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EXISTENCE OF ENTROPY SOLUTIONS FOR NONLINEAR ELLIPTIC PROBLEM HAVING LARGE MONOTONOCITY IN WEIGHTED ORLICZ-SOBOLEV SPACES

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We prove an existence result of entropy solution for a class of nonlinear elliptic problems of Leray-Lions type with large monotonicity condition in the framework of weighted Orlicz-Sobolev spaces and with right hand side $f \in L^1(\Omega)$.

1. Introduction

This paper deals with existence of solutions to the following nonlinear Dirichlet problem

$$\begin{cases} A(u) = f(x) - \operatorname{div} F(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^1(\Omega), F \in (C^0(\mathbb{R}))^N$ and $A(u) = -\operatorname{div}(\rho(x)a(x,u,\nabla u))$, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$.

Note that no growth hypothesis is assumed on the function F, which implies that the term $\operatorname{div} F(u)$ may be meaningless, even as a distribution. $a(x,u,\nabla u) = (a_i(x,u,\nabla u))_{1\leq i\leq N}$, $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory functions (that is

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measurable with respect to x in Ω for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, and continuous with respect to $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ for almost every $x \in \Omega$) such that for all ξ, ξ' in $\mathbb{R}^N$, $(x,s) \in \Omega \times \mathbb{R}$

$$|a_i(x,s,\xi)| \leq |\phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|s|)) + K_i(\overline{M}^{-1}M(c_1|\xi|)), \quad (2)$$

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') \geq 0, \quad (3)$$

$$a(x,s,\xi)\xi \geq M(\lambda_1|\xi|), \quad (4)$$

where c_1, c_2, $\lambda_1, K_i > 0$. Let M, P are two N-functions such that $P << M$. Moreover $\overline{M}, \overline{P}$ are the complementary functions of M and P respectively, ρ is a weight function on Ω (that is, measurable and positive a.e. on Ω) and ϕ_0, $\phi_i \in E_{\overline{M}}(\Omega,\rho)$, ($E_{\overline{M}}(\Omega,\rho)$ is introduced later.)

$$f \in L^1(\Omega), \quad (5)$$

and $F = (F_1,...,F_N)$ satisfies

$$F \in (C^0(\mathbb{R}))^N. \quad (6)$$

The notion of entropy solution, used in [17], allows us to give a meaning to a possible solution of (1)

In fact Boccardo proved in [17], for $\rho(x) = 1$ and p such that $2 - 1/N < p < N$, the existence and regularity of an entropy solution u of problem (1) . For the case $\rho(x) = 1$, $\phi = 0$ and f is a bounded measure, Benilan et al. proved in [7] the existence and uniqueness of entropy solutions,the same problem is treated using the notion of entropy solution introduced in [38] where $\rho(x) = 1$,$f \in L^1(\Omega)$, and $F \in L^{p'}(\Omega)^N$. We mention as a parallel development, the work of Lions and Murat [40] who consider similar problems in the context of the renormalized solutions introduced by Diperna and Lions [24] for the study of the Boltzmann equations. They can prove existence and uniqueness of renormalized solution,and afterwards by Dal Maso, Murat, Orsina and Prignet in [22] with general measure data.

In the general framework of weighted Orlicz-Sobolev spaces , the authors in [9] deal with the equation

$$-\operatorname{div}(\rho(x)a(x,u,\nabla u)) - \operatorname{div}\Phi(u) + g(x,u,\nabla u) = f$$

with $\Phi = 0$ and g is a nonlinear with the following (natural) growth condition:

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + M(|\xi|)\rho(x)),$$

which satisfies the sign condition

$$g(x,s,\xi)s \geq 0,$$

and the right-hand side $f \in W^{-1}E_M(\Omega,\rho)$,in the case where $f \in L^1(\Omega)$, $\phi = 0$ and $g = 0$ existence theorem have been proved by the authors in ([28]).

Several researches deals with the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts (see [4–6, 11, 12, 14–16, 18, 23, 26, 28–32, 46–49] for more details).

It is well known that the setting of weighted Orlicz–Sobolev spaces include many spaces as special spaces, such as Lebesgue spaces, weighted Lebesgue spaces and Orlicz spaces; see [44]. These spaces have many applications in various fields such as PDE, electrorheological fluids, and image restoration; see [21, 25, 39].

The feature of this paper, is to treat a class of problems for which the classical monotone operator methods (developed by Minty [43], Browder [20], Brézis [19] and Lions [41] in $W_0^{1,p}(\Omega)$ case) do not apply. The reason for this, is that $a(x,u,\nabla u)$ does not need to satisfy the strict monotonicity condition that is,

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') > 0, \text{ for all } \xi,\xi' \in \mathbb{R}^N, (\xi \neq \xi')$$

of a typical Leray–Lions operator but only a large monotonicity that is

$$\Big(a(x,s,\xi) - a(x,s,\xi')\Big)(\xi - \xi') \geq 0, \text{ for all } \xi,\xi' \in \mathbb{R}^N.$$

The aim of this paper is to prove the existence of solutions for (1) under the weaker assumption large monotonicity condition, without using the almost everywhere convergence of the gradients of the approximate equations, since this is impossible to prove in our setting. The main tools of our proof are a version of Minty's Lemma (here we use an idea of G. J. Minty [42]. We note that the techniques we used in the proof are different from those used in [8, 10].

The paper is organized as follows: We introduce some basic definitions and properties in the setting of weighted Orlicz–Sobolev spaces as well as an abstract theorem and we prepare some auxiliary results to prove our theorem, in the next section 2. In the final section 3, we state the main result and proofs.

2. Preliminary

This section present some definitions and well-known about N-functions, weighted Orlicz-Sobolev spaces (standard references see [1] and [27].

2.1. N-function

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with

$$M(t) > 0 \text{ for } t > 0, \quad \frac{M(t)}{t} \to 0 \text{ as } t \to 0 \text{ and } \frac{M(t)}{t} \to \infty \text{ as } t \to \infty.$$

Equivalently M admits the representation :

$$M(t) = \int_0^t m(\tau)d\tau,$$

where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $m(t) \to \infty$ as $t \to \infty$. The N-function $\overline{M}$ conjugate to M defined by

$$\overline{M}(t) = \int_0^t \overline{m}(\tau)d\tau,$$

where $\overline{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given bay $\overline{m}(t) = \sup\{s : m(s) \leq t\}$. It is well known that we can assume that m and $\overline{m}$ are continuous and strictly increasing. We will extend the N-functions into even functions on all $\mathbb{R}^+$.

Clearly $\overline{\overline{M}} = M$ and has Young's inequality

$$st \leq M(t) + \overline{M}(s) \text{ for all } s,t \geq 0.$$

The N-function M is said to satisfy the Δ_2-condition every-where (resp. infinity) if there exist $k > 0$ (resp. $t_0 > 0$) such that, $M(2t) \leq kM(t)$ for all $t \geq 0$ (resp. $t \geq t_0$).

Let P and Q be two N-functions, the notation $P << Q$ means that P grows essentially less rapidly than Q, that is to say for all $\varepsilon > 0$, $\frac{P(t)}{Q(\varepsilon t)} \to 0$ as $t \to +\infty$. That is the case if and only if $\frac{Q^{-1}(t)}{P^{-1}(t)} \to 0$ as $t \to \infty$.

2.2. Orlicz-Sobolev space

Let Ω be a open subset of $\mathbb{R}^N$ and M be an N-function.

The Orlicz classe $K_M(\Omega)$ (resp. the Orlicz spaces $L_M(\Omega)$) is the set of all (equivalence classes modulo equality a.e. in Ω of) real-valued measurable functions u defined in Ω and satisfying

$$\int_\Omega M(|u(x)|)dx < \infty \Big(\text{resp. } \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx < \infty \text{ for some } \lambda > 0\Big).$$

$L_M(\Omega)$ it is a Banach space under the norm:

$$\|u\|_{M,\Omega} = \inf\Big\{\lambda > 0 : \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)dx \leq 1\Big\}. \tag{7}$$

We denote by $B(\Omega)$ the set of bounded measurable functions with compact support in $\overline{\Omega}$ and by $E_M(\Omega)$ the closure of $B(\Omega)$ in $L_M(\Omega)$ (we have usual $E_M(\Omega) \subset K_M(\Omega) \subset L_M(\Omega)$).

The equality $L_M(\Omega) = E_M(\Omega)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or not .

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$ where $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$ and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large, according to whether Ω be infinite measure or note.

We return now to the Orlicz-Sobolev spaces $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all function u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. in $E_M(\Omega)$). It's Banach space under the norm :

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_{M,\Omega}. \tag{8}$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of $\prod L_M$ we have the weak topology $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ (resp. $W_0^1L_M(\Omega)$) is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega)$ (resp. $W^1L_M(\Omega)$) for the norm (8) (resp. for the topology $\sigma(\prod L_M, \prod E_{\overline{M}})$).

Definition 2.1. The sequence u_n converges to u in $L_M(\Omega)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega)$) if $\int_\Omega M\Big(\frac{|u_n - u|}{\lambda}\Big)dx \to 0$ as $n \to \infty$ for some $\lambda > 0$.

2.3. Weighted Orlicz–Sobolev space

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, M be an N-function and $\rho(x)$ be a weight function on Ω, i.e. measurable positive a.e. on Ω such that :

$$\rho \in L^1(\Omega). \tag{9}$$

The weighted Orlicz classe $K_M(\Omega,\rho)$ (resp. the weighted Orlicz space $L_M(\Omega,\rho)$) is the set of all (equivalence classes modulo equality a.e. in Ω) of real-valued measurable functions u defined in Ω and satisfying

$$m_\rho(u,M) = \int_\Omega M(|u(x)|)\rho(x)dx < \infty$$

$$\Big(\text{resp. } m_\rho\Big(\frac{u}{\lambda}, M\Big) = \int_\Omega M\Big(\frac{|u(x)|}{\lambda}\Big)\rho(x)dx < \infty \text{ for some } \lambda > 0\Big).$$

$L_M(\Omega,\rho)$ is a Banach space under the norm :

$$\|u\|_{M,\rho} = \inf\Big\{\lambda > 0; m\Big(\frac{u}{\lambda}, M\Big) \leq 1\Big\}. \tag{10}$$

We denote by $B(\Omega)$ the set of bounded measurable functions with compact support in $\overline{\Omega}$ and by $E_M(\Omega,\rho)$ the closure of $B(\Omega)$ in $L_M(\Omega,\rho)$ (we have usual $E_M(\Omega,\rho) \subset K_M(\Omega,\rho) \subset L_M(\Omega,\rho)$).

The equality $L_M(\Omega,\rho) = E_M(\Omega,\rho)$ hold if and only if M satisfies the Δ_2-condition, for all t or for t large according to whether Ω has a infinite measure or note.

The dual of $E_M(\Omega,\rho)$ can be identified with $L_{\overline{M}}(\Omega,\rho)$ by means of the pairing $\int_\Omega u(x)v(x)\rho(x)dx$ where $u \in L_M(\Omega,\rho)$ and $v \in L_{\overline{M}}(\Omega,\rho)$ and the dual norm on $L_{\overline{M}}(\Omega,\rho)$ is equivalent to $\|.\|_{\overline{M},\Omega}$ (see [9]).

The space $L_M(\Omega,\rho)$ is reflexive if and only if M an $\overline{M}$ satisfy the Δ_2-condition for all t or for t large according to whether Ω be infinite measure or note.

We return now to the weighted Orlicz-Sobolev spaces $W^1L_M(\Omega,\rho)$ (resp. $W^1E_M(\Omega,\rho)$) is the space of all function u such that $u \in L_M(\Omega)$ (resp. $u \in E_M(\Omega)$) and its distributional derivatives up to order 1 lie in $L_M(\Omega,\rho)$ (resp. $E_M(\Omega,\rho)$). It's Banach space under the norm

$$\|u\|_{1,M,\rho} = \|u\|_M + \|\nabla u\|_{M,\rho}, \tag{11}$$

(where $\|u\|_M = \|u\|_{M,\Omega}$). Thus $W^1L_M(\Omega,\rho)$ and $W^1E_M(\Omega,\rho)$) can be identified with subspaces of $\prod L_{M,\rho} = L_M \times \prod L_M(\Omega,\rho)$ we have the weak topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$ and $\sigma(\prod L_{M,\rho}, \prod L_{\overline{M},\rho})$. The space $W_0^1E_M(\Omega,\rho)$ (resp. $W_0^1L_M(\Omega,\rho)$) is defined by the closure of $D(\Omega)$ in $W^1E_M(\Omega,\rho)$) (resp. $W^1L_M(\Omega,\rho)$) for the norm (11) (resp. for the topology $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$).

Definition 2.2. The sequence u_n converges to u in $L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $L_M(\Omega,\rho)$) if $\int_\Omega M\Big(\frac{|u_n-u|}{\lambda}\Big)\rho(x)dx \to 0$ as $n \to \infty$ for some $\lambda > 0$.

Definition 2.3. The sequence u_n converges to u in $W^1L_M(\Omega,\rho)$ for the modular convergence (denoted by $u_n \to u$ (mod) $W^1L_M(\Omega,\rho)$) if for some $\lambda > 0$ $\int_\Omega M\Big(\frac{|u_n-u|}{\lambda}\Big)dx \to 0$ as $n \to \infty$ and $\int_\Omega M\Big(\frac{|D^\alpha(u_n-u)|}{\lambda}\Big)\rho(x)dx \to 0$ as $n \to \infty$ for $|\alpha| = 1$.

Lemma 2.4. *[3] Let M be an N-function. If $u_n \in L_M(\Omega)$ converges a.e. to u and u_n bounded in $L_M(\Omega)$, then $u \in L_M(\Omega)$ and $u_n \to u$ for the topology $\sigma(L_M(\Omega), E_{\overline{M}}(\Omega))$.*

Lemma 2.5. *[3] If the sequence $u_n \in L_M(\Omega,\rho)$ converges to u a.e. is bounded in $L_M(\Omega,\rho)$, then $u \in L_M(\Omega,\rho)$ and $u_n \to u$ for the topology $\sigma(L_M(\Omega,\rho), E_{\overline{M}}(\Omega,\rho))$.*

2.4. Compactness results

Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary, ρ the weight function and M an N-function. Let the following integrability assumptions :
There exists a real $s > 0$ such that :

$$(M(t))^{\frac{s}{s+1}} \text{ is } N\text{-function and an } \rho^{-s} \in L^1(\Omega), \tag{12}$$

$$\int_1^\infty \frac{t}{M(t)^{1+\frac{s}{N(s+1)}}} dM(t) = \infty, \tag{13}$$

$$\lim_{t\to\infty} \frac{1}{M^{-1}(t)} \int_0^{t^{\frac{s+1}{s}}} \frac{M^{-1}(u)}{u^{1+\frac{s}{N(s+1)}}} du = 0. \tag{14}$$

Remark 2.6. In the particular case where $M(t) = \frac{|t|^p}{p}$ $(1 < p < \infty)$, the first part of (12) is satisfied if $s > \frac{1}{p-1}$.

Theorem 2.7 (See[2] Theorem 9-5). *Let Ω an open bounded locally- lipschitzian boundary in $\mathbb{R}^N$ and M an N-function. Suppose that assumptions* (12)–(14) *are satisfied. So we have the following compact injection :*

$$W^1L_M(\Omega,\rho) \hookrightarrow\hookrightarrow E_M(\Omega).$$

Theorem 2.8. *(Weighted Poincaré inequality) Let Ω be a bounded open subset of $\mathbb{R}^N$ with locally Lipschitzian boundary, ρ a weight function, and M an N -function. If $u \in W_0^1L_M(\Omega)$, then*

$$\|u\|_M \le c\|\nabla u\|_{M,\rho}$$

where c is a positive constant, which implies that $\|\nabla u\|_{M,\rho}$ and $\|u\|_{1,M}$ are equivalent norms on $W_0^1L_{M,\rho}$.

Proof. Under the assumptions (12)-(14), the Sobolev conjugate N -function M_s^* of M_s is well defined by

$$M_s^{*-1} = \int_0^s \frac{M^{-1}(t)}{t^{1+\frac{1}{N}}} dt$$

and we have $W_0^1L_{M_s} \subset L_{M_s^*}$. And since $M \ll M_s^*$, we have $L_{M_s^*} \subset L_M$. Hence

$$\|u\|_M \le c_1\|u\|_{M_s^*} \le c_2\|u\|_{1,M_s},$$

where c_1 and c_2 are two positive constants. Then, by using the Poincaré inequality in the non-weighted Orlicz-Sobolev space, there exists a positive constant c' such that

$$\|u\|_{1,M_s} \leq c'\|\nabla u\|_{M_s}.$$

We will show that

$$\|\nabla u\|_{M_s} \leq c\|\nabla u\|_{M,\rho}.$$

For that we have

$$\begin{aligned}
\|v\|_{M_s} \leq \int_\Omega M_s(v(x))dx + 1 &= \int_\Omega M_s(v(x))\frac{1}{\rho(x)}\rho(x)dx + 1 \\
&\leq \int_\Omega S(M_s(v(x)))\,\rho(x)dx + \int_\Omega \frac{1}{\rho(x)}\rho(x)dx + 2 \\
&= \int_\Omega M(v(x))\rho(x)dx + \int_\Omega \rho^{-s}(x)dx + 1,
\end{aligned}$$

which implies that

$$\|v\|_{M_s} \leq c\|v\|_{M,\rho},$$

for some positive constant c. In fact, if this is not true, then there exists a sequence v_n such that $\|v_n\|_{M_s} \to \infty$ and for n large, $\|v_n\|_{M,\rho} \leq 1$. Hence, for n sufficiently large we get

$$\int_\Omega M\left(v_n(x)\right)\rho(x)dx \leq \|v_n\|_{M,\rho} \leq 1.$$

Then

$$\begin{aligned}
\|v_n\|_{M_s} &\leq \int_\Omega M(v(x))\rho(x)dx + \int_\Omega \rho^{-s}(x)dx + 1 \\
&\leq \|v_n\|_{M,\rho} + \int_\Omega \rho^{-s}(x)dx + 1,
\end{aligned}$$

which is a contradiction, since the left hand-side tends to infinity while the right hand-side is bounded. Finally, by taking $v = \nabla u$, we conclude the result.

□

We will also use the following technical lemmas.

2.5. Some technical lemmas

Lemma 2.9. *[35]Let $f_n, f \in L^1(\Omega)$ such that $f_n \geq 0$ a.e. in Ω, $f_n \to f$ a.e. in Ω et $\int_\Omega f_n(x)dx \to \int_\Omega f(x)dx$. Then $f_n \to f$ stongly in $L^1(\Omega)$.*

Lemma 2.10. *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^1 L_M(\Omega, \rho)$. Then $F(u) \in W_0^1 L_M(\Omega, \rho)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in \{x \in \Omega; u(x) \notin D\}, \\ 0 & a.e. \quad in \{x \in \Omega; u(x) \in D\}. \end{cases}$$

Proof. We suppose for the moment that F is also C^1, there exist a sequence $u_n \in D(\Omega)$ such that $u_n \to u$ (mod) $W^1 L_M(\Omega, \rho)$). Passing to subsequence, we can assume that

$$D^\alpha u_n \to D^\alpha u \quad \forall |\alpha| \leq 1 \text{ a.e. in } \Omega.$$

From the relation $|F(s)| \leq k|s|$, where k denote the Lipschitz constant for F, and
$\frac{\partial}{\partial x_i} F(u_n) = F'(u_n)\frac{\partial u_n}{\partial x_i}$, we deduce that $F(u_n)$ remains bounded in $W_0^1 L_M(\Omega, \rho)$. Thus going to a further subsequence, we obtain

$$F(u_n) \to w \in W_0^1 L_M(\Omega, \rho) \text{ for } \sigma(\prod L_M, \prod E_{\overline{M}}),$$

and also by a local application of the compact imbedding theorem, $F(u_n) \to w$ a.e. in Ω. Consequently $w = F(u)$, and $F(u) \in W_0^1 L_M(\Omega, \rho)$. Finally, by the usual chain rule for weak derivatives,

$$\frac{\partial}{\partial x_i} F(u) = F'(u) \frac{\partial u}{\partial x_i} \text{ a.e. in } \Omega. \tag{15}$$

For the general case. Taking convolution with the mollifiers, we get a sequence $F_n \in C^\infty(\mathbb{R})$ such that $F_n \to F$ uniformly on each compact, $F_n(0) = 0$ and $\left|F_n'\right| \leq k$. For each n, $F_n \in W_0^1 L_M(\Omega, \rho)$, and we have (15) with F replaced by F_n. Finally (15) follows from the generalized chain rule for weak derivatives. □

The following lemmas follow from the previous lemma.

Lemma 2.11. *Let $u, v \in W_0^1 L_M(\Omega, \rho)$ and let $w = min(u, v)$. Then $w \in W_0^1 L_M(\Omega, \rho)$ and*

$$\frac{\partial w}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} \; a.e. \; in \; \{x \in \Omega; u(x) \leq v(x)\}, \\ \frac{\partial v}{\partial x_i} \; a.e. \; in \; \{x \in \Omega; u(x) > v(x)\}. \end{cases}$$

Proof. Note that $\min(u, v) = u - (u - v)^+$ and apply Lemma 2.10 with $F(s) = s^+$. □

We introduce the truncate operator. For a given constant $k > 0$, we define the function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \begin{cases} s \text{ if } |s| \le k, \\ k\frac{s}{|s|} \text{ if } |s| > k. \end{cases}$$

Lemma 2.12. *Assume that* (12)–(14) *holds. Let $u \in W_0^1 L_M(\Omega, \rho)$, and let $T_k(u), k \in \mathbb{R}^+$, be the usual truncation then $T_k(u) \in W_0^1 L_M(\Omega, \rho)$. Moreover we have*

$$T_k(u) \to u \text{ strongly in } W_0^1 L_M(\Omega, \rho).$$

Lemma 2.13. *Let u_n be a sequence of $W_0^1 L_M(\Omega, \rho)$ such that $u_n \rightharpoonup u$ weakly in $W_0^1 L_M(\Omega, \rho)$ for the topology $\sigma(L_M(\Omega, \rho), E_{\overline{M}}(\Omega, \rho))$. Then $T_k(u_n) \rightharpoonup T_k(u)$.*

Proof. Since $u_n \rightharpoonup u$ and $W_0^1 L_M(\Omega, \rho) \hookrightarrow\hookrightarrow E_M(\Omega)$, we have $u_n \to u$ strongly in $E_M(\Omega)$ and a.e. in Ω, then $T_k(u_n) \to T_k(u)$ a.e. in Ω. On the other hand, for some $\lambda > 0$,

$$\int_\Omega M\Big(\frac{|T_k(u_n)|}{\lambda}\Big)dx \le \int_\Omega M\Big(\frac{|u_n|}{\lambda}\Big)dx$$

and

$$\int_\Omega M\Big(\frac{|\nabla T_k(u_n)|}{\lambda}\Big)\rho(x)dx = \int_\Omega M\Big(\frac{|T_k^{'}(u_n)|\,|\nabla u_n|}{\lambda}\Big)\rho(x)dx \le \int_\Omega M\Big(\frac{|\nabla u_n|}{\lambda}\Big)\rho(x)dx$$

imply that

$$\|T_k(u_n)\|_{1,M,\rho} \le \|u_n\|_{1,M,\rho}\,.$$

Then $(T_k(u))_n$ is bounded in $W_0^1 L_M(\Omega, \rho)$ hence by Lemma 2.5, we have $T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^1 L_M(\Omega, \rho)$ for $\sigma(L_M(\Omega, \rho), E_{\overline{M}}(\Omega, \rho))$. □

Lemma 2.14. *If the sequence $u_n \in E_M(\Omega, \rho)$ converges a.e. in Ω with $\rho \in L^1(\Omega)$, then it converges in norm in $E_M(\Omega, \rho)$ if and only if the norms are uniformly absolutely continuous, i.e. for each $\varepsilon > 0$ there exist $\delta > 0$ such that $\|u_n \chi_E\|_{M,\rho} < \varepsilon$, for all n and $E \subset \Omega$ with $|E| < \delta$.*

Proof. by the same argument introduced in the proof of Lemma 11.2 in [36] we find $E_{n,m} = \{x \in \Omega : |u_n(x) - u_n(x)| > \alpha\}$, where $\alpha = M^{-1}\Big(\frac{\varepsilon}{3\|\rho\|_1}\Big)$ and with $\delta > 0$ such that

$$\|u_n \chi_E\|_{M,\rho} < \frac{\varepsilon}{3}.$$

We denote by $H(E_M(\Omega), r)$ the set of functions $u \in L_M(\Omega)$ whose distance to $E_M(\Omega)$ (with respect to the Orlicz norm) is strictly less than r and by $B_{L_M}(\Omega)(0, r)$ the ball in $L_M(\Omega)$ (with respect to the Orlicz norm) of radius r and center 0. □

Lemma 2.15. *Let Ω be bounded subset of $\mathbb{R}^N$ with finite measure.let M, R and Q be N-functions such that $Q << R$, and let f be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:*

$$|f(x,s)| \leq b(x) + k_1 R^{-1}\left(\rho^{-1}(x) Q(k_2|s|)\right), \tag{16}$$

where $0 \leq b(x) \in E_M(\Omega,\rho), \rho \in L^1(\Omega)$ and $k_1, k_2 \in \mathbb{R}^+$. Then the Nemytskii operator

$$N_f(u)(x) = f(x,u(x))$$

satisfies :

1. *Sends $\left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p$ into $L_R(\Omega,\rho)$ and is continuous from $\left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p$ to the norm topology of $(L_Q(\Omega))^p$ into $L_R(\Omega,\rho)$ to the modular convergence;*

2. *It's uniformly bounded on $\left(B_{L_Q(\Omega)}(0,\frac{1}{k_2})\right)^p$;*

3. *If $b(x) \in E_{R_1}(\Omega,\rho)$ with $R_1 << R$, then N_f is continuous to the norm topology of $E_{R_1}(\Omega,\rho)$.*

Proof. (1) Let $u = (u_1, u_2, ..., u_p) \in \left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p$. Since $d(u_i, E_Q(\Omega)) < \frac{1}{k_2}$ $(1 \leq i \leq p)$, we have $\int_\Omega Q(k_2|u|)dx \leq 1$, (see Theorem 10.1 [36]). Let $\lambda \geq 2k_1$ such that $\frac{2b(x)}{\lambda} \in K_R(\Omega,\rho)$. By the growth condition (16) and the convexity of R, we get

$$\int_\Omega R\left(\frac{|f(x,s)|}{\lambda}\right)\rho(x)dx \leq \frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx + \frac{1}{2}\int_\Omega R\left(k_2|u(x)|\right)dx < \infty.$$

On the other hand, suppose that

$$u_n \to u \in \left(H(E_Q(\Omega),\frac{1}{k_2})\right)^p,$$

and let $\alpha > 0$ such that $d(k_2|u|, E_Q(\Omega)) < \alpha < 1$ and $d(k_2|u|, E_Q(\Omega)) < 1-\alpha < 1$.

We have $\frac{k_2}{\alpha}|u| \in K_Q(\Omega)$ and $\frac{k_2}{1-\alpha}|u| \in K_Q(\Omega)$ (see Theorem 10.1 [36]) and for $\lambda > 4k_1$ such that $\frac{4b(x)}{\lambda} \in K_R(\Omega,\rho)$, we get

$$\begin{aligned}
&\int_\Omega R\left(\frac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx \\
&\qquad \leq \int_\Omega R\left(\frac{2b(x)+k_1R^{-1}\left(\rho^{-1}(x)Q(k_2|u_n|)\right)+k_1R^{-1}\left(\rho^{-1}(x)Q(k_2|u|)\right)}{\lambda}\right)\rho(x)dx \\
&\qquad \leq \frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)dx+\frac{1-\alpha}{4}\int_\Omega Q\left(\frac{k_2}{1-\alpha}|u_n-u|\right)dx \\
&\qquad\quad +\frac{\alpha}{4}\int_\Omega Q\left(\frac{k_2}{\alpha}|u|\right)dx+\frac{1}{4}\int_\Omega Q(k_2|u|)\,dx.
\end{aligned}$$

Since $Q(k_2|u|) \leq Q\left(\frac{k_2}{\alpha}|u|\right)$, the last inequality becomes

$$\begin{aligned}
&\int_\Omega R\left(\frac{|f(x,u_n)-f(x,u)|}{\lambda}\right)\rho(x)dx \\
&\qquad \leq \frac{1}{2}\int_\Omega R\left(\frac{4b(x)}{\lambda}\right)\rho(x)dx+\int_\Omega Q\left(\frac{k_2}{1-\alpha}|u_n-u|\right)dx+\int_\Omega Q(k_2|u|)\,dx,
\end{aligned}$$

which implies by using the Vitali's theorem

$$f(x,u_n) \to f(x,u) \text{ (mod) in } L_R(\Omega,\rho),$$

for a subsequence denoted again u_n (which holds for the whole sequence).

(2) Let now $u \in \left(B_{L_Q(\Omega)}(0,\frac{1}{k_2})\right)^p$ and let $\lambda \geq 2k$ such that $\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx \leq 1$. By the growth condition (16) and the convexity of R, we get

$$\int_\Omega R\left(\frac{|f(x,u)|}{\lambda}\right)\rho(x)dx \leq \frac{1}{2}\int_\Omega R\left(\frac{2b(x)}{\lambda}\right)\rho(x)dx+\frac{1}{2}\int_\Omega Q(k_2|u(x)|)\,dx \leq 1,$$

which implies (2).

(3) Suppose that $b(x) \in E_{R_1}(\Omega,\rho)$ with $R_1 \prec\prec R$. As in (1), since $L_R(\Omega,\rho) \subset E_{R_1}(\Omega,\rho))$, we can show that $f(x,u) \in E_{R_1}(\Omega,\rho)$ for all $u \in \left(E_Q(\Omega),\frac{1}{k_2})\right)^p$. Suppose now that

$$u_n \to u \in \left(E_Q(\Omega),\frac{1}{k_2})\right)^p \text{ in } (L_Q(\Omega))^p,$$

We shall show that

$$f(x,u_n) \to f(x,u)\text{(mod) in } E_{R_1}(\Omega,\rho).$$

Fix $\varepsilon > 0$, we have as above

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\varepsilon}\right)\rho(x)dx \le \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\varepsilon}\right)\rho(x)dx$$
$$+\frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\varepsilon}R^{-1}\left(\rho^{-1}(x)Q(k_2|u_n|)\right)\right)\rho(x)dx+\frac{1}{4}\int_\Omega R_1\left(\frac{4k_1}{\varepsilon}R^{-1}\left(\rho^{-1}(x)Q(k_2|u|)\right)\right)\rho(x)dx$$

Since $R_1 \prec\prec R$, there exists K' such that $R_1(\frac{4k_1}{\varepsilon}t) \le R(t)+K'$ for all $t \ge 0$. Then, the last inequality can be written as the form

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\varepsilon}\right)\rho(x)dx \le \frac{1}{4}\int_\Omega R_1\left(\frac{4b(x)}{\varepsilon}\right)\rho(x)dx+\frac{1}{4}\int_\Omega Q(k_2|u_n|)dx$$
$$+\frac{1}{4}\int_\Omega Q(k_2|u|)dx+\frac{K'}{2}\int_\Omega \rho(x)dx.$$

As in (1) by using the Vitali's theorem, we get

$$\int_\Omega R_1\left(\frac{|f(x,u_n)-f(x,u)|}{\varepsilon}\right)\rho(x)dx \to 0 \text{ as } n\to\infty,$$

for a subsequence (which holds for the whole sequence). Since ε is arbitrary, we conclude (3). □

Definition 2.16. [1] **(Segment property) :** A domain Ω is said to satisfy the segment property, if there exist a finite open covering $\{\theta\}_{i=1}^k$ of $\bar{\Omega}$ and a corresponding nonzero vectors $z_i \in R^N$ such that $\left(\bar{\Omega}\cap\theta_i\right)+tz_i \subset \Omega$ for all $t \in (0,1)$ and $i = 1,\dots,k$.

Lemma 2.17. *[34] Suppose that Ω satisfies the segment property and let $u \in W_0^1L_M(\Omega,\rho)$. Then, there exists a sequence $u_n \in D(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1L_M(\Omega,\rho).$$

Furthermore, if $u \in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$ then $\|u_n\|_\infty \le (N+1)\|u\|_\infty$.

Lemma 2.18. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with the segment property. If $u \in (W_0^1L_{M,\rho}(\Omega))^N$ then $\int_\Omega \operatorname{div}(u)dx = 0$.*

Proof. Fix a vector $u = \left(u^1,\dots,u^N\right) \in \left(W_0^1L_{M,\rho}(\Omega)\right)^N$. since $W_0^1L_{M,\rho}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^1L_{M,\rho}(\Omega)$, then each term u^i can be approximated by a suitable sequence $u_k^i \in D(\Omega)$ such that u_k^i converges to u^i in $W_0^1L_{M,\rho}(\Omega)$. Moreover, due to the fact that $u_k^i \in C_0^\infty(\Omega)$, then the Green formula gives

$$\int_\Omega \frac{\partial u_k^i}{\partial x_i} = \int_{\partial\Omega} u_k^i\vec{n}ds = 0. \tag{17}$$

On the other hand, $\frac{\partial u_k^i}{\partial x_i} \to \frac{\partial u^i}{\partial x_i}$ in $L_{M,\rho}(\Omega)$. Thus $\frac{\partial u_k^i}{\partial x_i} \to \frac{\partial u^i}{\partial x_i}$ in $L^1(\Omega)$, which gives in view

of (17) that

$$\int_\Omega \operatorname{div}(u)dx = 0.$$

$\square$

3. Main results

Let Y be a closed subspace of $W^1L_M(\Omega,\rho)$ for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$ and let

$$Y_0 = Y \cap W^1L_M(\Omega,\rho),$$

such that Y is the closure of Y_0 for $\sigma(\prod L_{M,\rho}, \prod E_{\overline{M},\rho})$. In the next, we consider the complementary system (Y, Y_0, Z, Z_0) generated by Y i.e. Y_0^* can be identified to Z and Z_0^* can be identified to Y by the means $\langle .,. \rangle$. Let the mapping T (associated to the operator A) defined from $D(T) = \{u \in Y, a_0(x,u,\nabla u) \in L_{\overline{M}}(\Omega), a_i(x,u,\nabla u) \in L_{\overline{M}}($ into Z by the formula

$$a(u,v) = \int_\Omega a_0(x,u,\nabla u)v(x)dx + \sum_{1\le i\le N} \int_\Omega a_i(x,u,\nabla u)\frac{\partial v(x)}{\partial x_i}\rho(x)dx \quad \forall v \in Y_0.$$

We consider the complementary system

$$(Y, Y_0, Z, Z_0) = (W_0^1L_M(\Omega,\rho), W_0^1E_M(\Omega,\rho), W^{-1}E_{\overline{M}}(\Omega,\rho), W^{-1}L_{\overline{M}}(\Omega,\rho)).$$

As in [17], we define the entropy solution of our problem.

Definition 3.1. An entropy solution of the problem (1) is a measurable function u such that $T_k(u) \in W_0^1L_M(\Omega,\rho)$ for every $k > 0$ and such that

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\rho(x)dx \le \int_\Omega fT_k(u-\phi)dx + \int_\Omega F(u)\nabla T_k(u-\phi)$$

for every $\phi \in W_0^1E_M(\Omega,\rho) \cap L^\infty(\Omega)$.

Our main results are collected in the following theorem.

Theorem 3.2. *Under the assumptions (2)-(6),*(12)–(14) *and* $\rho(x)$ *be a weight function on* Ω *satisfy* (2.3) *there exist an entropy solution u of the problem* (1).

3.1. Main Lemma

Lemma 3.3. *Let u be a mesurable function such that $T_k(u)$ belongs to $W_0^1L_M(\Omega,\rho)$ for every $k>0$. Then*

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \leq \int_\Omega f\,T_k(u-\phi)dx+\int_\Omega F(u)\nabla T_k(u-\phi) \quad (18)$$

is equivalent to

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\phi)\,dx = \int_\Omega f\,T_k(u-\phi)dx+\int_\Omega F(u)\nabla T_k(u-\phi) \quad (19)$$

for every $\phi\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$, and for every $k>0$.

3.2. Proof of lemma 3.3

In fact (19) implies (18) is easily proved adding and subtracting

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx,$$

and then using assumption (3). Thus, it remains to prove that (18) implies (19). Let h and k be positive real numbers, let $\lambda\in\,]-1,1[$ and $\Psi\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$.

choose, $\phi=T_h(u-\lambda T_k(u-\Psi))\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$ as test function in (18), we have:

$$I_{hk}\leq J_{hk}, \quad (20)$$

with

$$I_{hk}=\int_\Omega a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\;dx,$$

and

$$J_{hk}=\int_\Omega f\,T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx+\int_\Omega F(u)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi))).$$

Put

$$A_{hk}=\{x\in\Omega, |u-T_h(u-\lambda T_k(u-\Psi))|\leq k\},$$

and

$$B_{hk}=\{x\in\Omega, |u-\lambda T_k(u-\Psi)|\leq h\}.$$

Then, we obtain

$$\begin{aligned}I_{hk} &= \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx\\ &+\int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx\\ &+\int_{A_{kh}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.\end{aligned}$$

Since $\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))$ is different to zero only on A_{kh}, we have

$$\int_{A_{kh}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx = 0. \quad (21)$$

Moreover, if $x\in B_{hk}^C$, we have $\nabla T_h(u-\lambda T_k(u-\Psi)) = 0$ and using (4), we deduce that,

$$\begin{aligned}&\int_{A_{kh}\cap B_{hk}^C} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx\\ &\quad=\int_{A_{kh}\cap B_{hk}^C} a(x,u,0)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx = 0. \quad (22)\end{aligned}$$

From (21) and (22), we obtain

$$I_{hk} = \int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Letting $h\to+\infty, |\lambda|\le 1$, we have

$$A_{kh}\to\{x,|\lambda||T_k(u-\Psi)|\le h\}=\Omega, \quad (23)$$

$$B_{hk}\to\Omega \quad which\,implies \quad A_{kh}\cap B_{hk}\to\Omega. \quad (24)$$

Which and using Lebesgue theorem, we conclude that

$$\begin{aligned}&\lim_{h\to+\infty}\int_{A_{kh}\cap B_{hk}} a(x,u,\nabla T_h(u-\lambda T_k(u-\Psi)))\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx\\ &\quad=\lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx. \quad (25)\end{aligned}$$

thus implies that,

$$\lim_{h\to+\infty} I_{hk} = \lambda\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\,dx. \quad (26)$$

On the other hand, we have,

$$J_{hk} = \int_\Omega f\,T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx+\int_\Omega F(u)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\,dx.$$

Then

$$\lim_{h\to+\infty}\int_\Omega f\ T_k(u-T_h(u-\lambda T_k(u-\Psi)))\ dx \tag{27}$$

$$+\int_\Omega F(u)\nabla T_k(u-T_h(u-\lambda T_k(u-\Psi)))\ dx \tag{28}$$

$$=\lambda(\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\ dx), \tag{29}$$

i.e.,

$$\lim_{h\to+\infty} J_{hk}=\lambda(\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\ dx),. \tag{30}$$

Together (26), (30) and passing to the limit in (20), we obtain,

$$\lambda\left(\int_\Omega a(x,u,\nabla(u-\lambda T_k(u-\Psi)))\nabla T_k(u-\Psi)\ dx\right) \tag{31}$$
$$\leq\lambda\left(\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\ dx\right)$$

for every $\Psi\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$, and for every $k>0$. Choosing $\lambda>0$ dividing by λ, and then letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\ dx\leq\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\ dx. \tag{32}$$

for $\lambda<0$, dividing by λ, and then letting λ tend to zero, we obtain

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\ dx\geq\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\ dx. \tag{33}$$

Combining (32) and (33), we conclude the following equality :

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-\Psi)\ dx=\int_\Omega fT_k(u-\Psi)dx+\int_\Omega F(u)\nabla T_k(u-\Psi)\ dx. \tag{34}$$

This completes the proof of Lemma 3.3.

3.3. Proof of Theorem 3.2

3.3.1. Approximate problem and a priori estimate

For $n\in\mathbb{N}$, define $f_n:=T_n(f),F_n=F(T_n)$. Let u_n be solution in $W_0^1L_\varphi(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(a(x,u_n,\nabla u_n))=f_n-\operatorname{div}F_n(u_n) & \text{in }\Omega\\ u_n=0 & \text{on }\partial\Omega, \end{cases} \tag{35}$$

which exists thanks to ([33],Proposition 1, Remark 2). Choosing $T_k(u_n)$ as test function in (35), we have

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx = \int_\Omega f_n T_k(u_n)\,dx + \int_\Omega F_n(u_n)\nabla T_k(u_n)\,dx,$$

We claim that:

$$\int_\Omega F_n(u_n)\nabla T_k(u_n)\,dx = 0, \tag{36}$$

using $\nabla T_k(u_n) = \nabla u_{n\chi\{|u_n|\leq k\}}$, define $\Theta(t) = F_n(t)\chi\{t| \leq k\}$, and $\tilde{\Theta}(t) = \int_0^t \Theta(\tau)\,d\tau$, We have by Lemma 2.18 $\tilde{\Theta}(u_n) \in (W_0^1 L_\varphi(\Omega))^N$

$$\int_\Omega F_n(u_n)\nabla T_k(u_n)\,dx = \int_\Omega F_n(u_n)\chi\{|u_n| \leq k\}\nabla u_n\,dx$$

$$= \int_\Omega \Theta(u_n)\nabla u_n\,dx = \int_\Omega \operatorname{div}(\tilde{\Theta}(u_n))\,dx = 0 \tag{37}$$

(by 2.18) which proves the claim.

Now thanks to assumption (4), we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx \geq \int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx,$$

then

$$\int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \leq k\|f\|_{L^1(\Omega)}. \tag{38}$$

Then

$$\int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \leq C_1 k, \tag{39}$$

where C_1 is a constant independently of n.

3.3.2. Locally convergence of u_n in measure

Taking $\lambda|T_k(u_n)|$ in (35) and using (39), one has

$$\int_\Omega \rho(x)M(\lambda_1\frac{|\nabla T_k(u_n)|}{\lambda})dx \leq \int_\Omega \rho(x)M(\lambda_1|\nabla T_k(u_n)|)dx \leq C_1 k. \tag{40}$$

Then we deduce by using (40), that

$$\begin{aligned} meas\{|u_n|>k\} &\leq \frac{1}{\inf\limits_k M(\frac{k}{\lambda})}\int_{\{|u_n|>k\}} M(\frac{|u_n(x)|}{\lambda})dx \\ &\leq \frac{1}{\inf\limits_k M(\frac{k}{\lambda})}\int_{\Omega} M(\frac{1}{\lambda}|T_k(u_n)|)dx \\ &\leq \frac{C_1 k}{\inf\limits_k M(\frac{k}{\lambda})} \quad \forall n, \quad \forall k\geq 0. \end{aligned} \tag{41}$$

For any $\beta>0$, we have

$$meas\{|u_n-u_m|>\beta\}\leq meas\{|u_n|>k\}+meas\{|u_m|>k\}+meas\{|T_k(u_n)-T_k(u_m)|>\beta\},$$

and so that

$$meas\{|u_n-u_m|>\beta\}\leq \frac{2C_1 k}{\inf\limits_{x\in\Omega} M(\frac{k}{\lambda})}+meas\{|T_k(u_n)-T_k(u_m)|>\beta\}. \tag{42}$$

By using (39) and Poincaré inequality in weighted Orlicz-Sobolev spaces (Theorem2.8), we deduce that $(T_k(u_n))$ is bounded in $W_0^1L_M(\Omega,\rho)$, and then there exists $\omega_k\in W_0^1L_M(\Omega,\rho)$ such that $T_k(u_n)\rightharpoonup \omega_k$ weakly in $W_0^1L_M(\Omega,\rho)$ for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$; strongly in $E_{\overline{M}}(\Omega,\rho)$ and a.e. in Ω.
Consequently, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon>0$, then by (42) and the fact that $\frac{2C_1k}{\inf\limits_{x\in\Omega} M(\frac{k}{\lambda})}\to 0$ as $k\to+\infty$ there exists some $k=k(\varepsilon)>0$ such that

$$meas\{|u_n-u_m|>\lambda\}<\varepsilon, \qquad \text{for all } n,m\geq h_0(k(\varepsilon),\lambda).$$

This proves that u_n is a Cauchy sequence in measure, thus, u_n converges almost everywhere to some measurable function u. Finally, there exist a subsequence of $\{u_n\}_n$, still indexed by n, and a function $u\in W_0^1L_M(\Omega,\rho)$ such that

$$\begin{cases} u_n\rightharpoonup u & \text{weakly in } W_0^1L_M(\Omega,\rho) \text{ for } \sigma(\Pi L_M,\Pi E_{\overline{M}}) \\ u_n\longrightarrow u & \text{strongly in } E_M(\Omega,\rho) \text{ and a.e. in } \Omega. \end{cases} \tag{43}$$

3.3.3. An intermediate Inequality

In this step, we shall prove that for $\phi\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx\leq \int_\Omega f_nT_k(u_n-\phi)\,dx+\int_\Omega F_n\nabla T_k(u_n-\phi)\,dx. \tag{44}$$

We choose now $T_k(u_n-\phi)$ as test function in (35), with ϕ in $W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$, we obtain

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\phi)\,dx=\int_\Omega f_nT_k(u_n-\phi)\,dx+\int_\Omega F_n\nabla T_k(u_n-\phi)\,dx. \quad (45)$$

Adding and substracting the term $\int_\Omega a(x,u_n,\nabla\phi),\nabla T_k(u_n-\phi)\,dx$ i.e.,

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n-\phi)\,dx+\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx \quad (46)$$

$$-\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx=\int_\Omega f_nT_k(u_n-\phi)dx+\int_\Omega F_n\nabla T_k(u_n-\phi)\,dx$$

.

Thanks to assumption (3) and the definition of truncation function, we have

$$\int_\Omega (a(x,u_n,\nabla u_n)-a(x,u_n,\nabla\phi))\nabla T_k(u_n-\phi)\,dx\geq 0. \quad (47)$$

Combining (46) and (47), we obtain (44).

3.3.4. Passing to the limit

We shall prove that for $\phi\in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$, we have

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx\leq\int_\Omega fT_k(u-\phi)dx+\int_\Omega F\nabla T_k(u-\phi)\,dx.$$

Firstly, we claim that

$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx\to\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \text{ as } n\to+\infty.$$

Since $T_M(u_n)\rightharpoonup T_M(u)$ weakly in $W_0^1L_M(\Omega,\rho)$, with $M=k+\|\phi\|_\infty$, then

$$T_k(u_n-\phi)\rightharpoonup T_k(u-\phi) \text{ in } W_0^1L_M(\Omega,\rho), \quad (48)$$

which gives

$$\frac{\partial T_k}{\partial x_i}(u_n-\phi)\rightharpoonup\frac{\partial T_k}{\partial x_i}(u-\phi) \text{ weakly in } L_M(\Omega,\rho) \qquad \forall i=1,..,N. \quad (49)$$

Show that

$$a(x,T_M(u_n),\nabla\phi)\to a(x,T_M(u),\nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega))^N.$$

Thanks to assumption (2), we obtain

$$|a_i(x,T_M(u_n),\nabla\phi)| \le |\phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u_n)|)) + K_i\overline{M}^{-1}M(c_1\,|\nabla\phi|),$$

with β and μ are positive constants. Since $T_M(u_n) \rightharpoonup T_M(u)$ weakly in $W_0^1L_M(\Omega,\rho)$ and $W_0^1L_M(\Omega,\rho) \hookrightarrow\hookrightarrow L_{\overline{M}}(\Omega,\rho)$, then $T_M(u_n) \rightharpoonup T_M(u)$ strongly in $L_M(\Omega,\rho)$ and a.e. in Ω, hence

$$|a(x,T_M(u_n),\nabla\phi)| \to |a(x,T_M(u),\nabla\phi)| \text{ a.e. in } \Omega.$$

and

$$|\phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u_n)|)) + K_i\overline{M}^{-1}M(c_1\,|\nabla\phi|) \to$$

$$|\phi_i(x)| + K_i\overline{P}^{-1}(\rho^{-1}(x)M(c_2|T_M(u)|)) + K_i\overline{M}^{-1}M(c_1\,|\nabla\phi|),$$

a.e. in Ω. Then, By Vitali's theorem, we deduce that

$$a(x,T_M(u_n),\nabla\phi) \to a(x,T_M(u),\nabla\phi) \text{ strongly in } (L_{\overline{M}}(\Omega,\rho))^N, \text{ as } n\to\infty. \quad (50)$$

Combining (49) and (50), we obtain

$$\int_\Omega a(x,u_n,\nabla\phi)\nabla T_k(u_n-\phi)\,dx \to \int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \text{ as } n\to+\infty. \quad (51)$$

Secondly, we show that

$$\int_\Omega f_nT_k(u_n-\phi)dx \to \int_\Omega fT_k(u-\phi)dx. \quad (52)$$

and

$$\int_\Omega F_n\nabla T_k(u_n-\phi)\,dx \to \int_\Omega F\nabla T_k(u-\phi)\,dx \quad (53)$$

We have $f_nT_k(u_n-\phi) \to fT_k(u-\phi)$ a.e. in Ω and $|fT_k(u_n-\phi)| \le k|f|$, and $F_n\nabla T_k(u_n-\phi) \to F\nabla T_k(u-\phi)$ a.e. in Ω, and $|F\nabla T_k(u_n-\phi)| \le k|F|$ then by using Vitali's theorem, we obtain (52)and (53) . Thanks to (51), (52)and (53) allow to pass to the limit in the inequality (44), so that $\forall\phi \in W_0^1L_M(\Omega,\rho)\cap L^\infty(\Omega)$, we deduce

$$\int_\Omega a(x,u,\nabla\phi)\nabla T_k(u-\phi)\,dx \le \int_\Omega fT_k(u-\phi)dx + \int_\Omega F\nabla T_k(u-\phi)dx.$$

In view of main Lemma, we can deduce that u is an entropy solution of the problem (1). This completes the proof of Theorem 3.2.

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