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THE CONDITIONS FOR BLOW-UP AND GLOBAL EXISTENCE OF SOLUTIONS FOR A DEGENERATE AND SINGULAR PARABOLIC EQUATION WITH A NON-LOCAL SOURCE

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In this paper, we consider the degenerate and singular porous medium equation with a non-local source: $v_\tau = \left(\xi^\beta (v^m)_\xi\right)_\xi + \int_0^a F(v^m)\, d\xi$. The conditions on the local and global existence of solutions are investigated. In the case of blow-up, the blow-up set is shown. Moreover the uniform blow-up profile of the blow-up solution is given.

1. Introduction

In this article, we find the conditions for the existence of global solution or blow-up solutions for the degenerate parabolic equation with a non-local source term as follows:

$$\left.\begin{aligned} v_\tau = \left(\xi^\beta (v^m)_\xi\right)_\xi + \int_0^a F(v^m)\, d\xi, (\xi,\tau) \in (0,a)\times(0,\infty), \\ v(0,\tau) = v(a,\tau) = 0, \tau > 0, \\ v(\xi,0) = v_0(\xi), \xi \in [0,a], \end{aligned}\right\} \tag{1}$$

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where β, m and a are constants with $a > 0$, $\beta \in [0,1)$ and $m > 1$, and F and v_0 are determined functions. Throughout this paper, we assume that $F(0) = 0$, $F(s) > 0$, $F'(s) > 0$ for $s > 0$ and the function v_0 satisfies the following:
(H1) $v_0 \in C^{2+\alpha}(0,a) \cap C[0,a]$ with $0 < \alpha < 1$,
(H2) $v_0 > 0$ on $(0,l), v_0(0) = v_0(a) = 0,$ and $v_{0\xi}(0) > 0$ and $v_{0\xi}(a) < 0$,
(H3) $\left(\xi^\beta \left(v_0^m(\xi)\right)_\xi\right)_\xi + \int_0^a F(v_0^m(\xi))d\xi > 0$ for $\xi \in (0,a)$,
(H4) (compatibility condition) $\lim\limits_{\xi \to 0^+} \left(\xi^\beta \left(v_0^m(\xi)\right)_\xi\right)_\xi = -\int_0^a F(v_0^m(\xi))d\xi$
and $\lim\limits_{\xi \to a^-} \left(\xi^\beta \left(v_0^m(\xi)\right)_\xi\right)_\xi = -\int_0^a F(v_0^m(\xi))d\xi$,
(H5) $\left(\xi^\beta \left(v_0^m\right)_\xi\right)_\xi \leq 0$ for $\xi \in (0,a)$.
We note that since$\beta \in [0,1)$, the coefficients of term $v_\xi, v_{\xi\xi}$ may tend to 0 or ∞ as ξ converges to 0^+, this implies that (1) is degenerate and singular. First of all, we introduce the definition of blow-up.

Definition 1.1. The solution v of (1) is said to blow up in a finite time at the point ξ_b if there are a time $\tau_b (> 0)$ and a sequence $\{(\xi_n, \tau_n)\}$ in $(0,a) \times (0,\infty)$ such that $(\xi_n, \tau_n) \to (\xi_b, \tau_b)$ as $n \to \infty$ and $\lim\limits_{n\to\infty} v(\xi_n, \tau_n) = \infty$. The point ξ_b and time τ_b are called a blow-up point and blow-up time, respectively. Furthermore, we call the set of all blow-up points to be the blow-up set. If the blow-up set contains every point of $[0,a]$, we say that the solution v of (1) is global blow-up.

In the past several decades, many research papers study the global existence or the blow-up property for solutions of various degenerate parabolic equations with a non-local source. (see [1, 4, 5, 8, 9, 11, 15–17] and reference there in). On the other hand, there are a few research papers of degenerate porous medium equations due to blow-up phenomena.

In 2001, W. Deng, Z. Duan and C. Xie [18] established the conditions to guarantee the occurrence for blow-up in a finite time for a porous medium problem with a non-local source:

$$\left.\begin{aligned} v_\tau = (v^m)_{\xi\xi} + a\int_{-l}^{l} v^q d\xi, \xi \in (-l,l), \tau > 0,\\ v(-l,\tau) = v(l,\tau) = 0, \tau > 0,\\ v(\xi,0) = v_0(\xi) \geq 0, \xi \in [-l,l], \end{aligned}\right\} \tag{2}$$

where $l > 0$, $a > 0$, $q > m > 1$ and v_0 is a specified function. In addition, they showed that the solution v of (2) blows up in a finite time.

In 2003,Q. Liu, Y. Chen and C. Xie [14] consider the non-local degenerate parabolic problem:

$$\left.\begin{array}{c} v_\tau = \xi^\alpha (v^m)_{\xi\xi} + a\displaystyle\int_0^l v^p d\xi - kv^q, \xi \in (0,l), \tau > 0, \\ v(0,\tau) = v(l,\tau) = 0, \tau > 0, \\ v(\xi,0) = v_0(\xi), \xi \in (0,l), \end{array}\right\} \tag{3}$$

where $l > 0$, $0 < \alpha < 2$, $p \geq q > m > 1$, and v_0 is a given function. They showed that the existence of a unique positive classical solution v of (3) and constructed conditions to blow-up in a finite time.

This paper is organized as follows. In section 2, we prove that the local existence and uniqueness of the solution v of (1). In section 3, we construct the condition on global existence for solutions of (1) and the condition which ensure the occurrence for blow-up in a finite time. The blow-up set and uniform blow up profile of the blow-up solution v of (1) are shown in last section.

2. Local existence

Since the (1) is degenerate and singular, the theory of partial differential equations in parabolic type can not apply directly. To investigate the local existence of the solution u of (1), we need to transform the (1) into the equivalent problem by letting $u = v^m, \tau = \frac{t}{ma^{\beta-2}}$ and $\xi = ax$. The problem associating to (1) is:

$$\left.\begin{array}{c} u_t = u^r\left[\left(x^\beta u_x\right)_x + a^{3-\beta}\displaystyle\int_0^1 F(u)\,dx\right], (x,t) \in (0,1)\times(0,\infty), \\ u(0,t) = u(1,t) = 0, t > 0, \\ u(x,0) = u_0(x), x \in [0,1], \end{array}\right\} \tag{4}$$

where $0 < r = \frac{m-1}{m} < 1, u_0 = v_0^m$ and the function u_0 has the following properties:

(A1) $u_0 \in C^{2+\alpha}(0,1) \cap C[0,1]$ with $0 < \alpha < 1$,

(A2) $u_0 > 0$ on $(0,1), u_0(0) = u_0(1) = 0,$ and $u_{0x}(0) \geq 0$ and $u_{0x}(1) \leq 0$,

(A3) $\left(x^\beta (u_0(x))_x\right)_x + a^{3-\beta}\displaystyle\int_0^1 F(u_0(x))dx > 0$ for $x \in (0,1)$,

(A4) $\displaystyle\lim_{x\to 0^+}\left(x^\beta (u_{0x}(x))_x\right)_x = -a^{3-\beta}\int_0^1 F(u_0(x))dx$ and $\displaystyle\lim_{x\to 1^-}\left(x^\beta (u_0(x))_x\right)_x = -a^{3-\beta}\int_0^1 F(u_0(x))dx$,

(A5) $\left(x^\beta (u_{0x})\right)_x \leq 0$ for $x \in (0,1)$.

To obtain the existence result of (4), we need to use the following lemma 2.1 in [13] that is the important tool and is used frequently from now on.

Lemma 2.1. *Let b_i is bounded and continuous, and $b_i \geq 0$ on $[0,1]\times[0,T]$ for $i=1,2,3,4$ and $d \geq 0$ on $[0,1]\times[0,T]$ with $0<T\leq\infty$. Suppose that $w \in C^{2,1}\left((0,1)\times(0,T)\right)\cap C\left([0,1]\times[0,T]\right)$ satisfies*

$$\left.\begin{array}{c} w-d\left(x,t\right)\left(x^{\beta}w_x\right)_x \geq b_1 w_x + b_2 w + b_3 \displaystyle\int_0^1 b_4 w\left(x,t\right)dx, \\ (x,t)\in(0,1)\times(0,T], \\ w(0,t)\geq 0, w(1,t)\geq 0, t\in(0,T], \\ w(x,0)\geq 0, x\in[0,1]. \end{array}\right\} \tag{5}$$

Then, $w \geq 0$ on $[0,1]\times[0,T]$.

To prove the existence of solutions for (4), we next consider the following auxiliary problem:

$$\left.\begin{array}{c} \theta_t = (\theta+\delta)^r\left[\left(x^{\beta}\theta_x\right)_x + a^{3-\beta}\displaystyle\int_0^1 F(\theta(x,t))dx\right], (x,t)\in(0,1)\times(0,\infty), \\ \theta(0,t;\delta)=\theta(1,t;\delta)=0, t>0, \\ \theta(x,0;\delta)=u_0(x), x\in[0,1], \end{array}\right\} \tag{6}$$

where δ is any positive constant and $\delta<1$. Let ε be any positive constant with $\varepsilon<\delta$. We next introduce a cut-off function, ρ. We refer to the Dunford and Schwartz book [12], there exists a non-decreasing function ρ such that $\rho=0$ if $x\leq 0$ and $\rho=1$ if $x\geq 1$. We construct the functions ρ_ε and $u_{0\varepsilon}$ by

$$\rho_\varepsilon(x)=\begin{cases} 0, & x\leq\varepsilon, \\ \rho\left(\dfrac{x}{\varepsilon}-1\right), & \varepsilon<x<2\varepsilon \\ 1, & x\geq 2\varepsilon, \end{cases}$$

and $u_0(x;\varepsilon)=\rho_\varepsilon(x)u_0(x)$. Then, we have that

$$\frac{\partial}{\partial\varepsilon}u_0(x;\varepsilon)=\begin{cases} 0, & x\leq\varepsilon, \\ -\dfrac{x}{\varepsilon^2}\rho'\left(\dfrac{x}{\varepsilon}-1\right)u_0(x), & \varepsilon<x<2\varepsilon, \\ 0, & x\geq 2\varepsilon. \end{cases}$$

It follows form the non-decreasing property of ρ that

$$\frac{\partial}{\partial\varepsilon}u_0(x;\varepsilon)\leq 0 \text{ and } \lim_{\varepsilon\to 0}u_0(x;\varepsilon)=u_0.$$

We next consider the regularized problem of (6):

$$\left.\begin{aligned}\tilde{\theta}_t = \left(\tilde{\theta}+\delta\right)^r \left[\left(x^\beta \tilde{\theta}_x\right)_x + a^{3-\beta}\int_\varepsilon^1 F\left(\tilde{\theta}(x,t)\right)dx\right], (x,t)\in(\varepsilon,1)\times(0,\infty),\\ \tilde{\theta}(0,t;\delta,\varepsilon)=\tilde{\theta}(1,t;\delta,\varepsilon)=0, t>0,\\ \tilde{\theta}(x,0)=u_0(x;\varepsilon), x\in[\varepsilon,1].\end{aligned}\right\} \tag{7}$$

We note that by $F(0)=0$, the zero function is a lower solution of (7). This implies that $\tilde{\theta}\geq 0$ for any $\varepsilon>0$. The next lemma show that the solution $\tilde{\theta}$ of the (7) is non-decreasing with respect to t.

Lemma 2.2. *If* $\left(x^\beta \frac{\mathrm{d}}{\mathrm{d}x}u_0(x;\varepsilon)\right)_x + a^{3-\beta}\int_\varepsilon^1 F\left(u_0(x;\varepsilon)\right)dx > 0$ *for any* $x\in(\varepsilon,1)$*. Then,* $\tilde{\theta}_t\geq 0$ *on* $[\varepsilon,1]\times[0,\infty]$*.*

Proof. Let $\phi=\tilde{\theta}_t$ on $[\varepsilon,1]\times[0,\infty]$. Then, we have that for any $(x,t)\in(\varepsilon,1)\times(0,\infty)$,

$$\begin{aligned}\phi_t &= \left(\tilde{\theta}+\delta\right)^r\left(x^\beta\phi_x\right)_x + r\left(\tilde{\theta}+\delta\right)^{r-1}(\tilde{\theta}_t)^2\\ &\quad+\left(\tilde{\theta}+\delta\right)^r a^{3-\beta}\int_\varepsilon^1 F'\left(\tilde{\theta}(x,t)\right)\phi(x,t)dx.\end{aligned}$$

It follows from (7) that the function ϕ satisfies:

$$\phi_t = \left(\tilde{\theta}+\delta\right)^r\left(x^\beta\phi_x\right)_x \geq (\tilde{\theta}+\delta)^r a^{3-\beta}\int_\varepsilon^1 F'\left(\tilde{\theta}(x,t)\right)\phi(x,t)dx,$$
$$(x,t)\in(\varepsilon,1)\times(0,\infty),$$

$$\phi(\varepsilon,t)=\tilde{\theta}_t(\varepsilon,t)=0, \phi(1,t)=\tilde{\theta}_t(1,t)=0, t>0,$$

$$\phi(x,0)=(u_0(x;\varepsilon)+\delta)^r\left[\left(x^\beta\frac{\mathrm{d}}{\mathrm{d}x}u_0(x;\varepsilon)\right)_x + a^{3-\beta}\int_\varepsilon^1 F\left(u_0(x;\varepsilon)\right)dx\right]>0,$$
$$x\in[\varepsilon,1].$$

Lemma 2.1 implies that $\tilde{\theta}_t\geq 0$ for any $(x,t)\in[\varepsilon,1]\times[0,\infty)$. □

We next show the boundedness property of $\tilde{\theta}$ on some time interval.

Lemma 2.3. *There exist a time t_1 and a function $k\in C^1[0,t_1]$ such that* (7) *has a unique classical solution $\tilde{\theta}$ on $[\varepsilon,1]\times[0,t_1]$ for all $\varepsilon>0$ and $0\leq\tilde{\theta}\leq k$.*

Proof. Consider the ordinary differential equation:

$$\left.\begin{aligned}k'(t)=a^{3-\beta}F(k(t))(k(t)+1)^r, t>0\\ k(0)=\max_{x\in[0,1]} u_0(x).\end{aligned}\right\} \tag{8}$$

By the theory of ordinary differential equation, there exists a positive constant t_1 such that (8) has a unique positive solution k on $[0,t_1]$. In the following, we show that for all $\varepsilon > 0$, $k(t) \geq \tilde{\theta}(x,t)$ for any $(x,t) \in [\varepsilon,1] \times [0,t_1]$. Set $\phi(x,t) = k(t) - \tilde{\theta}(x,t)$ for $(x,t) \in [\varepsilon,1] \times [0,t_1]$. We then obtain that for any $(x,t) \in (\varepsilon,1) \times (0,t_1]$,

$$\begin{aligned}
\phi_t \geq\; & (k(t)+\delta)^r a^{3-\beta} \int_\varepsilon^1 F(k(t))dx \\
& -(\tilde{\theta}+\delta)^r \left[\left(x^\beta \tilde{\theta}_x\right)_x + a^{3-\beta} \int_\varepsilon^1 F(\tilde{\theta}(x,t))dx \right] \\
=\; & (k(t)+\delta)^r \left(x^\beta \phi_x\right)_x + r\eta_1^{r-1} \frac{\tilde{\theta}_t}{(\theta_\varepsilon+\delta)^r}\phi \\
& +(k(t)+\delta)^r a^{3-\beta} \int_\varepsilon^1 F'(\eta_2)dx,
\end{aligned}$$

where η_1 and η_2 are some constants between k and $\tilde{\theta}$. Therefore, $\phi_t \geq (k(t)+\delta)^r \left(x^\beta \phi_x\right)_x + r\eta_1^{r-1} \frac{\tilde{\theta}_t}{(\tilde{\theta}+\delta)^r}\phi + (k(t)+\delta)^r a^{3-\beta} \int_\varepsilon^1 F'(\eta_2)dx$ for $(x,t) \in (\varepsilon,1) \times (0,t_1]$.

Next, we consider on the parabolic boundary and then we have that
$\phi(\varepsilon,t) = k(t) > 0, \phi(1,t) = k(t) > 0, t \in (0,t_1]$ and
$\phi(x,0) = k(0) - \tilde{\theta}(x,0) = \max_{x\in[0,1]} u_0(x) - u_0(x;\varepsilon) \geq 0, x \in [\varepsilon,1]$.
Lemma 2.1 implies that $\phi \geq 0$, that is $\tilde{\theta} \leq k$ for any $[\varepsilon,1] \times [0,t_1]$. Based on the proof of Theorem A.1. in [18], we can conclude that (7) has a unique classical solution $\tilde{\theta}$ on $[\varepsilon,1] \times [0,t_1]$. The proof of this lemma is completed. □

We next show that the function $\tilde{\theta}$ has the monotonicity property.

Lemma 2.4. *Assume that $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are solutions of* (7) *with* $0 < \varepsilon_1 < \varepsilon_2 < 1$. *Then $\tilde{\theta}_1 \geq \tilde{\theta}_2$ on* $[\varepsilon_2,1] \times [0,t_1]$.

Proof. Set $\phi(x,t) = \tilde{\theta}_1 - \tilde{\theta}_2$ on $[\varepsilon_2,1] \times [0,t_1]$. For any $(x,t) \in (\varepsilon_2,1) \times (0,t_1]$, we have that

$$\begin{aligned}
\phi_t =\; & (\tilde{\theta}_1+\delta)^r \left[\left(x^\beta (\tilde{\theta}_1)_x\right)_x + a^{3-\beta} \int_{\varepsilon_2}^1 F\left(\tilde{\theta}_1(x,t)\right) dt \right] \\
& -(\tilde{\theta}_2+\delta)^r \left[\left(x^\beta (\tilde{\theta}_2)_x\right)_x + a^{3-\beta} \int_{\varepsilon_2}^1 F\left(\tilde{\theta}_2(x,t)\right) dt \right] \\
=\; & \left(\tilde{\theta}_1+\delta\right)^r \left(x^\beta \phi_x\right)_x + r\eta_3^{r-1} \left(\tilde{\theta}_2+\delta\right)^{-r} (\tilde{\theta}_2)_t \phi \\
& +a^{3-\beta} \left(\tilde{\theta}_1+\delta\right)^r \int_{\varepsilon_2}^1 F'(\eta_4)\, \phi(x,t) dx
\end{aligned}$$

where η_3 and η_4 are some constants between $\tilde{\theta}_1$ and $\tilde{\theta}_2$.

By a fact that $\frac{\partial}{\partial \varepsilon} u_0(x;\varepsilon) \leq 0$, the function ϕ satisfies

$$\phi_t - \left(\tilde{\theta}_1 + \delta\right)^r \left(x^\beta \phi_x\right)_x \geq \frac{r\eta_3^{r-1}(\tilde{\theta}_2)_t}{\left(\tilde{\theta}_2 + \delta\right)^r}\phi + a^{3-\beta}\left(\tilde{\theta}_1 + \delta\right)^r \int_{\varepsilon_2}^1 F'(\eta_4)\,\phi\,dx,$$
$$(x,t) \in (\varepsilon_2, 1) \times (0, t_1],$$

$$\phi(\varepsilon_2, t) = \tilde{\theta}_1(\varepsilon_2, t; \delta, \varepsilon_1) \geq 0, \phi(1,t) = 0, t \in (0, t_1],$$

$$\phi(x,0) = u_0(x;\varepsilon_1) - u_0(x;\varepsilon_2) \geq 0, x \in [\varepsilon_2, 1].$$

By Lemma 2.1, $\tilde{\theta}_1 \geq \tilde{\theta}_2$ on $[\varepsilon_2, 1] \times [0, t_1]$. $\square$

Lemma 2.3 and Lemma 2.4 ensure that $\lim_{\varepsilon \to 0} \tilde{\theta}(x,t)$ exists and then we construct the function θ which is a good candidate for the solution for (6) by

$$\theta(x,t) = \begin{cases} \lim_{\varepsilon \to 0} \tilde{\theta}(x,t), & (x,t) \in (\varepsilon, 1] \times [0, t_1] \\ 0, & (x,t) \in \{0\} \times [0, t_1]. \end{cases} \tag{9}$$

By modifying the proofs of Theorem 2.3 in [10] and Lemma 10 and Theorem 12 in [6], we obtain the existence result for (6).

Theorem 2.5. *The function θ defined by* (9) *is a unique classical solution of* (6)*, on* $[0,1] \times [0, t_1]$.

In order to prove the existence of solutions for (4). By using the same technique as in Lemma 2.2 and Lemma 2.3, we can show that the solution θ of (6) satisfies that $\theta_t \geq 0$ for all δ and $u_0(x) \leq \theta(x,t) \leq k(t)$ for any $(x,t) \in [0,1] \times [0,t_1]$ where the function k is given in Lemma 2.3. The next lemma deals with an additional property of θ.

Lemma 2.6. *Assume that θ_1 and θ_2 are solutions of* (6) *with* $0 < \delta_1 < \delta_2 < 1$. *Then,* $v_{\delta_2} \geq v_{\delta_1}$ *on* $[0,1] \times [0,t_1]$.

Proof. Set $\phi = \theta_2 - \theta_1$ on $[0,1] \times [0,t_1]$. From (6), we have that for any $(x,t) \in (0,1) \times [0,t_1]$,

$$\begin{aligned} \phi_t \;\; &\geq \;\; (\theta_2 + \delta_2)^r \left[\left(x^\beta (\theta_2)_x\right)_x + a^{3-\beta}\int_0^1 F(\theta_2(x,t))\,dx\right] \\ &\quad - (\theta_1 + \delta_2)^r \left[\left(x^\beta (\theta_1)_x\right)_x + a^{3-\beta}\int_0^1 F(\theta_2(x,t))\,dx\right] \\ &= \;\; (\theta_2 + \delta_2)^r \left(x^\beta \phi_x\right)_x + r\eta_5^{r-1}(\theta_1 + \delta_1)^{-r}(\theta_1)_t \phi \\ &\quad + a^{3-\beta}(\theta_2 + \delta_2)^r \int_0^1 F'(\eta_6)\,\phi\,dx, \end{aligned}$$

where η_5 and η_6 are some constants between θ_1 and θ_2. Then, the function ϕ satisfies
$\phi_t - ((\theta_2)_t + \delta_2)^r \left(x^\beta \phi_x\right)_x \geq \frac{r\eta_5^{r-1}(\theta_1)_t}{(\theta_1+\delta_1)^r}\phi + a^{3-\beta}(\theta_2+\delta_2)^r \int_0^1 F'(\eta_6)\,\phi dx$ for $(x,t) \in (0,1) \times (0,t_1]$.
On the parabolic boundary, we obtain that $\phi(0,t) = 0, \phi(1,t) = 0, t \in (0,t_1]$ and $\phi(x,0) = 0, x \in [0,1]$. By Lemma 2.1, $\theta_2 \geq \theta_1$ on $[0,1] \times [0,t_1]$. □

It follows from Lemma 2.6 that we define the function u by

$$u(x,t) = \lim_{\delta \to 0} \theta(x,t), (x,t) \in (0,1) \times (0,t_1]. \tag{10}$$

By modifying Lemma 2.7 in [18], and Lemma 10 and Theorem 12 in [6], we obtain the existence theorem for the equivalent problem (4).

Theorem 2.7. *The function u given by* (10) *is a unique non-negative classical solution of the equivalent problem* (4) *on* $[0.1] \times [0,t_1]$ *for some positive constant* t_1.

Note that by the transformation techniques, $u = v^m, \tau = \frac{t}{ma^{\beta-2}}$ and $\xi = ax$, and Theorem 2.7, we obtain the existence result of (1).

Corollary 2.8. *There exists a time $\tilde{t_1} > 0$ such that* (1) *admits a unique non-negative classical solution on* $[0,a] \times [0,\tilde{t_1}]$ *for some positive constant* $\tilde{t_1}$.

3. Blow-up and non-blow-up conditions

First, we give the sufficient condition for blow-up in a finite time for the solution u of (1). Let us consider the following boundary value problem:

$$\left.\begin{aligned} -\left(x^\beta \Phi'(x)\right)' &= \lambda \Phi(x), x \in (0,1), \\ \Phi(0) = \Phi(1) &= 0. \end{aligned}\right\} \tag{11}$$

The boundary value problem (11) is solvable by [19]. Let the first eigenvalue and its corresponding eigenfunction denote by $\lambda_1 > 0$ and Φ_1 respectively. Without loss of generality, we assume that $\max_{x \in [0,1]} \Phi_1(x) = 1$. The next theorem shows the condition that guarantees for the occurrence of blow-up in a finite time depending on the value of the constant a.

Theorem 3.1. *Assume that there exists a positive constant c_1 such that $F(s) \geq c_1 s^q$ for $s > 0$ and $q > 1$. If*

$$a > \max\left\{ \left(\frac{\lambda_1}{c_1 \int_0^1 \Phi_1(x)dx} \right)^{\frac{q}{3-\beta}}, \left(\frac{1}{\int_0^1 u_0^q(x)dx} \right)^{\frac{1}{3-\beta}} \right\},$$

then the solution u of (4) *blows up in a finite time.*

Proof. Let $\Pi(t) = \int_0^1 u^{1-r}(x,t)\Phi_1(x)dx$. We obtain that

$$\begin{aligned} \frac{1}{1-r}\Pi'(t) &= \int_0^1 \left(x^\beta u_x\right)_x \Phi_1(x)dx + a^{3-\beta}\int_0^1 F(u(x,t))dx \int_0^1 \Phi_1(x)dx \\ &= -\lambda_1 \int_0^1 u(x,t)\Phi_1(x)dx + a^{3-\beta}\int_0^1 F(u(x,t))dx \int_0^1 \Phi_1(x)dx. \end{aligned}$$

From

$$\begin{aligned} \int_0^1 u(x,t)\Phi_1(x)dx &\leq \frac{1}{a^{(3-\beta)/q}} \left(\int_0^1 a^{3-\beta}u^q(x,t)dx\right)\left(\int_0^1 \Phi_1^{\frac{q}{q-1}}(x)dx\right)^{1-\frac{1}{q}} \\ &\leq \frac{1}{a^{(3-\beta)/q}} \left(\int_0^1 a^{3-\beta}u^q(x,t)dx\right)^{\frac{1}{q}}, \end{aligned}$$

we obtain that

$$\begin{aligned} \frac{1}{1-r}\Pi'(t) \geq \frac{-\lambda_1}{a^{(3-\beta)/q}} &\left(\int_0^1 a^{3-\beta}u^q(x,t)dx\right)^{\frac{1}{q}} \\ &+ a^{3-\beta}\int_0^1 c_1 u^q(x,t)dx \int_0^1 \Phi_1(x)dx. \end{aligned}$$

From $u_t \geq 0$ and $a^{3-\beta}\int_0^1 u_0^q(x)dx \geq 1$, this implies that $a^{3-\beta}\int_0^1 u^q(x)dx \geq 1$.

It follows that $\left(\int_0^1 a^{3-\beta}u^q(x,t)dx\right)^{\frac{1}{q}} \leq a^{3-\beta}\int_0^1 u^q(x,t)dx$ with $q > 1$. Then

$$\begin{aligned} \frac{1}{1-r}\Pi'(t) &\geq -\frac{\lambda_1 a^{3-\beta}}{a^{(3-\beta)/q}}\int_0^1 u^q(x,t)dx + a^{3-\beta}c_1\int_0^1 u^q(x,t)dx \int_0^1 \Phi_1 dx \\ &= a^{3-\beta}\int_0^1 u^q(x,t)dx\left[\frac{-\lambda_1}{a^{(3-\beta)/q}} + c_1\int_0^1 \Phi_1(x)dx\right]. \end{aligned}$$

By the definition of the constant a, we get that $\frac{-\lambda_1}{a^{(3-\beta)/q}} + c_1\int_0^1 \Phi_1(x)dx \geq \eta_7$ where η_7 is a positive constant. Then,

$$\frac{1}{1-r}\Pi'(t) \geq \eta_7 a^{3-\beta}\int_0^1 u^q(x,t)dx.$$

Since

$$\int_0^1 u^{1-r}(x,t)\Phi_1(x)dx \le \left(\int_0^1 u^q(x,t)dx\right)^{\frac{1-r}{q}} \left(\int_0^1 \Phi_1^{\frac{q}{q+r-1}}(x)dx\right)^{\frac{q+r-1}{q}},$$

we obtain that

$$\int_0^1 u^q(x,t)dx \ge \left(\int_0^1 u^{1-r}(x,t)\Phi_1(x)dx\right)^{\frac{q}{1-r}} \Big/ \left(\int_0^1 \Phi_1^{\frac{q}{q+r-1}}(x)dx\right)^{\frac{q+r-1}{1-r}}.$$

We then obtain that

$$\begin{aligned} \frac{1}{1-r}\Pi'(t) &\ge \eta_7 a^{3-\beta}\left(\int_0^1 u^{1-r}(x,t)\Phi_1(x)dx\right)^{\frac{q}{1-r}} \Big/ \left(\left(\int_0^1 \Phi_1^{\frac{q}{q+r-1}}(x)dx\right)^{\frac{q+r-1}{1-r}}\right. \\ &\ge \eta_7 a^{3-\beta}\Pi^{\frac{q}{1-r}}(t), \end{aligned}$$

or

$$\left(\Pi^{1-q/(1-r)}(t)\right)' \le \eta_7 a^{3-\beta}(1-r-q). \tag{12}$$

Integrating (12) over $(0,t)$, we have that

$$\Pi^{1-q/(1-r)}(t) - \Pi^{1-q/(1-r)}(0) \le \eta_7 a^{3-\beta}(1-r-q)t.$$

or

$$\Pi^{\frac{q}{1-r}-1}(t) \ge \frac{\Pi^{\frac{q}{1-r}-1}(0)}{1-\eta_7 a^{3-\beta}(q+r-1)\Pi^{\frac{q}{1-r}-1}(0)t}.$$

We see that $\Pi^{\frac{q}{1-r}-1}(t)$ exists for $t \in [0,T_b)$ but $\Pi^{\frac{q}{1-r}-1}(t)$ is unbounded as t converges to T_b where

$$T_b = \frac{\Pi^{1-\frac{q}{1-r}}(0)}{\eta_7 a^{3-\beta}(q+r-1)} = \frac{1}{\eta_7 a^{3-\beta}(q+r-1)}\left(\int_0^1 u_0^{1-r}(x)\Phi_1(x)dx\right)^{\frac{-(q+r-1)}{1-r}} \tag{13}$$

is called the blow-up time. Therefore, Π blows up in a finite time. This implies that the solution u of the equivalent problem (4) blows up in a finite time. Then, the proof of this theorem is completed. □

By Theorem 3.1 and the transformation technique, we can obtain the following result.

Corollary 3.2. The solution v of (1) blows up in finite time if the constant a is sufficiently large.

Next, we will show that under some conditions, the solution u of (4) can exist globally. To obtain the desired results, we need the following comparison theorem.

Lemma 3.3. *Let u and u_0 be the solution and the initial function of* (4)*, respectively. Assume that a non-negative function $\varphi \in C^{2,1}((0,1)\times(0.T))\cap C([0,1]\times[0,T])$ satisfies*

$$\varphi_t \geq (\leq)\varphi^r\left[\left(x^\beta\varphi_x\right)_x + a^{3-\beta}\int_0^1 F(\varphi(x,t))dx\right], (x,t)\in(0,1)\times(0,T],$$
$$\varphi(0,t)\geq(=)0, \varphi(1,t)\geq(=)0, t\in(0,T],$$
$$\varphi(x,0)\geq(\leq)u_0(x), x\in[0,1].$$

Then, $\varphi \geq (\leq) u$ on $[0,1]\times[0,T]$.

Proof. We first consider in the case " $\geq$ ". Let $z(x,t)=\varphi(x,t)-u(x,t)$ on $[0,1]\times[0,T]$. By Lemma 2.1 and property $(A2)$, $u>0$ in $(0,1)\times(0,T]$. For any $(x,t)\in(0,1)\times(0,T]$, we have

$$z_t = \varphi^r\left(x^\beta z_x\right)_x + r\eta_8^{r-1}u^{-r}u_t z + a^{3-\beta}\varphi^r\int_0^1 F'(\eta_9)z(x,t)dx,$$

where η_8 and η_9 are some constants between φ and u. Hence, z satisfies:

$$z_t - \varphi^r\left(x^\beta z_x\right)_x = r\eta_8^{r-1}u^{-r}u_t z + a^{3-\beta}\varphi^r\int_0^1 F'(\eta_9)z(x,t)dx,$$
$$(x,t)\in(0,1)\times(0,T],$$
$$z(0,t)\geq 0, z(1,t)\geq 0, t\in(0,T],$$
$$z(x,0)\geq 0, x\in[0,1].$$

Lemma 2.1 yields that $\varphi \geq u$ on $[0,1]\times[0,T]$. Similarly, we can get the result in the case " $\leq$ ". The proof of this lemma is completed. □

Let us consider the boundary value problem:

$$-\left(x^\beta\Lambda'(x)\right)' = 1, x\in(0,1) \text{ and } \Lambda(0)=\Lambda(1)=0.$$

The solution Λ of the above boundary value problem is given by $\Lambda(x)=\frac{1}{2-\beta}x^{1-\beta}(1-x)$ for $x\in(0,1)$. The next theorem deals with the non-blow-up result.

Theorem 3.4. *The solution u of the equivalent* (4) *exists globally if the value of the constant a is small enough.*

Proof. Let $z(x,t) = c_2\Lambda(x)$ on $[0,1]\times[0,\infty)$ where c_2 is a positive constant and $c_2\Lambda \geq u_0$. Choose $a \leq \left(\dfrac{c_2}{F(c_2\max_{x\in[0,1]}\Lambda(x))}\right)^{\frac{1}{3-\beta}}$. Then,

$$\begin{aligned}
& z_t - z^r\left[\left(x^\beta z_x\right)_x + a^{3-\beta}\int_0^1 F(z(x,t))dx\right] \\
& \geq c_2^r\Lambda^r(x)\left[c_2 - a^{3-\beta}\int_0^1 F(c_2\max_{x\in[0,1]}\Lambda(x))dx\right] \\
& \geq 0
\end{aligned}$$

for any $(x,t)\in(0,1)\times(0,\infty)$. Moreover, $z(0,t) = z(1,t) = 0$ for $t>0$ and $z(x,0) = c_2\Lambda(x) \geq u_0(x)$ for $x\in[0,1]$. By Lemma 3.3, $z\geq u$ on $[0,1]\times[0,\infty]$. We can conclude that the solution u of the equivalent problem (4) exists globally. □

By the transformation technique and Theorem 3.4 that we obtain the following result.

Corollary 3.5. *The solution v of* (1) *does not blow up if a is small enough.*

4. Blow-up set and the uniform blow-up profile

In this section, we show that the set of blow-up points and the blow-up profile for the blow-up solution v of (1) at the blow-up time T_b. From the hypothesises $(H1)-(H5)$, we know that there are a sufficiently small positive constant ε_1 and a non-negative function $\varphi_{0\varepsilon}(x)$ such that

(H1*) $\varphi_{0\varepsilon}\in C^{2+\alpha}(\varepsilon,1-\varepsilon)\cap C[\varepsilon,1-\varepsilon]$ with $\alpha\in(0,1)$ and $\varepsilon\in(0,\varepsilon_1]$,

(H2*) $\varphi_{0\varepsilon}(\varepsilon)=0$ and $\varphi_{0\varepsilon}(1-\varepsilon)=0$,

(H3*) $\varphi_{0\varepsilon}(x)<u_0(x)$ for $x\in(\varepsilon,2\varepsilon)\cup(1-2\varepsilon,1-\varepsilon)$ and $\varphi_{0\varepsilon}(x)=u_0(x)$ for $x\in(2\varepsilon,1-2\varepsilon)$,

(H4*) $\left(x^\beta(\varphi_{0\varepsilon})_x\right)_x\leq 0$ for $x\in(\varepsilon,1-\varepsilon)$,

(H5*) $\varphi_{0\varepsilon}$ is non-increasing with respect to $\varepsilon\in(0,\varepsilon_1]$,

$\lim\limits_{x\to\varepsilon^+}\left(x^\beta(\varphi_{0\varepsilon})_x\right)_x = -a^{3-\beta}\displaystyle\int_\varepsilon^{1-\varepsilon}F(\varphi_{0\varepsilon}(x))dx$

and $\lim\limits_{x\to(1-\varepsilon)^-}\left(x^\beta(\varphi_{0\varepsilon})_x\right)_x = -a^{3-\beta}\displaystyle\int_\varepsilon^{1-\varepsilon}F(\omega_{0\varepsilon}(\xi))d\xi$

(H6*) $\left(x^\beta(\varphi_{0\varepsilon})_x\right)_x + a^{3-\beta}\displaystyle\int_\varepsilon^{1-\varepsilon}F(\varphi_{0\varepsilon}(x))dx \geq 0$ for $\varepsilon\in(0,\varepsilon_1]$ and $x\in(\varepsilon,1-\varepsilon)$.

Clearly, $\lim_{\varepsilon\to 0}\varphi_{0\varepsilon}=u_0$. In following, we consider the following regularized problem:

$$\left.\begin{array}{c}\varphi_{\varepsilon t}=(\varphi_\varepsilon+\delta)^r\left[\left(x^\beta(\varphi_\varepsilon)_x\right)_x+a^{3-\beta}\int_\varepsilon^{1-\varepsilon}F\left(\varphi_\varepsilon(x,t)\right)dx\right],\\ (x,t)\in(\varepsilon,1-\varepsilon)\times(0,\infty),\\ \varphi_\varepsilon(\varepsilon,t)=\varphi_\varepsilon(1-\varepsilon,t)=0,t>0,\\ \varphi_\varepsilon(x,0)=\varphi_{0\varepsilon}(x),x\in[\varepsilon,1-\varepsilon].\end{array}\right\}\tag{14}$$

As discussed before, we obtain that the regularized problem (14) has a unique positive solution φ_ε and

$$\lim_{\delta\to 0,\varepsilon\to 0}\varphi_\varepsilon=u$$

where u is the solution of the equivalent problem (4). In order to find the blow-up set and the blow-up profile of the blow-up solution v of (1) when time t near the blow-up time T_b, we need the following lemma.

Lemma 4.1. *Before blow-up occurs,* $\left(x^\beta u_x\right)_x\le 0$ *for* $(x,t)\in(0,1)\times[0,T_b)$.

Proof. It follows from $(H6^*)$ that $\varphi_{\varepsilon t}\ge 0$ on $[0,1]\times[0,T_b)$.
Let $z(x,t)=\left(x^\beta(\varphi_\varepsilon)_x\right)_x$ on $[0,1]\times[0,T_b)$.
We obtain that for any $(x,t)\in(\varepsilon,1-\varepsilon)\times(0,T_b)$,

$$\begin{aligned}z_t-(\varphi_\varepsilon+\delta)^r(x^\beta z_x)_x-2r(\varphi_\varepsilon+\delta)^{r-1}x^\beta\varphi_{\varepsilon x}z_x-r(\varphi_\varepsilon+\delta)^{-1}\varphi_{\varepsilon t}z\\ =r(r-1)(\varphi_\varepsilon+\delta)^{-2}x^\beta\varphi_{\varepsilon t}(\varphi_{\varepsilon x})^2\end{aligned}$$

or

$$z_t-(\varphi_\varepsilon+\delta)^r(x^\beta z_x)_x-2r(\varphi_\varepsilon+\delta)^{r-1}x^\beta\varphi_{\varepsilon x}z_x-r(\varphi_\varepsilon+\delta)^{-1}\varphi_{\varepsilon t}z\le 0.$$

On the boundary conditions, we obtain that by $(H5^*)$,

$$z(\varepsilon,t)=\left(x^\beta\varphi_x\right)_x|_{x=\varepsilon}=-a^{3-\beta}\int_\varepsilon^{1-\varepsilon}F(\varphi_\varepsilon(x,t))dx<0$$

and

$$z(1-\varepsilon,t)=\left(x^\beta\varphi_x\right)_x|_{x=1-\varepsilon}=-a^{3-\beta}\int_\varepsilon^{1-\varepsilon}F(\varphi_\varepsilon(x,t))dx<0.$$

For the initial data, it follows from $(H4^*)$ that $z(x,0)\le 0$ for $x\in[\varepsilon,1-\varepsilon]$. By lemma 2.1, we get that $z\le 0$ on $[\varepsilon,1-\varepsilon]\times[0,T_b)$. Since ε and δ are arbitrary, and $\lim_{\delta\to 0,\varepsilon\to 0}\varphi_\varepsilon=u$, we have that $\left(x^\beta u_x\right)_x\le 0$ for $(x,t)\in(0,1)\times(0,T_b]$. The proof of Lemma 4.1 is complete. $\square$

It follows from Lemma 4.1 that we obtain the following corollary which is used in the part of showing the blow-up set of the blow-up solution v of (1).

Corollary 4.2. *Before blow-up occurs,* $(\xi^\beta v_\xi^m)_\xi \leq 0$ *for* $(\xi,\tau) \in (0,a)\times[0,T_b)$.

The next theorem states about the set of all blow-up points of the blow-up solution v of (1).

Theorem 4.3. *Assume that the solution v of* (1) *blows up in a finite time T_b and there exits a positive constant c_1 such that $F(s) \geq c_1 s^q$ for $s > 0$ and $q > 1$. Then, the set of all blow-up points is the whole interval* $[0,a]$.

Proof. Let ε be any positive constant. Construct functions ψ and Ψ by $\psi(\tau) = \int_0^a F(v^m(\xi,\tau))\,d\xi$ and $\Psi(\tau) = \int_0^\tau \psi(s)ds$. We let $c_3 = \inf_{\xi\in(\varepsilon,a-\varepsilon)} \mu(\xi)$ where μ is a positive solution of the following boundary value problem:

$$-\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^\beta \frac{\mathrm{d}}{\mathrm{d}x}\mu^m(\xi)\right) = 1, \xi \in (0,a),$$
$$\mu(0) = \mu(a) = 0.$$

From Corollary 4.2, we have that for $\tau \in (0,T_b)$,

$$\begin{aligned}\int_0^a v^m(\xi,\tau)dx &= -\int_0^a v^m(\xi,\tau)\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^\beta \frac{\mathrm{d}}{\mathrm{d}\xi}\mu^m(\xi)\right)d\xi \\ &\geq -c_3^m \int_0^{a-\varepsilon}\left(\xi^\beta v_\xi^m\right)_\xi d\xi.\end{aligned}$$

We then obtain

$$0 \leq \lim_{\tau\to T_b} \frac{-c_3^m \int_\varepsilon^{a-\varepsilon}\left(\xi^\beta v_\xi^m\right)_\xi}{\psi(\tau)} d\xi \leq \lim_{\tau\to T_b} \frac{\int_0^a v^m(\xi,\tau)d\xi}{c_1\int_0^a v^{mq}(\xi,\tau)d\xi} = 0.$$

This means that $\lim_{\tau\to T_b} \dfrac{\int_\varepsilon^{a-\varepsilon}\left(\xi^\beta v_\xi^m\right)_\xi}{\psi(\tau)} d\xi = 0$. As $\varepsilon \to 0$, we obtain that

$$\lim_{\tau\to T_b} \frac{\left(\xi^\beta v_\xi^m\right)_\xi}{\psi(\tau)} = 0 \text{ for } \xi \in (0,a). \tag{15}$$

By integrating the first equation in (1) both sides with respect to τ from 0 to τ, we get that

$$v(\xi,\tau) - v_0(\xi) = \int_0^\tau \left(\xi^\beta v_\xi^m(\xi,s)\right)_\xi ds + \Psi(\tau). \tag{16}$$

From an assumption that v blow up at the finite time T_b, this implies that $\lim_{\tau\to T_b} v(\xi_b,\tau)=\infty$ for some $\xi_b\in(0,a)$ and then we obtain that

$$\lim_{t\to T_b} v(\xi_b,\tau)-\lim_{\tau\to T_b} v_0(\xi_b)=\lim_{\tau\to T_b}\int_0^\tau \left(\xi^\beta v_\xi^m(\xi_b,s)\right)_\xi ds+\lim_{\tau\to T_b}\Psi(\tau).$$

We therefore have that

$$\lim_{\tau\to T_b}\Psi(\tau)=\infty. \tag{17}$$

By (15), we have that

$$\lim_{\tau\to T_b}\frac{\int_0^\tau \left(\xi^\beta v_\xi^m(\xi,s)\right)_\xi ds}{\Psi(\tau)}=0 \text{ for } \xi\in(0,a). \tag{18}$$

Let ξ^* be a fixed point in (0,a). We have that by (16),

$$\lim_{\tau\to T_b}\frac{v(\xi^*,\tau)}{\Psi(\tau)}-\lim_{\tau\to T_b}\frac{v_0(\xi^*)}{\Psi(\tau)}=\lim_{\tau\to T_b}\frac{\int_0^\tau \left((\xi^*)^\beta v_\xi^m(\xi^*,s)\right)_\xi ds}{\Psi(\tau)}+1.$$

(17) and (18) yield that

$$\lim_{\tau\to T_b}\frac{v(\xi^*,\tau)}{\Psi(\tau)}=1, \tag{19}$$

which means that the solution v of (1) blows up at the point ξ^*. By the arbitrariness of ξ^*, we can conclude that the solution v of (1) blows up everywhere in $(0,a)$. For $\xi^*\in\{0,a\}$, we can always find a sequence $\{(\xi_n,\tau_n)\}$ in $(0,a)\times(0,T_b)$ such that $(\xi_n,\tau_n)\to(\xi^*,T_b)$ and $\lim_{n\to\infty} v(\xi_n,\tau_n)=\infty$. Therefore, the blow-up set is $[0,a]$. The proof of Theorem 4.3 is completed. □

The last theorem shows the blow-up profile of the blow-up solution v of (1) when τ approaches to the blow-up time T_b.

Theorem 4.4. *If $f(s)=s^q$ with $q>1$. Then, $v(\xi,\tau)\sim[a(mq-1)(T_b-\tau)]^{\frac{-1}{mq-1}}$ for any $\xi\in(0,a)$ as $\tau\to T_b$.*

Proof. (19) tells us that for any $\xi\in(0,a)$,

$$v(\xi,\tau)\sim\Psi(\tau) \text{ as } \tau\to T_b. \tag{20}$$

Then, by (20), we obtain that

$$\Psi'(\tau)=\int_0^a v^{mq}(\xi,\tau)d\xi\sim a\Psi^{mq}(\tau) \text{ as } \tau\to T_b. \tag{21}$$

Integrating (21) over (τ,T_b), we have that by (17),

$$\Psi(t)\sim[a(mq-1)(T_b-\tau)]^{\frac{-1}{mq-1}} \text{ as } \tau\to T_b. \tag{22}$$

Then, we conclude that as τ approaches to the blow-up time T_b, $v(\xi,\tau)\sim[a(mq-1)(T_b-\tau)]^{\frac{-1}{mq-1}}$ for any $\xi\in(0,a)$. □

5. Conclusion

In this article, we consider the degenerate and singular porous medium problem with a non-local source term. We prove that the such problem has a local classical solution before blow-up occur. The conditions for blow-up and non-blow-up of solutions of the problem depend on the value of constant a. In the blow-up case, we prove that the solution of the problem blows up completely. Finally, the uniform blow-up profile of the blow-up solution near the blow-up time is shown.

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