25) into the first relations in (3.19); here $F(t, y)$ is excluded by crossing differentiation from the second and third boundary conditions.

## 4. Linear instability of the shock wave.

Here we prove that the flow of a viscous gas with the shock front from the previous section is unstable. To make the explanation more informative we firstly turn our attention to the problem (3.12), (3.16), the simplest case of the linear IBVP on stability of a shock wave in a viscous gas. We look for solutions of a special form:

$$\varphi = \hat{\varphi}\, \mathrm{e}^{n(\frac{S}{r}t + \frac{\Lambda}{r}x)} \ \text{ at } \ x \geq 0, \tag{4.1}$$

where $\hat{\varphi}$, $S$, and $\Lambda$ are some constants such that

$$\operatorname{Re} S > 0, \operatorname{Re} \Lambda < 0; \tag{4.2}$$

$n$ is an integer number. Substituting (4.1) into (3.12) and taking $\hat{\varphi} \neq 0$, we obtain:

$$(S + \Lambda)\, \Lambda^2 = \varepsilon^2 \left( (S + \Lambda)^2 - \left(\frac{\Lambda}{M}\right)^2 \right), \varepsilon^2 = \frac{1}{n}. \tag{4.3}$$

We find roots of the equation (4.3), assuming that $n$ is large, the value $S$ is known and can be decomposed into the series:

$$S = S_0 + \varepsilon\, S_1 + \dots .$$

So, assuming that

$$\Lambda = \Lambda^{(0)} + \varepsilon\, \Lambda^{(1)} + \dots ,$$

we successively find:

$$\Lambda_1 = \Lambda_1^{(0)} + \varepsilon\,\Lambda_1^{(1)} + \dots,\ \Lambda_2 = \varepsilon\,\Lambda_2^{(1)} + \dots,\ \Lambda_3 = \varepsilon\,\Lambda_3^{(1)} + \dots.$$

Here $\Lambda_1^{(0)} = -S_0$, $\Lambda_1^{(1)} = -S_1$, $\Lambda_2^{(1)} = -\sqrt{S_0}$, $\Lambda_3^{(1)} = \sqrt{S_0}$. Therefore, if $S_0 > 0$, then $\Lambda_1^{(0)} < 0$, $\Lambda_2^{(1)} < 0$, and (4.2) is valid for the roots $\Lambda_{1,2}$, $S$, and a sufficiently large $n$.

Finally, we look for the solution to (3.12), (3.16) in the following form:

$$\varphi = \Big(\hat{\varphi}_1\, \mathrm{e}^{n\frac{\Lambda_1}{r}x} + \hat{\varphi}_2\, \mathrm{e}^{n\frac{\Lambda_2}{r}x}\Big)\mathrm{e}^{n\frac{S}{r}t}. \tag{4.4}$$

Here the constants $\hat{\varphi}_{1,2}$ are determined by a single (!) relation:

$$\Big\{\gamma(\varepsilon^2\beta^2\Lambda_1 + M^2\Lambda_1^2) - \varepsilon^2 S\Big\}\,\hat{\varphi}_1 + \Big\{\gamma(\varepsilon^2\beta^2\Lambda_2 + M^2\Lambda_2^2) - \varepsilon^2 S\Big\}\,\hat{\varphi}_2 = 0.$$

So, (3.12), (3.16) always have a nontrivial (4.4)-like solution; the constants $S$, $\Lambda_1$, and $\Lambda_2$ satisfy (4.2). As known, existence of such a solution proves instability of the flow with a shock wave. From the mathematical point of view, we have proved ill-posedness of (3.12), (3.16) since the consequence of solutions

$$\varphi_n(t,x) = \mathrm{e}^{-\sqrt{n}+n\frac{S}{r}t}\Big(\hat{\varphi}_1\, \mathrm{e}^{n\frac{\Lambda_1}{r}x} + \hat{\varphi}_2\, \mathrm{e}^{n\frac{\Lambda_2}{r}x}\Big)$$

is a Hadamard-type example (on the Hadamard example see [17]).

Now we turn to a more difficult problem (3.12), (3.14), (3.15). The solution to (3.12) is again sought in the form (4.4). We write the solution to (3.14) in the form:

$$\psi = \hat{\psi}\, \mathrm{e}^{n(\frac{S}{r}t + \frac{\widetilde{\Omega}}{r}x)} \quad \text{for} \ \ x \le 0. \tag{4.5}$$

Here $\hat{\psi}$, $\widetilde{\Omega}$ are constants,

$$\operatorname{Re}\widetilde{\Omega} > 0. \tag{4.6}$$

To define $\widetilde{\Omega}$ we obtain the algebraic equation

$$(\widetilde{S} + \widetilde{\Omega})\,\widetilde{\Omega}^2 = \varepsilon^2\left\{(\widetilde{S} + \widetilde{\Omega})^2 - \Big(\frac{\widetilde{\Omega}}{M_\infty}\Big)^2\right\}\frac{r}{r_\infty},\ \widetilde{S} = \frac{S}{\hat{u}}. \tag{4.7}$$

The needed root $\widetilde{\Omega}$ is sought as the expansion into series

$$\widetilde{\Omega} = \varepsilon\,\widetilde{\Omega}^{(1)} + \dots.$$

Here $\widetilde{\Omega}^{(1)} = \sqrt{\frac{S_0}{\hat{u}} \frac{r}{r_\infty}}$. There are no other roots of (4.7) with the property (4.6). Consequently, the problem (3.12), (3.14), (3.15) has the solution (4.4), (4.5); moreover, we have a single relation to determine three constants $\hat{\varphi}_{1,2}$, and $\hat{\psi}$ which has been derived from (3.15) by substituting (4.4), (4.5). Thus, we have demonstrated instability of the viscous flow with a shock wave in the 1-D case.

It is easy to show instability of the discontinuous viscous flow in the case of the planar symmetry using the one-dimensional examples of instability from above. Indeed, for the system (3.22) we seek the function $\varphi(t, x, y)$ in the form (4.4) with the already known $\Lambda_1$ and $\Lambda_2$. The function $\Phi(t, x, y)$ is determined as follows:

$$\Phi = \widehat{\Phi}\, \mathrm{e}^{n(\frac{S}{r}t + \frac{Q}{r}x)} \text{ for } x \geq 0. \tag{4.8}$$

Here $\hat{\Phi}$ and $Q$ are constants,

$$\operatorname{Re} Q < 0\,. \tag{4.9}$$

To define $Q$ we obtain the algebraic equation

$$Q^2 = \varepsilon^2 R_1 r (S + Q). \tag{4.10}$$

The needed root $Q$, satisfying (4.9), is sought as the expansion into the series

$$Q = \varepsilon\, Q^{(1)} + \dots.$$

Here $Q^{(1)} = -\sqrt{R_1 r S_0}$. For (3.24) the function $\psi(t, x, y)$ is sought in the form (4.5) with the already known $\widetilde{\Omega}$, the function $\Psi(t, x, y)$ is determined as follows:

$$\Psi = \widehat{\Psi}\, \mathrm{e}^{n(\frac{\widetilde{S}}{r}t + \frac{G}{r}x)} \text{ for } x \leq 0. \tag{4.11}$$

Here $\widehat{\Psi}$ and $G$ are constants,

$$\operatorname{Re} G > 0. \tag{4.12}$$

To determine $G$ we use the following algebraic relation:

$$G^2 = \varepsilon^2 R_{1\infty} r (\widetilde{S} + G). \tag{4.13}$$

Accounting (4.12), we seek $G$ as the expansion into the series:

$$G = \varepsilon\, G^{(1)} + \dots.$$

Here $G^{(1)} = -\sqrt{R_{1\infty}\, r\, \frac{S_0}{\hat{u}}}$. Thus, we seek the solution to (3.22), (3.24) in the form (4.4), (4.8), (4.5), (4.11); we have only two relations to determine the constants $\hat{\varphi}_{1,2}$, $\widehat{\Phi}$, $\hat{\psi}$, and $\widehat{\Psi}$. Thus we can write down a nontrivial solution to this system. Using this solution and (3.23), (3.25), we construct a solution of the original linear IBVP (3.17)–(3.19). Existence of such a solution proves instability of the discontinuous viscous flow for the case of the planar symmetry.

In this connection, the result from [19] that small perturbations decrease exponentially in time despite of the form of boundary conditions takes aback. The reason of the wrong conclusion in [19] is that the problem (3.17)–(3.19) has been substituted by a problem which has no relation to the issue.

In addition, we note that in each considered variant the function $F(t)$ was represented as follows:

$$F(t) = F_0\, \mathrm{e}^{n \frac{S}{r} t},$$

where the constant $F_0$ is determined easily from an appropriate boundary condition (see (3.8) and (3.19)).

## 5. Concluding remarks.

The main conclusion of the present work is that, in the framework of the Navier-Stokes mathematical model, the shock wave in a viscous gas can not be considered as a surface of a strong discontinuity since instability of the corresponding discontinuous flow which has been stated above means that such a flow can not be realized physically. This instability also means that application of the stabilization method in calculation of steady-state flows around blunt bodies in a viscous gas with a bow compression shock has no sufficient ground.

The author is indebted to Yu. L. Trakhinin, for valuable advises, proposals, and useful discussions, and E. V. Mishchenko for the help in preparation of this work.

This work was partially supported by Russian Foundation for Basic Research (01−01−00781 and 02−01−00641), by INTAS, project "Conservation laws of mechanics of continua: waves and fronts", 868.

## REFERENCES


[1] A.M. Blokhin , *Estimation of the energy integral of a mixed problem for gas dynamics equations with boundary conditions on the shock wave,* Siberian Math. J., 22 - 4 (1981), pp. 501–523; translation from Sibirsk. Mat. Zh., 22 - 4, (1981), pp. 23–51.

[2] A.M. Blokhin, *Uniqueness of the classical solution of a mixed problem for equations of gas dynamics with boundary conditions on a shock wave,* Siberian Math. J., 23 - 5 (1982), pp. 604–615; translation from Sibirsk. Mat. Zh., 23 - 5 (1982), pp. 17–30 .

[3] A.M. Blokhin, *Energy integrals and their applications to gas dynamics problems,* (in Russian), Novosibirsk: Nauka 1986.

[4] A.M. Blokhin, *Strong discontinuities in Magnetohydrodynamics,* Nova Science Publishers, N. Y., 1993.

[5] S.P. Dyakov, *On stability of shock waves,* (in Russian), Journal of Experimental and Theoretical Physics, 27 3(9) (1954), pp. 288–295.

[6] D. Hoff, *Global well-posedness of the Cauchy problem for the Navier-Stokes equations of nonisentropic flow with discontinuous initial data,* J. Differ. Equations, 95 (1992), pp. 33–74.

[7] A.V. Kazhikhov, *About global solvability of boundary value problems for one-dimensional equations of a viscous heat gas,* (in Russian), Dynamics of continuum media, 24, pp. 45–61.

[8] J.U. Kim, *Global existence of solutions of the equations of one-dimensional thermoviscoelasticity with initial data in BV an $L^1$,* Ann. Sc. Norm. Super. Pisa, 10 (1983), pp. 357–427 .

[9] L.D. Landau - E.M. Lifshiz, *Fluid Mechanics. Course of Theoretical Physics,* Pergamon Press, Vol. 6, New York-Oxford, 1997.

[10] L.D. Landau - E.M. Lifshiz, *Electrodynamics of Continuous Media. Course of Theoretical Physics,* Oxford-London-New York-Paris: Pergamon Press, 1960.

[11] T.P. Liu, *Shock waves for compressible Navier-Stokes equations are stable,* Comm. Pure Appl. Math., 39 (1986), pp. 565–594.

[12] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables,* Springer-Verlag, New York, 1984.

[13] L.V. Ovsyannikov, *Lecture on gas dynamics,* (in Russian) Nauka, Moscow, 1981.

[14] B.L. Rozhdestvenskii - N.N. Janenko, *Systems of quasilinear equations and their applications to gas dynamics,* Translations of Mathematical Monographs, 55. American Mathematical Society, Providence, 1983.

[15] L.I. Sedov, *Mechanics of Continuous Media,* (in Russian) Nauka, Vol. 1, Moscow, 1983.

[16] D. Serre, *Sur l'équation monodimensionnelle d'un fluide visqueux, compressible et conducteur de chaleur,* C. R. Acad. Sci. Paris, Sér. I, 303 (1986), pp. 703–706 .

[17] S.L. Sobolev, *Equations of Mathematical Physics,* (in Russian) Nauka, Moscow, 1966.

[18] J.C. Tannehill - T.L. Holst - J.V. Rakich, *Numerical computation of two-dimensional viscous blunt body flows with an impinging shock,* AIAA J., 14 (1976), pp. 204–211.

[19] A.M. Zajdel', *Evolution of perturbations for planar shock waves,* (in Russian), Applied Mechanics and Technical Physics, 8 - 4 (1967), pp. 30–39.

*Sobolev Institute of Mathematics,*
*Russian Academy of Science,*
*Koptyuga pr. 4, 630090*
*Novosibirsk (RUSSIA)*
*e-mail:*blokhin@math.nsc.ru