EXISTENCE OF ENTROPY SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEM HAVING LARGE MONOTONOCITY IN WEIGHTED ORLICZ-SOBOLEV SPACES

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We prove an existence result of entropy solution for a class of nonlinear elliptic problems of Leray-Lions type with large monotonicity condition in the framework of weighted Orlicz-Sobolev spaces and with right hand side $f \in L^1(\Omega)$.

1. Introduction

This paper deals with existence of solutions to the following nonlinear Dirichlet problem

\[
\begin{cases}
A(u) = f(x) - \text{div} F(u) & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $f \in L^1(\Omega), F \in (C^0(\mathbb{R}))^N$ and $A(u) = - \text{div}(\rho(x)a(x,u,\nabla u)), \Omega$ is a bounded domain of $\mathbb{R}^N, N \geq 2$.

Note that no growth hypothesis is assumed on the function $F$, which implies that the term $\text{div} F(u)$ may be meaningless, even as a distribution. $a(x,u,\nabla u) = (a_i(x,u,\nabla u))_{1 \leq i \leq N}$, $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory functions (that is
measurable with respect to $x$ in $\Omega$ for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ for almost every $x \in \Omega$) such that for all $\xi, \xi' \in \mathbb{R}^N, (x, s) \in \Omega \times \mathbb{R}$

$$|a_i(x, s, \xi)| \leq |\phi_i(x)| + K_i \bar{P}^{-1}(\rho^{-1}(x) M(c_2 |s|)) + K_i (\bar{M}^{-1} M(c_1 |\xi|)),$$  \hfill (2)

$$\left( a(x, s, \xi) - a(x, s, \xi') \right) (\xi - \xi') \geq 0,$$  \hfill (3)

$$a(x, s, \xi) \xi \geq M(\lambda_1 |\xi|),$$  \hfill (4)

where $c_1, c_2, \lambda_1, K_i > 0$. Let $M, P$ are two N-functions such that $P \ll M$. Moreover $\bar{M}, \bar{P}$ are the complementary functions of $M$ and $P$ respectively, $\rho$ is a weight function on $\Omega$ (that is, measurable and positive a.e. on $\Omega$) and $\phi_0, \phi_i \in E_M(\Omega, \rho)$, $(E_M(\Omega, \rho)$ is introduced later.)

$$f \in L^1(\Omega),$$  \hfill (5)

and $F = (F_1, \ldots, F_N)$ satisfies

$$F \in (C^0(\mathbb{R}))^N.$$  \hfill (6)

The notion of entropy solution, used in [17], allows us to give a meaning to a possible solution of (1)

In fact Boccardo proved in [17], for $\rho(x) = 1$ and $p$ such that $2 - 1/N < p < N$, the existence and regularity of an entropy solution $u$ of problem (1). For the case $\rho(x) = 1$, $\phi = 0$ and $f$ is a bounded measure, Benilan et al. proved in [7] the existence and uniqueness of entropy solutions, the same problem is treated using the notion of entropy solution introduced in [38] where $\rho(x) = 1, f \in L^1(\Omega)$, and $F \in L^p'(\Omega)^N$. We mention as a parallel development, the work of Lions and Murat [40] who consider similar problems in the context of the renormalized solutions introduced by Diperna and Lions [24] for the study of the Boltzmann equations. They can prove existence and uniqueness of renormalized solution, and afterwards by Dal Maso, Murat, Orsina and Prignet in [22] with general measure data.

In the general framework of weighted Orlicz-Sobolev spaces, the authors in [9] deal with the equation

$$- \text{div}(\rho(x)a(x, u, \nabla u)) - \text{div} \Phi(u) + g(x, u, \nabla u) = f$$

with $\Phi = 0$ and $g$ is a nonlinear with the following (natural) growth condition:

$$|g(x, s, \xi)| \leq b(|s|)(c(x) + M(|\xi|) \rho(x)),$$
which satisfies the sign condition
\[ g(x,s,\xi)s \geq 0, \]
and the right-hand side \( f \in W^{-1}E_M(\Omega,\rho), \) in the case where \( f \in L^1(\Omega), \phi = 0 \) and \( g = 0 \) existence theorem have been proved by the authors in ([28]).

Several researches deals with the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts (see [4–6, 11, 12, 14–16, 18, 23, 26, 28–32, 46–49] for more details).

It is well known that the setting of weighted Orlicz–Sobolev spaces include many spaces as special spaces, such as Lebesgue spaces, weighted Lebesgue spaces and Orlicz spaces; see [44]. These spaces have many applications in various fields such as PDE, electrorheological fluids, and image restoration; see [21, 25, 39].

The feature of this paper, is to treat a class of problems for which the classical monotone operator methods (developed by Minty [43], Browder [20], Brézis [19] and Lions [41] in \( W^{1,p}_0(\Omega) \) case) do not apply. The reason for this, is that \( a(x,u,\nabla u) \) does not need to satisfy the strict monotonicity condition that is,
\[
\left( a(x,s,\xi) - a(x,s,\xi') \right)(\xi - \xi') > 0, \text{ for all } \xi, \xi' \in \mathbb{R}^N, (\xi \neq \xi')
\]
of a typical Leray–Lions operator but only a large monotonicity that is
\[
\left( a(x,s,\xi) - a(x,s,\xi') \right)(\xi - \xi') \geq 0, \text{ for all } \xi, \xi' \in \mathbb{R}^N.
\]

The aim of this paper is to prove the existence of solutions for (1) under the weaker assumption large monotonicity condition, without using the almost everywhere convergence of the gradients of the approximate equations, since this is impossible to prove in our setting. The main tools of our proof are a version of Minty’s Lemma (here we use an idea of G. J. Minty [42]. We note that the techniques we used in the proof are different from those used in [8, 10].

The paper is organized as follows: We introduce some basic definitions and properties in the setting of weighted Orlicz–Sobolev spaces as well as an abstract theorem and we prepare some auxiliary results to prove our theorem, in the next section 2. In the final section 3, we state the main result and proofs.

2. Preliminary

This section present some definitions and well-known about N-functions, weighted Orlicz-Sobolev spaces (standard references see [1] and [27].
2.1. N-function

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. $M$ is continuous, convex, with

$$M(t) > 0 \text{ for } t > 0, \quad \frac{M(t)}{t} \to 0 \text{ as } t \to 0 \text{ and } \frac{M(t)}{t} \to \infty \text{ as } t \to \infty.$$ 

Equivalently, $M$ admits the representation:

$$M(t) = \int_0^t m(\tau) d\tau,$$

where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to $M$ defined by

$$\overline{M}(t) = \int_0^t \overline{m}(\tau) d\tau,$$

where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\overline{m}(t) = \sup \{ s : m(s) \leq t \}$. It is well known that we can assume that $m$ and $\overline{m}$ are continuous and strictly increasing. We will extend the N-functions into even functions on all $\mathbb{R}^+$.

Clearly $M = \overline{M}$ and has Young’s inequality:

$$st \leq M(t) + \overline{M}(s) \text{ for all } s, t \geq 0.$$ 

The N-function $M$ is said to satisfy the $\Delta_2$-condition everywhere (resp. infinity) if there exist $k > 0$ (resp. $t_0 > 0$) such that, $M(2t) \leq kM(t)$ for all $t \geq 0$ (resp. $t \geq t_0$).

Let $P$ and $Q$ be two N-functions, the notation $P << Q$ means that $P$ grows essentially less rapidly than $Q$, that is to say, for all $\varepsilon > 0$, $\frac{P(t)}{Q(\varepsilon t)} \to 0$ as $t \to +\infty$.

That is the case if and only if $\frac{Q^{-1}(t)}{P^{-1}(t)} \to 0$ as $t \to \infty$.

2.2. Orlicz-Sobolev space

Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $M$ be an N-function.

The Orlicz classe $K_M(\Omega)$ (resp. the Orlicz spaces $L_M(\Omega)$) is the set of all (equivalence classes modulo equality a.e. in $\Omega$ of) real-valued measurable functions $u$ defined in $\Omega$ and satisfying

$$\int_{\Omega} M(|u(x)|) dx < \infty \quad \text{(resp. } \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right) dx < \infty \text{ for some } \lambda > 0) \text{.}$$

$L_M(\Omega)$ is a Banach space under the norm:

$$\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}. \quad (7)$$
We denote by $B(\Omega)$ the set of bounded measurable functions with compact support in $\Omega$ and by $E_M(\Omega)$ the closure of $B(\Omega)$ in $L_M(\Omega)$ (we have usual $E_M(\Omega) \subset K_M(\Omega) \subset L_M(\Omega)$).

The equality $L_M(\Omega) = E_M(\Omega)$ hold if and only if $M$ satisfies the $\Delta_2$-condition, for all $t$ or for $t$ large according to whether $\Omega$ has a infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_M(\Omega)$ by means of the pairing

$$\int_\Omega u(x)v(x)\,dx$$

where $u \in L_M(\Omega)$ and $v \in L_M(\Omega)$ and the dual norm on $L_M(\Omega)$ is equivalent to $\|\cdot\|_M$.

The space $L_M(\Omega)$ is reflexive if and only if $M$ and $\overline{M}$ satisfy the $\Delta_2$-condition for all $t$ or for $t$ large according to whether $\Omega$ be infinite measure or note.

We return now to the Orlicz-Sobolev spaces $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all function $u$ such that $u$ and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. in $E_M(\Omega)$). It’s Banach space under the norm :

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}. \quad (8)$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of $\prod L_M$ we have the weak topology $\sigma(\prod L_M, \prod E_M)$ and $\sigma(\prod L_M, \prod L_M)$.

The space $W^1_0 E_M(\Omega)$ (resp. $W^1_0 L_M(\Omega)$) is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega)$ (resp. $W^1L_M(\Omega)$) for the norm (8) (resp. for the topology $\sigma(\prod L_M, \prod E_M)$).

**Definition 2.1.** The sequence $\{u_n\}$ converges to $u$ in $L_M(\Omega)$ for the modular convergence (denoted by $u_n \to u \pmod{L_M(\Omega)}$) if $\int_\Omega M(\frac{|u_n - u|}{\lambda})\,dx \to 0$ as $n \to \infty$ for some $\lambda > 0$.

### 2.3. Weighted Orlicz–Sobolev space

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $M$ be an N-function and $\rho(x)$ be a weight function on $\Omega$, i.e. measurable positive a.e. on $\Omega$ such that :

$$\rho \in L^1(\Omega). \quad (9)$$

The weighted Orlicz classe $K_M(\Omega, \rho)$ (resp. the weighted Orlicz space $L_M(\Omega, \rho)$) is the set of all (equivalence classes modulo equality a.e. in $\Omega$) of real-valued measurable functions $u$ defined in $\Omega$ and satisfying

$$m_\rho(u, M) = \int_\Omega M(|u(x)|)\rho(x)\,dx < \infty$$

(resp. $m_\rho\left(\frac{u}{\lambda}, M\right) = \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right)\rho(x)\,dx < \infty$ for some $\lambda > 0$).
\( L_M(\Omega, \rho) \) is a Banach space under the norm:

\[
\|u\|_{M, \rho} = \inf\left\{ \lambda > 0; m\left( \frac{u}{\lambda}, M \right) \leq 1 \right\}.
\] (10)

We denote by \( B(\Omega) \) the set of bounded measurable functions with compact support in \( \Omega \) and by \( E_M(\Omega, \rho) \) the closure of \( B(\Omega) \) in \( L_M(\Omega, \rho) \) (we have usual \( E_M(\Omega, \rho) \subset K_M(\Omega, \rho) \subset L_M(\Omega, \rho) \)).

The equality \( L_M(\Omega, \rho) = E_M(\Omega, \rho) \) hold if and only if \( M \) satisfies the \( \Delta_2 \)-condition, for all \( t \) or for \( t \) large according to whether \( \Omega \) has a infinite measure or not.

The dual of \( E_M(\Omega, \rho) \) can be identified with \( L_M(\Omega, \rho) \) by means of the pairing

\[
\int_\Omega u(x) v(x) \rho(x) \, dx \text{ where } u \in L_M(\Omega, \rho) \text{ and } v \in L_M(\Omega, \rho)
\]

and the dual norm on \( L_M(\Omega, \rho) \) is equivalent to \( \|\cdot\|_{M, \Omega} \) (see [9]).

The space \( L_M(\Omega, \rho) \) is reflexive if and only if \( M \) is a Banach space under the norm \( \|\cdot\|_M \).

We return now to the weighted Orlicz-Sobolev spaces \( W^1L_M(\Omega, \rho) \) (resp. \( W^1E_M(\Omega, \rho) \)) is the space of all function \( u \) such that \( u \in L_M(\Omega) \) (resp. \( u \in E_M(\Omega) \)) and its distributional derivatives up to order 1 lie in \( L_M(\Omega, \rho) \) (resp. \( E_M(\Omega, \rho) \)). It’s Banach space under the norm

\[
\|u\|_{1, M, \rho} = \|u\|_M + \|\nabla u\|_{M, \rho},
\] (11)

(\( \|u\|_M = \|u\|_{M, \Omega} \)). Thus \( W^1L_M(\Omega, \rho) \) and \( W^1E_M(\Omega, \rho) \) can be identified with subspaces of \( \prod L_M, \rho = L_M \times \prod L_M(\Omega, \rho) \) we have the weak topology \( \sigma(\prod L_M, \rho, \prod E_M, \rho) \) and \( \sigma(\prod L_M, \rho, \prod L_M, \rho) \). The space \( W^1L_M(\Omega, \rho) \) (resp. \( W^1E_M(\Omega, \rho) \)) is defined by the closure of \( D(\Omega) \) in \( W^1E_M(\Omega, \rho) \) (resp. \( W^1L_M(\Omega, \rho) \)) for the norm (11) (resp. for the topology \( \sigma(\prod L_M, \rho, \prod E_M, \rho) \)).

**Definition 2.2.** The sequence \( u_n \) converges to \( u \) in \( L_M(\Omega, \rho) \) for the modular convergence (denoted by \( u_n \to u \) (mod) \( L_M(\Omega, \rho) \)) if

\[
\int_\Omega M\left( \frac{|u_n - u|}{\lambda} \right) \rho(x) \, dx \to 0 \text{ as } n \to \infty \text{ for some } \lambda > 0
\]


**Definition 2.3.** The sequence \( u_n \) converges to \( u \) in \( W^1L_M(\Omega, \rho) \) for the modular convergence (denoted by \( u_n \to u \) (mod) \( W^1L_M(\Omega, \rho) \)) if

\[
\int_\Omega M\left( \frac{|u_n - u|}{\lambda} \right) \rho(x) \, dx \to 0 \text{ as } n \to \infty \text{ and } \int_\Omega M\left( \frac{|D^\alpha (u_n - u)|}{\lambda} \right) \rho(x) \, dx \to 0 \text{ as } n \to \infty \text{ for } |\alpha| = 1.
\]

**Lemma 2.4.** [3] Let \( M \) be an \( N \)-function. If \( u_n \in L_M(\Omega) \) converges a.e. to \( u \) and \( u_n \) bounded in \( L_M(\Omega) \), then \( u \in L_M(\Omega) \) and \( u_n \to u \) for the topology \( \sigma(L_M(\Omega), E_M(\Omega)) \).
Lemma 2.5. [3] If the sequence \( u_n \in L_M(\Omega, \rho) \) converges to \( u \) a.e. is bounded in \( L_M(\Omega, \rho) \), then \( u \in L_M(\Omega, \rho) \) and \( u_n \to u \) for the topology \( \sigma(L_M(\Omega, \rho), E_M(\Omega, \rho)) \).

2.4. Compactness results

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with locally Lipschitzian boundary, \( \rho \) the weight function and \( M \) an N-function. Let the following integrability assumptions:

There exists a real \( s > 0 \) such that:

\[
(M(t))^{\frac{s}{s+1}} \text{ is } N\text{-function and an } \rho^{-s} \in L^1(\Omega),
\]

\[
\int_{1}^{\infty} \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}} dM(t) = \infty,
\]

\[
\lim_{t \to \infty} \frac{1}{M^{-1}(t)} \int_{0}^{t^{s+1}} \frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}} du = 0.
\]

Remark 2.6. In the particular case where \( M(t) = |t|^p \) (1 < \( p \) < \( \infty \)), the first part of (12) is satisfied if \( s > \frac{1}{p-1} \).

Theorem 2.7 (See[2] Theorem 9-5). Let \( \Omega \) an open bounded locally- Lipschitzian boundary in \( \mathbb{R}^N \) and \( M \) an N-function. Suppose that assumptions (12)–(14) are satisfied. So we have the following compact injection:

\[
W^1_{L_M}(\Omega, \rho) \hookrightarrow E_M(\Omega).
\]

Theorem 2.8. (Weighted Poincaré inequality) Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with locally Lipschitzian boundary, \( \rho \) a weight function, and \( M \) an N-function. If \( u \in W^1_{L_M}(\Omega) \), then

\[
\|u\|_M \leq c \|\nabla u\|_{M, \rho}
\]

where \( c \) is a positive constant, which implies that \( \|\nabla u\|_{M, \rho} \) and \( \|u\|_{1, M} \) are equivalent norms on \( W^1_{L_M}(\Omega) \).

Proof. Under the assumptions (12)-(14), the Sobolev conjugate N-function \( M_s^* \) of \( M_s \) is well defined by

\[
M_s^{*-1} = \int_{0}^{s} \frac{M^{-1}(t)}{t^{1+\frac{s}{N}}} dt
\]

and we have \( W^1_{L_M}(\Omega) \subset L_{M_s^*} \). And since \( M \ll M_s^* \), we have \( L_{M_s^*} \subset L_M \). Hence

\[
\|u\|_M \leq c_1 \|u\|_{M_s^*} \leq c_2 \|u\|_{1, M_s},
\]
where $c_1$ and $c_2$ are two positive constants. Then, by using the Poincaré inequality in the non-weighted Orlicz-Sobolev space, there exists a positive constant $c'$ such that
\[ \|u\|_{1, M_s} \leq c' \|\nabla u\|_{M_s}. \]
We will show that
\[ \|\nabla u\|_{M_s} \leq c \|\nabla u\|_{M, \rho}. \]

For that we have
\[
\|v\|_{M_s} \leq \int_{\Omega} M_s(v(x)) \, dx + 1 = \int_{\Omega} M_s(v(x)) \frac{1}{\rho(x)} \rho(x) \, dx + 1 \\
\leq \int_{\Omega} S(M_s(v(x))) \rho(x) \, dx + \int_{\Omega} \frac{1}{\rho(x)} \rho(x) \, dx + 2 \\
= \int_{\Omega} M(v(x)) \rho(x) \, dx + \int_{\Omega} \rho^{-s}(x) \, dx + 1,
\]
which implies that
\[ \|v\|_{M_s} \leq c \|v\|_{M, \rho}, \]
for some positive constant $c$. In fact, if this is not true, then there exists a sequence $v_n$ such that $\|v_n\|_{M_s} \to \infty$ and for $n$ large, $\|v_n\|_{M, \rho} \leq 1$. Hence, for $n$ sufficiently large we get
\[
\int_{\Omega} M(v_n(x)) \rho(x) \, dx \leq \|v_n\|_{M, \rho} \leq 1.
\]
Then
\[
\|v_n\|_{M_s} \leq \int_{\Omega} M(v(x)) \rho(x) \, dx + \int_{\Omega} \rho^{-s}(x) \, dx + 1 \\
\leq \|v_n\|_{M, \rho} + \int_{\Omega} \rho^{-s}(x) \, dx + 1,
\]
which is a contradiction, since the left hand-side tends to infinity while the right hand-side is bounded. Finally, by taking $v = \nabla u$, we conclude the result.

We will also use the following technical lemmas.

2.5. Some technical lemmas

Lemma 2.9. [35] Let $f_n, f \in L^1(\Omega)$ such that $f_n \geq 0$ a.e. in $\Omega$, $f_n \to f$ a.e. in $\Omega$ et $\int_{\Omega} f_n(x) \, dx \to \int_{\Omega} f(x) \, dx$. Then $f_n \to f$ strongly in $L^1(\Omega)$. 

Lemma 2.10. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W^{1}_{0}L_{M}(\Omega, \rho)$. Then $F(u) \in W^{1}_{0}L_{M}(\Omega, \rho)$. Moreover, if the set $D$ of discontinuity points of $F'$ is finite, then

$$
\frac{\partial}{\partial x_{i}} F(u) = \begin{cases} 
F'(u) \frac{\partial u}{\partial x_{i}} & \text{a.e. in } \{x \in \Omega; u(x) \notin D\}, \\
0 & \text{a.e. in } \{x \in \Omega; u(x) \in D\}.
\end{cases}
$$

Proof. We suppose for the moment that $F$ is also $C^{1}$, there exist a sequence $u_{n} \in D(\Omega)$ such that $u_{n} \to u$ (mod $W^{1}_{0}L_{M}(\Omega, \rho)$). Passing to subsequence, we can assume that

$$
D^{\alpha} u_{n} \to D^{\alpha} u \quad \forall |\alpha| \leq 1 \text{ a.e. in } \Omega.
$$

From the relation $|F(s)| \leq k |s|$, where $k$ denote the Lipschitz constant for $F$, and

$$
\frac{\partial}{\partial x_{i}} F(u_{n}) = F'(u_{n}) \frac{\partial u_{n}}{\partial x_{i}},
$$

we deduce that $F(u_{n})$ remains bounded in $W^{1}_{0}L_{M}(\Omega, \rho)$. Thus going to a further subsequence, we obtain

$$
F(u_{n}) \to w \in W^{1}_{0}L_{M}(\Omega, \rho) \text{ for } \sigma(\prod L_{M}, \prod E_{M}),
$$

and also by a local application of the compact imbedding theorem, $F(u_{n}) \to w$ a.e. in $\Omega$. Consequently $w = F(u)$, and $F(u) \in W^{1}_{0}L_{M}(\Omega, \rho)$. Finally, by the usual chain rule for weak derivatives,

$$
\frac{\partial}{\partial x_{i}} F(u) = F'(u) \frac{\partial u}{\partial x_{i}} \text{ a.e. in } \Omega. \quad (15)
$$

For the general case. Taking convolution with the mollifiers, we get a sequence $F_{n} \in C^{\infty}(\mathbb{R})$ such that $F_{n} \to F$ uniformly on each compact, $F_{n}(0) = 0$ and $\left|F_{n}'\right| \leq k$. For each $n$, $F_{n} \in W^{1}_{0}L_{M}(\Omega, \rho)$, and we have (15) with $F$ replaced by $F_{n}$. Finally (15) follows from the generalized chain rule for weak derivatives.

The following lemmas follow from the previous lemma.

Lemma 2.11. Let $u, v \in W^{1}_{0}L_{M}(\Omega, \rho)$ and let $w = \min(u, v)$. Then $w \in W^{1}_{0}L_{M}(\Omega, \rho)$ and

$$
\frac{\partial w}{\partial x_{i}} = \begin{cases} 
\frac{\partial u}{\partial x_{i}} & \text{a.e. in } \{x \in \Omega; u(x) \leq v(x)\}, \\
\frac{\partial v}{\partial x_{i}} & \text{a.e. in } \{x \in \Omega; u(x) > v(x)\}.
\end{cases}
$$

Proof. Note that $\min(u, v) = u - (u - v)^{+}$ and apply Lemma 2.10 with $F(s) = s^{+}$. \qed
We introduce the truncate operator. For a given constant \( k > 0 \), we define the function \( T_k : \mathbb{R} \to \mathbb{R} \) as

\[
T_k(s) = \begin{cases} 
  s & \text{if } |s| \leq k, \\
  k \frac{s}{|s|} & \text{if } |s| > k.
\end{cases}
\]

**Lemma 2.12.** Assume that (12)–(14) holds. Let \( u \in W^1_0L_M(\Omega, \rho) \), and let \( T_k(u), k \in \mathbb{R}^+ \), be the usual truncation then \( T_k(u) \in W^1_0L_M(\Omega, \rho) \). Moreover we have

\[
T_k(u) \to u \text{ strongly in } W^1_0L_M(\Omega, \rho).
\]

**Lemma 2.13.** Let \( u_n \) be a sequence of \( W^1_0L_M(\Omega, \rho) \) such that \( u_n \to u \) weakly in \( W^1_0L_M(\Omega, \rho) \) for the topology \( \sigma(L_M(\Omega, \rho), E_M(\Omega, \rho)) \). Then \( T_k(u_n) \to T_k(u) \).

**Proof.** Since \( u_n \to u \) and \( W^1_0L_M(\Omega, \rho) \) is compact in \( E_M(\Omega) \), we have \( u_n \to u \) strongly in \( E_M(\Omega) \) and a.e. in \( \Omega \), then \( T_k(u_n) \to T_k(u) \) a.e. in \( \Omega \). On the other hand, for some \( \lambda > 0 \),

\[
\int_{\Omega} M \left( \frac{|T_k(u_n)|}{\lambda} \right) dx \leq \int_{\Omega} M \left( \frac{|u_n|}{\lambda} \right) dx
\]

and

\[
\int_{\Omega} M \left( \frac{|
abla T_k(u_n)|}{\lambda} \right) \rho(x) dx = \int_{\Omega} M \left( \frac{|T_k'(u_n)||\nabla u_n|}{\lambda} \right) \rho(x) dx \leq \int_{\Omega} M \left( \frac{|
abla u_n|}{\lambda} \right) \rho(x) dx
\]

imply that

\[
||T_k(u_n)||_{1,M,\rho} \leq ||u_n||_{1,M,\rho}.
\]

Then \( (T_k(u_n)) \) is bounded in \( W^1_0L_M(\Omega, \rho) \) hence by Lemma 2.5, we have \( T_k(u_n) \to T_k(u) \) in \( W^1_0L_M(\Omega, \rho) \) for \( \sigma(L_M(\Omega, \rho), E_M(\Omega, \rho)) \).

**Lemma 2.14.** If the sequence \( u_n \in E_M(\Omega, \rho) \) converges a.e. in \( \Omega \) with \( \rho \in L^1(\Omega) \), then it converges in norm in \( E_M(\Omega, \rho) \) if and only if the norms are uniformly absolutely continuous, i.e. for each \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that \( \|u_n \chi_E\|_{M,\rho} < \varepsilon \), for all \( n \) and \( E \subset \Omega \) with \( |E| < \delta \).

**Proof.** by the same argument introduced in the proof of Lemma 11.2 in [36] we find \( E_{n,m} = \{ x \in \Omega : |u_n(x) - u_n(x)| > \alpha \} \), where \( \alpha = M^{-1} \left( \frac{\varepsilon}{3 \|\rho\|_1} \right) \) and with \( \delta > 0 \) such that

\[
\|u_n \chi_E\|_{M,\rho} < \frac{\varepsilon}{3}.
\]

We denote by \( H(E_M(\Omega), r) \) the set of functions \( u \in L_M(\Omega) \) whose distance to \( E_M(\Omega) \) (with respect to the Orlicz norm) is strictly less than \( r \) and by \( B_{L_M(\Omega)}(0, r) \) the ball in \( L_M(\Omega) \) (with respect to the Orlicz norm) of radius \( r \) and center 0. 

\[\square\]
Lemma 2.15. Let \( \Omega \) be a bounded subset of \( \mathbb{R}^N \) with finite measure. Let \( M, R \) and \( Q \) be \( N \)-functions such that \( Q \ll R \), and let \( f \) be a Carathéodory function such that for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R} \):

\[
|f(x, s)| \leq b(x) + k_1 R^{-1} \left( \rho^{-1}(x) Q(k_2 |s|) \right),
\]

where \( 0 \leq b(x) \in E_M(\Omega, \rho) \), \( \rho \in L^1(\Omega) \) and \( k_1, k_2 \in \mathbb{R}^+ \). Then the Nemytskii operator

\[ N_f(u)(x) = f(x, u(x)) \]

satisfies:

1. Sends \( \left( H(E_Q(\Omega), \frac{1}{k_2}) \right)^p \) into \( L_R(\Omega, \rho) \) and is continuous from \( \left( H(E_Q(\Omega), \frac{1}{k_2}) \right)^p \) to the norm topology of \( (L_Q(\Omega))^p \) into \( L_R(\Omega, \rho) \) to the modular convergence;

2. It’s uniformly bounded on \( \left( B_{L_Q(\Omega)}(0, \frac{1}{k_2}) \right)^p \);

3. If \( b(x) \in E_{R_1}(\Omega, \rho) \) with \( R_1 \ll R \), then \( N_f \) is continuous to the norm topology of \( E_{R_1}(\Omega, \rho) \).

Proof. (1) Let \( u = (u_1, u_2, \ldots, u_p) \in \left( H(E_Q(\Omega), \frac{1}{k_2}) \right)^p \). Since \( d(u_i, E_Q(\Omega)) < \frac{1}{k_2} \) (1 \( \leq i \leq p \)), we have \( \int_{\Omega} Q(k_2 |u|)dx \leq 1 \), (see Theorem 10.1 [36]). Let \( \lambda \geq 2k_1 \) such that \( \frac{2b(x)}{\lambda} \in K_{R}(\Omega, \rho) \). By the growth condition (16) and the convexity of \( R \), we get

\[
\int_{\Omega} R \left( \frac{|f(x, s)|}{\lambda} \right) \rho(x)dx \leq \frac{1}{2} \int_{\Omega} R \left( \frac{2b(x)}{\lambda} \right) \rho(x)dx + \frac{1}{2} \int_{\Omega} R(k_2 |u(x)|)dx < \infty.
\]

On the other hand, suppose that

\[ u_n \to u \in \left( H(E_Q(\Omega), \frac{1}{k_2}) \right)^p, \]

and let \( \alpha > 0 \) such that \( d(k_2 |u|, E_Q(\Omega)) < \alpha < 1 \) and \( d(k_2 |u|, E_Q(\Omega)) < 1 - \alpha < 1 \).

We have \( \frac{k_2}{\alpha} |u| \in K_Q(\Omega) \) and \( \frac{k_2}{1-\alpha} |u| \in K_Q(\Omega) \) (see Theorem 10.1 [36]) and
for $\lambda > 4k_1$ such that $\frac{4b(x)}{\lambda} \in K_R(\Omega, \rho)$, we get

$$\int_{\Omega} R \left( \frac{|f(x,u_n) - f(x,u)|}{\lambda} \right) \rho(x) dx$$

$$\leq \int_{\Omega} R \left( \frac{2b(x) + k_1 R^{-1} (\rho^{-1}(x) Q(k_2 |u_n|))}{\lambda} + k_1 R^{-1} (\rho^{-1}(x) Q(k_2 |u|)) \right) \rho(x) dx$$

$$\leq \frac{1}{2} \int_{\Omega} R \left( \frac{4b(x)}{\lambda} \right) dx + \frac{1 - \alpha}{4} \int_{\Omega} Q \left( \frac{k_2}{1 - \alpha} |u_n - u| \right) dx$$

$$+ \frac{\alpha}{4} \int_{\Omega} Q \left( \frac{k_2}{\alpha} |u| \right) dx + \frac{1}{4} \int_{\Omega} Q(k_2 |u|) dx.$$  

Since $Q(k_2 |u|) \leq Q \left( \frac{k_2}{\alpha} |u| \right)$, the last inequality becomes

$$\int_{\Omega} R \left( \frac{|f(x,u_n) - f(x,u)|}{\lambda} \right) \rho(x) dx$$

$$\leq \frac{1}{2} \int_{\Omega} R \left( \frac{4b(x)}{\lambda} \right) \rho(x) dx + \int_{\Omega} Q \left( \frac{k_2}{1 - \alpha} |u_n - u| \right) dx + \int_{\Omega} Q(k_2 |u|) dx,$$

which implies by using the Vitali’s theorem

$$f(x,u_n) \to f(x,u) \text{ (mod) in } L_R(\Omega, \rho),$$

for a subsequence denoted again $u_n$ (which holds for the whole sequence).

(2) Let now $u \in \left( B_{L_Q(\Omega)}(0, \frac{1}{k_2}) \right)^p$ and let $\lambda \geq 2k$ such that $\int_{\Omega} R \left( \frac{2b(x)}{\lambda} \right) \rho(x) dx \leq 1.$

By the growth condition (16) and the convexity of $R$, we get

$$\int_{\Omega} R \left( \frac{|f(x,u)|}{\lambda} \right) \rho(x) dx \leq \frac{1}{2} \int_{\Omega} R \left( \frac{2b(x)}{\lambda} \right) \rho(x) dx + \frac{1}{2} \int_{\Omega} Q(k_2 |u(x)|) dx \leq 1,$$

which implies (2).

(3) Suppose that $b(x) \in E_{R_1}(\Omega, \rho)$ with $R_1 \prec R$. As in (1), since $L_R(\Omega, \rho) \subset E_{R_1}(\Omega, \rho)$, we can show that $f(x,u) \in E_{R_1}(\Omega, \rho)$ for all $u \in \left( E_Q(\Omega), \frac{1}{k_2} \right)^p$.

Suppose now that

$$u_n \to u \in \left( E_Q(\Omega), \frac{1}{k_2} \right)^p \text{ in } (L_Q(\Omega))^p,$$

We shall show that

$$f(x,u_n) \to f(x,u) \text{ (mod) in } E_{R_1}(\Omega, \rho).$$
Fix \( \varepsilon > 0 \), we have as above
\[
\int_{\Omega} R_1 \left( \frac{|f(x,u_n) - f(x,u)|}{\varepsilon} \right) \rho(x)dx \leq \frac{1}{4} \int_{\Omega} R_1 \left( \frac{4b(x)}{\varepsilon} \right) \rho(x)dx \\
+ \frac{1}{4} \int_{\Omega} R_1 \left( \frac{4k_1}{\varepsilon} R^{-1} (\rho^{-1}(x)Q(k_2|u_n|)) \right) \rho(x)dx + \frac{1}{4} \int_{\Omega} \left( \frac{4k_1}{\varepsilon} R^{-1} (\rho^{-1}(x)Q(k_2|u|)) \right) \rho(x)dx
\]

Since \( R_1 \ll R \), there exists \( K' \) such that \( R_1(\frac{4k_1}{\varepsilon}t) \leq R(t) + K' \) for all \( t \geq 0 \). Then, the last inequality can be written in the form
\[
\int_{\Omega} R_1 \left( \frac{|f(x,u_n) - f(x,u)|}{\varepsilon} \right) \rho(x)dx \leq \frac{1}{4} \int_{\Omega} R_1 \left( \frac{4b(x)}{\varepsilon} \right) \rho(x)dx + \frac{1}{4} \int_{\Omega} Q(k_2|u_n|)dx \\
+ \frac{1}{4} \int_{\Omega} Q(k_2|u|)dx + \frac{K}{2} \int_{\Omega} \rho(x)dx.
\]

As in (1) by using the Vitali’s theorem, we get
\[
\int_{\Omega} R_1 \left( \frac{|f(x,u_n) - f(x,u)|}{\varepsilon} \right) \rho(x)dx \to 0 \text{ as } n \to \infty,
\]
for a subsequence (which holds for the whole sequence). Since \( \varepsilon \) is arbitrary, we conclude (3). \( \square \)

**Definition 2.16.** [1] **(Segment property):** A domain \( \Omega \) is said to satisfy the segment property, if there exist a finite open covering \( \{\theta_i\}_{i=1}^k \) of \( \bar{\Omega} \) and a corresponding nonzero vectors \( z_i \in \mathbb{R}^N \) such that \( (\bar{\Omega} \cap \theta_i) + tz_i \subset \Omega \) for all \( t \in (0,1) \) and \( i = 1, \ldots, k \).

**Lemma 2.17.** [34] Suppose that \( \Omega \) satisfies the segment property and let \( u \in W^1_0 L_M(\Omega, \rho) \). Then, there exists a sequence \( u_n \in D(\Omega) \) such that
\[
u_n \to u \text{ for modular convergence in } W^1_0 L_M(\Omega, \rho).
\]
Furthermore, if \( u \in W^1_0 L_M(\Omega, \rho) \cap L^\infty(\Omega) \), then \( \|u_n\|_\infty \leq (N+1)\|u\|_\infty \).

**Lemma 2.18.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with the segment property. If \( u \in (W^1_0 L_M, \rho)(\Omega))^N \) then \( \int_{\Omega} \text{div}(u)dx = 0 \).

**Proof:** Fix a vector \( u = (u^1, \ldots, u^N) \in (W^1_0 L_M, \rho)(\Omega))^N \), since \( W^1_0 L_M, \rho)(\Omega) \) is the closure of \( C^\infty_0 (\Omega) \) in \( W^1_0 L_M, \rho)(\Omega) \), then each term \( u^i \) can be approximated by a suitable sequence \( u_k^i \in D(\Omega) \) such that \( u_k^i \) converges to \( u^i \) in \( W^1_0 L_M, \rho)(\Omega) \). Moreover, due to the fact that \( u_k^i \in C^\infty_0 (\Omega) \), then the Green formula gives
\[
\int_{\Omega} \frac{\partial u_k^i}{\partial x_i} = \int_{\partial \Omega} u_k^i n_i ds = 0.
\]
On the other hand, \( \frac{\partial u_k}{\partial x_i} \rightarrow \frac{\partial u'}{\partial x_i} \) in \( L_{M, \rho}(\Omega) \). Thus \( \frac{\partial u_k}{\partial x_i} \rightarrow \frac{\partial u'}{\partial x_i} \) in \( L^1(\Omega) \), which gives in view of (17) that

\[
\int_{\Omega} \text{div}(u) dx = 0.
\]

\[\square\]

3. Main results

Let \( Y \) be a closed subspace of \( W^1 L_M(\Omega, \rho) \) for \( \sigma(\prod L_{M, \rho}, \prod E_M, \rho) \) and let

\[
Y_0 = Y \cap W^1 L_M(\Omega, \rho),
\]

such that \( Y \) is the closure of \( Y_0 \) for \( \sigma(\prod L_{M, \rho}, \prod E_M, \rho) \). In the next, we consider the complementary system \( (Y, Y_0, Z, Z_0) \) generated by \( Y \) i.e. \( Y_0^* \) can be identified to \( Z \) and \( Z_0^* \) can be identified to \( Y \) by the means \( \langle ., . \rangle \). Let the mapping \( T \) (associated to the operator \( A \)) defined from \( D(T) = \{ u \in Y, a_0(x, u, \nabla u) \in L_M(\Omega), a_i(x, u, \nabla u) \in L_M(\Omega) \} \) into \( Z \) by the formula

\[
a(u, v) = \int_{\Omega} a_0(x, u, \nabla u)v(x)dx + \sum_{1 \leq i \leq N} \int_{\Omega} a_i(x, u, \nabla u)\frac{\partial v(x)}{\partial x_i} \rho(x)dx \quad \forall v \in Y_0.
\]

We consider the complementary system

\[
(Y, Y_0, Z, Z_0) = (W^1_0 L_M(\Omega, \rho), W^1_0 E_M(\Omega, \rho), W^{-1} E_M(\Omega, \rho), W^{-1} L_M(\Omega, \rho)).
\]

As in [17], we define the entropy solution of our problem.

**Definition 3.1.** An entropy solution of the problem (1) is a measurable function \( u \) such that \( T_k(u) \in W^1_0 L_M(\Omega, \rho) \) for every \( k > 0 \) and such that

\[
\int_{\Omega} a(x, u, \nabla u)\nabla T_k(u - \phi)\rho(x)dx \leq \int_{\Omega} f T_k(u - \phi)dx + \int_{\Omega} F(u)\nabla T_k(u - \phi)
\]

for every \( \phi \in W^1_0 E_M(\Omega, \rho) \cap L^\infty(\Omega) \).

Our main results are collected in the following theorem.

**Theorem 3.2.** Under the assumptions (2)-(6),(12)-(14) and \( \rho(x) \) be a weight function on \( \Omega \) satisfy (2.3) there exist an entropy solution \( u \) of the problem (1).
3.1. Main Lemma

Lemma 3.3. Let $u$ be a measurable function such that $T_k(u)$ belongs to $W^1_0 L_M(\Omega, \rho)$ for every $k > 0$. Then

$$\int_{\Omega} a(x, u, \nabla \phi) \nabla T_k(u - \phi) \, dx \leq \int_{\Omega} f T_k(u - \phi) \, dx + \int_{\Omega} F(u) \nabla T_k(u - \phi)$$  \hspace{1cm} (18)

is equivalent to

$$\int_{\Omega} a(x, u, \nabla u) \nabla T_k(u - \phi) \, dx = \int_{\Omega} f T_k(u - \phi) \, dx + \int_{\Omega} F(u) \nabla T_k(u - \phi)$$  \hspace{1cm} (19)

for every $\phi \in W^1_0 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, and for every $k > 0$.

3.2. Proof of lemma 3.3

In fact (19) implies (18) is easily proved adding and subtracting

$$\int_{\Omega} a(x, u, \nabla \phi) \nabla T_k(u - \phi) \, dx,$$

and then using assumption (3). Thus, it remains to prove that (18) implies (19). Let $h$ and $k$ be positive real numbers, let $\lambda \in ]-1, 1[$ and $\Psi \in W^1_0 L_M(\Omega, \rho) \cap L^\infty(\Omega)$.

choose, $\phi = T_h(u - \lambda T_k(u - \Psi)) \in W^1_0 L_M(\Omega, \rho) \cap L^\infty(\Omega)$ as test function in (18), we have:

$$I_{hk} \leq J_{hk},$$  \hspace{1cm} (20)

with

$$I_{hk} = \int_{\Omega} a(x, u, \nabla T_h(u - \lambda T_k(u - \Psi))) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx,$$

and

$$J_{hk} = \int_{\Omega} f T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx + \int_{\Omega} F(u) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))).$$

Put

$$A_{hk} = \{ x \in \Omega, |u - T_h(u - \lambda T_k(u - \Psi)| \leq k \},$$

and

$$B_{hk} = \{ x \in \Omega, |u - \lambda T_k(u - \Psi)| \leq h \}.$$
Then, we obtain

\[ I_{hk} = \int_{A_{kh} \cap B_{hk}} a(x, u, \nabla T_h(u - \lambda T_k(u - \Psi))) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx \]

\[ + \int_{A_{kh} \cap B_{hk}^C} a(x, u, \nabla T_h(u - \lambda T_k(u - \Psi))) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx \]

\[ + \int_{A_{kh}^C} a(x, u, \nabla T_h(u - \lambda T_k(u - \Psi))) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx. \]

Since \( \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \) is different to zero only on \( A_{kh} \), we have

\[ \int_{A_{kh}^C} a(x, u, \nabla T_h(u - \lambda T_k(u - \Psi))) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx = 0. \]  \hfill (21)

Moreover, if \( x \in B_{hk}^C \), we have \( \nabla T_h(u - \lambda T_k(u - \Psi)) = 0 \) and using (4), we deduce that,

\[ \int_{A_{kh} \cap B_{hk}^C} a(x, u, \nabla T_h(u - \lambda T_k(u - \Psi))) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx = 0. \]  \hfill (22)

From (21) and (22), we obtain

\[ I_{hk} = \int_{A_{kh} \cap B_{hk}} a(x, u, \nabla T_h(u - \lambda T_k(u - \Psi))) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx. \]

Letting \( h \to +\infty, |\lambda| \leq 1 \), we have

\[ A_{kh} \to \{ x, |\lambda| |T_k(u - \Psi)| \leq h \} = \Omega, \] \hfill (23)

\[ B_{hk} \to \Omega \quad \text{which implies} \quad A_{kh} \cap B_{hk} \to \Omega. \] \hfill (24)

Which and using Lebesgue theorem, we conclude that

\[ \lim_{h \to +\infty} \int_{A_{kh} \cap B_{hk}} a(x, u, \nabla T_h(u - \lambda T_k(u - \Psi))) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx \]

\[ = \lambda \int_{\Omega} a(x, u, \nabla (u - \lambda T_k(u - \Psi))) \nabla T_k(u - \Psi) \, dx. \] \hfill (25)

Thus implies that,

\[ \lim_{h \to +\infty} I_{hk} = \lambda \int_{\Omega} a(x, u, \nabla (u - \lambda T_k(u - \Psi))) \nabla T_k(u - \Psi) \, dx. \] \hfill (26)

On the other hand, we have,

\[ J_{hk} = \int_{\Omega} f T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx + \int_{\Omega} F(u) \nabla T_k(u - T_h(u - \lambda T_k(u - \Psi))) \, dx. \]
Then
\[
\lim_{h \to +\infty} \int \nabla T_k (u - \lambda T_k (u - \Psi)) \, dx
\]
\[
+ \int \nabla F(u) \nabla T_k (u - \lambda T_k (u - \Psi)) \, dx
\]
\[
= \lambda \left( \int f T_k (u - \Psi) \, dx + \int F(u) \nabla T_k (u - \Psi) \, dx \right),
\]
i.e.,
\[
\lim_{h \to +\infty} J_{hk} = \lambda \left( \int f T_k (u - \Psi) \, dx + \int F(u) \nabla T_k (u - \Psi) \, dx \right).
\]
Together (26), (30) and passing to the limit in (20), we obtain,
\[
\lambda \left( \int a(x, u, \nabla (u - \lambda T_k (u - \Psi))) \nabla T_k (u - \Psi) \, dx \right)
\]
\[
\leq \lambda \left( \int f T_k (u - \Psi) \, dx + \int F(u) \nabla T_k (u - \Psi) \, dx \right)
\]
for every \( \Psi \in \text{W}_{0}^{1} L_{\text{ad}}(\Omega, \rho) \cap L_{\infty}(\Omega) \), and for every \( k > 0 \). Choosing \( \lambda > 0 \) dividing by \( \lambda \), and then letting \( \lambda \) tend to zero, we obtain
\[
\int a(x, u, \nabla u) \nabla T_k (u - \Psi) \, dx \leq \int f T_k (u - \Psi) \, dx + \int F(u) \nabla T_k (u - \Psi) \, dx.
\]
for \( \lambda < 0 \), dividing by \( \lambda \), and then letting \( \lambda \) tend to zero, we obtain
\[
\int a(x, u, \nabla u) \nabla T_k (u - \Psi) \, dx \geq \int f T_k (u - \Psi) \, dx + \int F(u) \nabla T_k (u - \Psi) \, dx.
\]
Combining (32) and (33), we conclude the following equality :
\[
\int a(x, u, \nabla u) \nabla T_k (u - \Psi) \, dx = \int f T_k (u - \Psi) \, dx + \int F(u) \nabla T_k (u - \Psi) \, dx.
\]
This completes the proof of Lemma 3.3.

3.3. Proof of Theorem 3.2

3.3.1. Approximate problem and a priori estimate

For \( n \in \mathbb{N} \), define \( f_n := T_n(f), F_n = F(T_n) \). Let \( u_n \) be solution in \( \text{W}_{0}^{1} L_{\varphi}(\Omega) \) of the problem
\[
\begin{cases} 
- \text{div}(a(x, u_n, \nabla u_n)) = f_n - \text{div} F_n(u_n) & \text{in } \Omega \\
\quad \quad u_n = 0 & \text{on } \partial\Omega,
\end{cases}
\]
which exists thanks to ([33],Proposition 1, Remark 2). Choosing $T_k(u_n)$ as test function in (35), we have

$$\int_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n) \, dx = \int_{\Omega} f_nT_k(u_n) \, dx + \int_{\Omega} F_n(u_n)\nabla T_k(u_n) \, dx,$$

We claim that:

$$\int_{\Omega} F_n(u_n)\nabla T_k(u_n) \, dx = 0, \quad (36)$$

using $\nabla T_k(u_n) = \nabla u_n\chi_{\{|u_n| \leq k\}}$, define $\Theta(t) = F_n(t)\chi_{\{t \leq k\}}$, and $\tilde{\Theta}(t) = \int_0^t \Theta(\tau) \, d\tau$,

We have by Lemma 2.18 $\tilde{\Theta}(u_n) \in (W^1_0 L^q(\Omega))^N$

$$\int_{\Omega} F_n(u_n)\nabla T_k(u_n) \, dx = \int_{\Omega} F_n(u_n)\chi_{\{|u_n| \leq k\}}\nabla u_n \, dx$$

$$= \int_{\Omega} \Theta(u_n)\nabla u_n \, dx = \int_{\Omega} \text{div}(\tilde{\Theta}(u_n)) \, dx = 0 \quad (37)$$

(by 2.18) which proves the claim.

Now thanks to assumption (4), we obtain

$$\int_{\Omega} a(x,u_n,\nabla u_n)\nabla T_k(u_n) \, dx \geq \int_{\Omega} \rho(x)M(\lambda T_k(u_n)) \, dx,$$

then

$$\int_{\Omega} \rho(x)M(\lambda T_k(u_n)) \, dx \leq k\|f\|_{L^1(\Omega)}. \quad (38)$$

Then

$$\int_{\Omega} \rho(x)M(\lambda T_k(u_n)) \, dx \leq C_1 k, \quad (39)$$

where $C_1$ is a constant independently of $n$.

3.3.2. Locally convergence of $u_n$ in measure

Taking $\lambda |T_k(u_n)|$ in (35) and using (39), one has

$$\int_{\Omega} \rho(x)M(\lambda T_k(u_n)) \, dx \leq \int_{\Omega} \rho(x)M(\lambda T_k(u_n)) \, dx \leq C_1 k. \quad (40)$$
Then we deduce by using (40), that
\[
\text{meas}\{|u_n| > k\} \leq \frac{1}{\inf_{k} M(\frac{k}{\lambda})} \int_{\{|u_n| > k\}} M\left(\frac{|u_n(x)|}{\lambda}\right) dx
\]
\[
\leq \frac{1}{\inf_{k} M(\frac{k}{\lambda})} \int_{\Omega} M\left(\frac{1}{\lambda} |T_k(u_n)|\right) dx
\]
\[
\leq \frac{C_1k}{\inf_{k} M(\frac{k}{\lambda})} \forall n, \forall k \geq 0. \tag{41}
\]

For any $\beta > 0$, we have
\[
\text{meas}\{|u_n - u_m| > \beta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \beta\},
\]
and so that
\[
\text{meas}\{|u_n - u_m| > \beta\} \leq \frac{2C_1k}{\inf_{k} M(\frac{k}{\lambda})} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \beta\}. \tag{42}
\]

By using (39) and Poincaré inequality in weighted Orlicz-Sobolev spaces (Theorem 2.8), we deduce that $(T_k(u_n))$ is bounded in $W^1_0 L_M(\Omega, \rho)$, and then there exists $\omega_k \in W^1_0 L_M(\Omega, \rho)$ such that $T_k(u_n) \rightharpoonup \omega_k$ weakly in $W^1_0 L_M(\Omega, \rho)$ for $\sigma(\Pi L_M, \Pi E_M)$; strongly in $E_M(\Omega, \rho)$ and a.e. in $\Omega$.

Consequently, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in $\Omega$.

Let $\varepsilon > 0$, then by (42) and the fact that $\frac{2C_1k}{\inf_{k} M(\frac{k}{\lambda})} \to 0$ as $k \to +\infty$, there exists some $k = k(\varepsilon) > 0$ such that
\[
\text{meas}\{|u_n - u_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).
\]

This proves that $u_n$ is a Cauchy sequence in measure, thus, $u_n$ converges almost everywhere to some measurable function $u$. Finally, there exist a subsequence of $\{u_n\}_n$, still indexed by $n$, and a function $u \in W^1_0 L_M(\Omega, \rho)$ such that
\[
\begin{cases}
  u_n \rightharpoonup u \quad \text{weakly in } W^1_0 L_M(\Omega, \rho) \text{ for } \sigma(\Pi L_M, \Pi E_M) \\
  u_n \rightarrow u \quad \text{strongly in } E_M(\Omega, \rho) \text{ and a.e. in } \Omega.
\end{cases} \tag{43}
\]

### 3.3.3. An intermediate Inequality

In this step, we shall prove that for $\phi \in W^1_0 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, we have
\[
\int_{\Omega} a(x, u_n, \nabla \phi) \nabla T_k(u_n - \phi) \, dx \leq \int_{\Omega} f_n T_k(u_n - \phi) \, dx + \int_{\Omega} F_n \nabla T_k(u_n - \phi) \, dx. \tag{44}
\]
We choose now $T_k(u_n - \phi)$ as test function in (35), with $\phi$ in $W^1_0 L^1 M(\Omega, \rho) \cap L^\infty(\Omega)$, we obtain
\[
\int_{\Omega} a(x,u_n,\nabla u_n) \nabla T_k(u_n - \phi) \, dx = \int_{\Omega} f_n T_k(u_n - \phi) \, dx + \int_{\Omega} F_n \nabla T_k(u_n - \phi) \, dx. \tag{45}
\]

Adding and subtracting the term $\int_{\Omega} a(x,u_n,\nabla \phi) \nabla T_k(u_n - \phi) \, dx$ i.e.,
\[
\int_{\Omega} a(x,u_n,\nabla u_n) \nabla T_k(u_n - \phi) + \int_{\Omega} a(x,u_n,\nabla \phi) \nabla T_k(u_n - \phi) \, dx \tag{46}
\]
\[- \int_{\Omega} a(x,u_n,\nabla \phi) \nabla T_k(u_n - \phi) \, dx = \int_{\Omega} f_n T_k(u_n - \phi) \, dx + \int_{\Omega} F_n \nabla T_k(u_n - \phi) \, dx.
\]

Thanks to assumption (3) and the definition of truncation function, we have
\[
\int_{\Omega} (a(x,u_n,\nabla u_n) - a(x,u_n,\nabla \phi)) \nabla T_k(u_n - \phi) \, dx \geq 0. \tag{47}
\]

Combining (46) and (47), we obtain (44).

### 3.3.4. Passing to the limit

We shall prove that for $\phi \in W^1_0 L^1 M(\Omega, \rho) \cap L^\infty(\Omega)$, we have
\[
\int_{\Omega} a(x,u,\nabla \phi) \nabla T_k(u - \phi) \, dx \leq \int_{\Omega} f T_k(u - \phi) \, dx + \int_{\Omega} F \nabla T_k(u - \phi) \, dx.
\]

Firstly, we claim that
\[
\int_{\Omega} a(x,u_n,\nabla \phi) \nabla T_k(u_n - \phi) \, dx \to \int_{\Omega} a(x,u,\nabla \phi) \nabla T_k(u - \phi) \, dx \text{ as } n \to +\infty.
\]

Since $T_M(u_n) \to T_M(u)$ weakly in $W^1_0 L^1 M(\Omega, \rho)$, with $M = k + \|\phi\|_\infty$, then
\[
T_k(u_n - \phi) \to T_k(u - \phi) \text{ in } W^1_0 L^1 M(\Omega, \rho), \tag{48}
\]

which gives
\[
\frac{\partial T_k}{\partial x_i} (u_n - \phi) \to \frac{\partial T_k}{\partial x_i} (u - \phi) \text{ weakly in } L^1(\Omega, \rho) \quad \forall i = 1, \ldots, N. \tag{49}
\]

Show that
\[
a(x,T_M(u_n),\nabla \phi) \to a(x,T_M(u),\nabla \phi) \text{ strongly in } (L^1(\Omega))^N.
\]
Thanks to assumption (2), we obtain

$$|a_i(x,T_M(u_n),\nabla \phi)| \leq |\phi_i(x)| + K_i P^{-1}(\rho^{-1}(x)M(c_2|T_M(u_n)|)) + K_i M^{-1}M(c_1|\nabla \phi|),$$

with $\beta$ and $\mu$ are positive constants. Since $T_M(u_n) \to T_M(u)$ weakly in $W^1_0 L_M(\Omega, \rho)$ and $W^1_0 L_M(\Omega, \rho)$, then $T_M(u_n) \to T_M(u)$ strongly in $L_M(\Omega, \rho)$ and a.e. in $\Omega$, hence

$$|a(x,T_M(u_n),\nabla \phi)| \to |a(x,T_M(u),\nabla \phi)| \text{ a.e. in } \Omega.$$

and

$$|\phi_i(x)| + K_i P^{-1}(\rho^{-1}(x)M(c_2|T_M(u)|)) + K_i M^{-1}M(c_1|\nabla \phi|) \to$$

$$|\phi_i(x)| + K_i P^{-1}(\rho^{-1}(x)M(c_2|T_M(u)|)) + K_i M^{-1}M(c_1|\nabla \phi|),$$

a.e. in $\Omega$. Then, By Vitali’s theorem, we deduce that

$$a(x,T_M(u_n),\nabla \phi) \to a(x,T_M(u),\nabla \phi) \text{ strongly in } (L_M(\Omega, \rho))^N, \text{ as } n \to \infty. \quad (50)$$

Combining (49) and (50), we obtain

$$\int_{\Omega} a(x,u_n,\nabla \phi)\nabla T_k(u_n - \phi) \, dx \to \int_{\Omega} a(x,u,\nabla \phi)\nabla T_k(u - \phi) \, dx \text{ as } n \to +\infty. \quad (51)$$

Secondly, we show that

$$\int_{\Omega} f_n T_k(u_n - \phi) \, dx \to \int_{\Omega} f T_k(u - \phi) \, dx. \quad (52)$$

and

$$\int_{\Omega} F_n \nabla T_k(u_n - \phi) \, dx \to \int_{\Omega} F \nabla T_k(u - \phi) \, dx \quad (53)$$

We have $f_n T_k(u_n - \phi) \to f T_k(u - \phi)$ a.e. in $\Omega$ and $|f T_k(u_n - \phi)| \leq k|f|$, and $F_n \nabla T_k(u_n - \phi) \to F \nabla T_k(u - \phi)$ a.e. in $\Omega$, and $|F \nabla T_k(u_n - \phi)| \leq k|F|$ then by using Vitali’s theorem, we obtain (52) and (53). Thanks to (51), (52) and (53) allow to pass to the limit in the inequality (44), so that $\forall \phi \in W^1_0 L_M(\Omega, \rho) \cap L^\infty(\Omega)$, we deduce

$$\int_{\Omega} a(x,u,\nabla \phi)\nabla T_k(u - \phi) \, dx \leq \int_{\Omega} f T_k(u - \phi) \, dx + \int_{\Omega} F \nabla T_k(u - \phi) \, dx.$$

In view of main Lemma, we can deduce that $u$ is an entropy solution of the problem (1). This completes the proof of Theorem 3.2.
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