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ON THE PROOF OF HÖRMANDER'S HYPOELLIPTICITY THEOREM

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This is a survey paper about the proof of the hypoellipticity theorem by Hörmander (Acta Math. 1967). We will compare three different proofs of this result: the original one by Hörmander, the proof given by Kohn (Proc. Sympos. Pure Math., 1973) and independently by Oleĭnik and Radkevič in their 1973 monograph, and the more recent proof of a special case of this result, concerning sublaplacians on Carnot groups, given by Bramanti and Brandolini (Nonlinear Analysis, 2015).

This is a survey paper about the proof of the celebrated hypoellipticity theorem by Hörmander (see [7]). More precisely, we will compare three different proofs of this result: the original one by Hörmander, the proof given a few years later by Kohn [8] and independently by Oleĭnik and Radkevič[1] [9], and the more recent proof of a special case of this result, concerning sublaplacians on Carnot groups, given in [2].

To put into context our discussion, let us recall that a linear differential operator $\mathcal{L}$ with smooth coefficients is said to be hypoelliptic in an open set $\Omega \subseteq \mathbb{R}^N$ if, for every open set $\Omega' \subseteq \Omega$, whenever a distribution $u \in \mathcal{D}'(\Omega')$ is such that $\mathcal{L}u \in C^\infty(\Omega')$ then $u \in C^\infty(\Omega')$. The aforementioned Hörmander's theorem provides an almost complete characterization of second order hypoelliptic operators with real coefficients.

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[1]However, from now on we will simply refer to "Kohn's proof" to deal with this proof.

Hörmander's preliminary analysis consists in proving that every hypoelliptic second order differential operator has necessarily semi-definite principal part. From this fact he deduces that in any open set where the rank of the coefficient matrix is constant, the operator (or its opposite) can be rewritten in the form

$$\mathcal{L} = \sum_{j=1}^{q} X_j^2 + X_0 + c \tag{1}$$

where $X_0, X_1, \ldots, X_q$ are real smooth vector fields (that is, first order differential operators) and c is a smooth function. Therefore Hörmander considers operators already written in the form (1) and proves that:

Theorem 0.1. *An operator $\mathcal{L}$ as in (1) is hypoelliptic in $\Omega \subseteq \mathbb{R}^N$ if the vector fields of the Lie algebra generated by $X_0, X_1, \ldots, X_q$ span the whole $\mathbb{R}^N$ at every point $x \in \Omega$.*

The algebraic assumption appearing in the above statement has been labeled "Hörmander's condition". Explicitly, this condition means that if we define the commutator of two vector fields X, Y as the vector field

$$[X,Y] = XY - YX,$$

then among the iterated commutators X_{j_1}, $[X_{j_1}, X_{j_2}]$, $[X_{j_1}, [X_{j_2}, X_{j_3}]]$, ... at every point of the domain Ω there exist N which are linearly independent.

Conversely, if in an open set $U \subset \Omega$ the rank of the Lie algebra (that is, the dimension of the vector space spanned by iterated commutators) is constant and strictly less than N, then the operator $\mathcal{L}$ is not hypoelliptic in U. Hence Hörmander's condition is "almost necessary" for hypoellipticity.

For a discussion of some motivations to study operators of type (1), the reader is referred for instance to [1, Chap. 2].

From now on, in this paper, we will focus on the special class of Hörmander operators

$$\mathcal{L} = \sum_{j=1}^{q} X_j^2$$

with $X_1, \ldots, X_q$ satisfying Hörmander's condition in a domain Ω of $\mathbb{R}^N$. These operators are known as "sum of squares of Hörmander's vector fields" or "sub-laplacians", and the proof of Hörmander's theorem is substantially less difficult and more transparent in this situation, that is when the term X_0 in (1) is lacking.

The interesting case is when $q < N$, so that the operator is degenerate elliptic. If the vector fields X_i were constant, the operator $\mathcal{L}$ could not be regularizing, since it would control only q fixed directions. The geometric idea behind Hörmander's theorem is that if the missing directions in the operator $\mathcal{L}$ can be recovered by the commutators of the generators X_i, then the possible smoothness of $\mathcal{L}u$ propagates in every direction to u.

Example 0.2. In $\mathbb{R}^3 \ni (x,y,t)$, let

$$X_1 = \partial_x + 2y\partial_t$$
$$X_2 = \partial_y - 2x\partial_t.$$

Then the operator

$$\mathcal{L} = X_1^2 + X_2^2 = (\partial_x + 2y\partial_t)^2 + (\partial_y - 2x\partial_t)^2$$

is degenerate elliptic in $\mathbb{R}^3$. Nevertheless, since

$$[X_1, X_2] = X_1X_2 - X_2X_1 = -4\partial_t,$$

the vector fields

$$X_1, X_2, [X_1, X_2]$$

span $\mathbb{R}^3$ at every point of the space. Therefore Hörmander's condition holds, and $\mathcal{L}$ is hypoelliptic.

1. The role of subelliptic estimates

To explore the properties of a "sum of squares" operator, let us start with the following elementary computation. For a smooth vector field X in Ω, let X^* be its transposed, defined by the identity

$$\int_\Omega (Xu)\, v dx = \int_\Omega u\,(X^* v)\, dx \text{ for any } u, v \in C_0^\infty(\Omega).$$

Integration by parts shows that $X^* = -X + c$ for some smooth function c, hence for $u \in C_0^\infty(\Omega)$, denoting by K the support of u, we can write

$$\begin{aligned}
\sum_{j=1}^q \left\|X_j u\right\|_{L^2}^2 &= \sum_{j=1}^q \int \left(X_j^* X_j u\right) u dx \\
&= \sum_{j=1}^q \left\{ -\int \left(X_j^2 u\right) u dx + \int \left(c_j X_j u\right) u dx \right\} \\
&\leqslant |\langle \mathcal{L}u, u\rangle| + c_K \sum_{j=1}^q \left\|X_j u\right\|_{L^2} \|u\|_{L^2} \\
&\leqslant |\langle \mathcal{L}u, u\rangle| + \frac{1}{2} \sum_{j=1}^q \left\|X_j u\right\|_{L^2}^2 + c\,\|u\|_{L^2}^2
\end{aligned}$$

hence

$$\sum_{j=1}^{q} \left\|X_j u\right\|_{L^2}^2 \leqslant c_K \left\{ |\langle \mathcal{L}u, u\rangle| + \|u\|_{L^2}^2 \right\}$$

or also

$$\|u\|_{W_X^{1,2}} \leqslant c_K \left\{ \|\mathcal{L}u\|_{L^2} + \|u\|_{L^2} \right\}, \tag{2}$$

where we have introduced the *Sobolev norm induced by the vector fields*,

$$\|u\|_{W_X^{1,2}}^2 = \sum_{j=1}^{q} \left\|X_j u\right\|_{L^2}^2 + \|u\|_{L^2}^2 .$$

Now, (2) is a very natural a priori estimate: the differential operator $\mathcal{L}$, sum of squares of the vector fields X_i, controls the derivatives along the directions of the same vector fields. Note that this estimate holds independently of the validity of Hörmander's condition. If we want to prove a regularity result, however, we have to show that actually the norm $W_X^{1,2}$ somewhat controls the behavior of a function in *every* direction, provided Hörmander's condition holds. However, for $q < N$ there is no hope of controlling the usual Sobolev norm $W^{1,2}$ with the $W_X^{1,2}$ norm, so, a priori, two possible strategies appear: either we try to bound a Sobolev norm of fractional order of u with the right hand side of (2), or we bound some Sobolev norm of integral order of u with an even stronger Sobolev norm of $\mathcal{L}u$. Both the original proof by Hörmander and Kohn's proof choose the first possibility, while the proof in [2] chooses the second one, as we will see later. Introducing the (standard) Sobolev norm of real order r, defined via Fourier transform by

$$\|u\|_{H^r}^2 = \int |\widehat{u}(\xi)|^2 \left(1 + |\xi|^2\right)^r d\xi,$$

a key step consists in the proof of the following:

Proposition 1.1 (Hörmander [7, (3.4)], Kohn [8, (5)]). *For every $K \Subset \Omega$ there exist $\varepsilon \in (0,1)$, $c > 0$ such that for every $u \in C_0^\infty(K)$,*

$$\|u\|_{H^\varepsilon} \leqslant c \|u\|_{W_X^{1,2}} \tag{3}$$

provided the vector fields $X_1, X_2, ..., X_q$ satisfy Hörmander's condition.

Let us first discuss how this result implies a full regularity estimate, then we will describe how this proposition is actually proved.

Combining (2) and (3) we can write, for every $u \in C_0^\infty(K)$, the *basic subelliptic estimate*:

$$\|u\|_{H^\varepsilon} \leqslant c_K \left\{ \|\mathcal{L}u\|_{L^2} + \|u\|_{L^2} \right\}. \tag{4}$$

This estimate is then localized and iterated to higher order.

Before stating the result, let us introduce a shorthand notation which will be used throughout the paper. We will write

$$\eta_1 \prec \eta_2$$

to say that

$$\eta_1, \eta_2 \in C_0^\infty(\Omega), 0 \leqslant \eta_1 \leqslant \eta_2 \leqslant 1, \eta_2 = 1 \text{ on } \operatorname{sprt} \eta_1.$$

Theorem 1.2 (Subelliptic estimates, Kohn, [8, (14)]). *Under the above assumptions, for every couple of cutoff functions $\eta_1 \prec \eta_2$, for every $\sigma, m > 0$, there exists $c > 0$ such that, for every $u \in C^\infty(\Omega)$,*

$$\|\eta_1 u\|_{H^{s+\varepsilon}} \leqslant c\left\{\|\eta_2 \mathcal{L} u\|_{H^s} + \|\eta_2 u\|_{H^{-m}}\right\}.$$

(The number $\varepsilon \in (0,1)$ is the same appearing in (3)).

Note that, by a well-known result about the local structure of distributions, for every distribution $u \in D'(\Omega)$ and every $\eta_2 \in C_0^\infty(\Omega)$ there exists $m > 0$ such that $\eta_2 u \in H^{-m}$. Hence, as soon as we know that the estimate in the above theorem actually holds for every distribution u such that the right hand side is finite, we can conclude that u is smooth on every open subset of Ω where $\mathcal{L} u$ is smooth, that is $\mathcal{L}$ is hypoelliptic. Therefore proving Hörmander's theorem from Theorem 1.2 is a matter of smoothing of distributions (see [8, pp.64-65]).

An analogous result is proved by Hörmander in the following form:

Theorem 1.3 (Hörmander, [7, Prop. 3.2]). *For every $\sigma > 0$, if $u \in D'(\Omega)$ and $\mathcal{L} u \in H_{loc}^\sigma(\Omega)$, then $u \in H_{loc}^{\sigma+\varepsilon}(\Omega)$. The same is true when Ω is replaced with any open subset of Ω. In particular, $\mathcal{L}$ is hypoelliptic.*

In the next two sections we will briefly describe the different way in which Hörmander and Kohn prove Proposition 1.1 and Theorem 1.2 or 1.3. We will concentrate on the proof of a priori estimates for smooth functions, disregarding the issue of showing how a *distributional* solution can be regularized.

The previous discussion shows that, despite the differences between the two approaches, which will be enlightened in the next two sections, subelliptic estimates are a key step for both. These estimates have a twofold feature. On the one hand, the use of isotropic (and fractional) Sobolev norms H^σ can look unnatural, in relation to the study of the operator $\mathcal{L}$, which is strongly anisotropic. This reflects in the poor regularization that these estimates imply: a gain of ε derivatives. On the other hand, in view of the final result that we want to prove (smoothness with respect to *all* the variables) it is exactly an isotropic norm that we need. Let us compare this result, however, with the estimates proved some years later by Folland [5] (for Hörmander operators on homogeneous groups) and by Rothschild-Stein [10] (in the general case). Restricting ourselves, again, to the case of operators "sum of squares", these results read as follows:

Theorem 1.4. *Under the above assumptions, for every couple of cutoff functions* $\eta_1 \prec \eta_2$*, every nonnegative integer* k *and* $p \in (1,\infty)$*, there exists* $c > 0$ *such that, for every* $u \in C^\infty(\Omega)$*,*

$$\|\eta_1 u\|_{W_X^{k+2,p}} \leqslant c \left\{ \|\eta_2 \mathcal{L} u\|_{W_X^{k,p}} + \|\eta_2 u\|_{L^p} \right\}.$$

Here the Sobolev norms are those induced by the vector fields,

$$\|u\|_{W_X^{k,p}} = \|u\|_{L^p} + \sum_{h=1}^{k} \sum_{j_1,j_2,\dots,j_h=1,2,\dots,q} \left\| X_{j_1} X_{j_2} \dots X_{j_h} u \right\|_{L^p}.$$

Now, these estimates are more natural than subelliptic estimates, since they measure the regularity of a function along the directions of the vector fields, and imply a gain of regularity of exactly 2 derivatives, in this scale of spaces. At the same time, since by Hörmander's condition every Cartesian derivative ∂_{x_i} can be expressed as a linear combination of iterated commutators of $X_1,\dots,X_q$, a function belonging to $W_X^{k,p}$ for every k also belongs to all the (isotropic) Sobolev spaces $W^{k,p}$, and then is smooth, by the standard Sobolev embedding theorems. We note, however, that in order to prove the last theorem, a wealth of new results had to be proved. In particular, Hörmander's hypoellipticity theorem is used by Folland in [5] to prove the existence of a homogeneous fundamental solution for $\mathcal{L}$, smooth outside the pole. This is a key tool for the proof of these estimates, both in [5] and in [10]. Therefore these estimates cannot be seen as an alternative path towards the hypoellipticity theorem; instead, they represent one of its far consequences.

Disclaimer. Sections 2, 3, 4 contain an informal discussion of some points contained in the papers [8], [7], [2], respectively. We refer to these papers for details and complete proofs.

2. Kohn's proof of the subelliptic estimates

Let us fix some notation which will be used throughout the paper. For a multi-index

$$I = (i_1, \dots i_k),\ i_j \in \{1,2,\dots,q\},$$

we let

$$|I| = k,$$
$$X_{[I]} = [[\dots[[X_{i_1}, X_{i_2}], X_{i_3}], \dots], X_{i_k}].$$

Hörmander's condition can be expressed saying that for some positive integer s (called the step of the Lie algebra), at every point of the domain the vectors

$$\left\{X_{[I]}\right\}_{|I| \leqslant s}$$

span $\mathbb{R}^N$.

The comparison between the proofs by Hörmander and Kohn is more easily done starting with Kohn's approach to the proof of subelliptic estimates (Theorem 1.2). Recall that we are always considering the case

$$\mathcal{L} = \sum_{j=1}^{q} X_j^2.$$

To prove (3), one can first of all check, exploiting the definition of fractional Sobolev norm, that for every $\varepsilon \in (0,1)$:

$$\|u\|_{H^\varepsilon}^2 \leqslant C_\varepsilon \left(\|u\|_2^2 + \sum_{j=1}^{N} \left\| \frac{\partial u}{\partial x_j} \right\|_{H^{\varepsilon-1}}^2 \right). \tag{5}$$

Then, exploiting Hörmander's condition, one can write $\frac{\partial u}{\partial x_j}$ in terms of commutators, and prove that

$$\left\| \frac{\partial u}{\partial x_j} \right\|_{H^{\varepsilon-1}}^2 \leqslant c \sum_{|I| \leqslant s} \left\| X_{[I]} u \right\|_{H^{\varepsilon-1}}^2. \tag{6}$$

Now, the hard part of the proof, exploiting techniques of *pseudodifferential operators*, amounts to showing that, for a suitable $\varepsilon \in (0,1)$,

$$\sum_{|I| \leqslant s} \left\| X_{[I]} u \right\|_{H^{\varepsilon-1}}^2 \leqslant c \left(\sum_{j=1}^{q} \left\| X_j u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \tag{7}$$

This is done iteratively. For $X_{[I]} = X_j X_{[I']} - X_{[I']} X_j$ one proves, first of all, that

$$\left\| X_{[I]} u \right\|_{H^{\varepsilon-1}}^2 \leqslant c \left(\left\| X_j u \right\|_{L^2}^2 + \left\| X_{[I']} u \right\|_{H^{2\varepsilon-1}}^2 + \|u\|_{L^2}^2 \right). \tag{8}$$

Applying recursively this bound one gets

$$\left\|X_{[I]}u\right\|_{H^{\varepsilon-1}}^2 \leqslant c\left(\sum_{j=1}^q \left\|X_j u\right\|_{L^2}^2 + \sum_{j=1}^q \left\|X_j u\right\|_{H^{2^{|I|-1}\varepsilon-1}}^2 + \|u\|_{L^2}^2\right),$$

and since $|I| \leqslant s$, choosing $\varepsilon = 2^{1-s}$, so that $2^{|I|-1}\varepsilon - 1 \leqslant 0$, we have $\left\|X_j u\right\|_{H^{2^{|I|-1}\varepsilon-1}}^2 \leqslant \left\|X_j u\right\|_{L^2}^2$ and

$$\left\|X_{[I]}u\right\|_{H^{\varepsilon-1}}^2 \leqslant c\left(\sum_{j=1}^q \left\|X_j u\right\|_{L^2}^2 + \|u\|_{L^2}^2\right),$$

that is (7), which together with (5)-(6) gives (3) and so (4).

To give a taste of the techniques of pseudodifferential operators, let us look into the proof of (8), which is the hard step in the above argument. In the rest of this section we will present some ideas about the techniques used by Kohn, without giving the explicit definition of pseudodifferential operator. In this presentation, we follow [4] and [3, Chap.5].

Let $\mathcal{S}\left(\mathbb{R}^N\right)$ be the Schwartz space of rapidly decreasing smooth functions. An *operator of type* $m \in \mathbb{R}$ is, by definition, a linear operator

$$T : \mathcal{S}\left(\mathbb{R}^N\right) \to \mathcal{S}\left(\mathbb{R}^N\right)$$

such that for every $\sigma \in \mathbb{R}$ there exists $c > 0$ such that

$$\|Tu\|_{H^\sigma} \leqslant c\,\|u\|_{H^{\sigma+m}} \text{ for every } u \in \mathcal{S}\left(\mathbb{R}^N\right).$$

With some toil one can prove the following:

Proposition 2.1. *According to the previous definition of "operator of type m",*

(i) a differential operator of order m $(= 1, 2, 3, ...)$ *with coefficients in* $\mathcal{S}\left(\mathbb{R}^N\right)$ *is actually an operator of type m;*

(ii) the multiplication operator by a function $a \in \mathcal{S}\left(\mathbb{R}^N\right)$ *is an operator of type* 0*;*

(iii) the fractional differentiation operator of order $\sigma \in \mathbb{R}$*, denoted by* Λ^σ *and defined via Fourier transform letting*

$$\widehat{\Lambda^\sigma u}(\xi) = \left(1 + |\xi|^2\right)^{\sigma/2} \widehat{u}(\xi) \textit{ for } u \in \mathcal{S}\left(\mathbb{R}^N\right),$$

is an operator of type σ*;*

(iv) the composition of two operators T_1, T_2 *of type* m_1, m_2*, respectively, is an operator of type* $m_1 + m_2$*;*

(v) the commutator

$$[T_1, T_2] = T_1 T_2 - T_2 T_1$$

of two operators T_1, T_2 *of order* m_1, m_2*, respectively, obtained composing operators of the kinds (i)-(ii)-(iii), is an operator of type* $m_1 + m_2 - 1$*.*

The last statement in the above proposition is perhaps the strongest tool from the theory of pseudodifferential operators which is involved in the proof of subelliptic estimates: given the composition $T_1 T_2$ of two noncommuting operators of types m_1, m_2, the "error term" that we insert replacing $T_1 T_2$ with $T_2 T_1$ behaves like a *lower order term*, with respect to $T_1 T_2$, from the point of view of the regularity estimates in the scale of spaces H^σ.

The smooth vector fields $X_1, X_2, ..., X_q$, originally defined on a bounded domain Ω, can be thought as defined on the whole $\mathbb{R}^N$ with coefficients in $\mathcal{S}\left(\mathbb{R}^N\right)$, suitably extending their coefficients and modifying them near the boundary of Ω. Hence they can be regarded as operators of type 1.

The following properties of the operators Λ^σ are easily checked:

Lemma 2.2. *For every $\sigma, \tau \in \mathbb{R}$, denoting by $\langle \cdot, \cdot \rangle$ the usual scalar product in L^2, we have:*
(a) $\|\varphi\|_{H^\sigma} = \|\Lambda^\sigma \varphi\|_{L^2}$
(b) $\langle \Lambda^\sigma \varphi, \Lambda^\tau \psi \rangle = \langle \Lambda^{\sigma+\tau} \varphi, \psi \rangle$
(c) $\|\varphi\|_{H^\sigma}^2 = \left\langle \varphi, \Lambda^{2\sigma} \varphi \right\rangle.$

With this background, we can now describe the proof of (8). To fix ideas, let $I = (j, i)$, $X_{[I]} = X_j X_i - X_i X_j$ (assuming $|I| = 2$ just simplifies notation; the proof in the general case is very similar). Then, by Lemma 2.2 (c),

$$\begin{aligned}
\left\|X_{[I]} u\right\|_{H^{\varepsilon-1}}^2 &= \left\langle X_{[I]} u, \Lambda^{2\varepsilon-2} X_{[I]} u \right\rangle \\
&= \left\langle X_j X_i u, \Lambda^{2\varepsilon-2} X_{[I]} u \right\rangle - \left\langle X_i X_j u, \Lambda^{2\varepsilon-2} X_{[I]} u \right\rangle \\
&\equiv A - B.
\end{aligned}$$

The terms A and B are similar, so let us show how to estimate A. Since $X_j^* = -X_j + g$ for some $g \in \mathcal{S}\left(\mathbb{R}^N\right)$,

$$\begin{aligned}
\left\langle X_j X_i u, \Lambda^{2\varepsilon-2} X_{[I]} u \right\rangle &= -\left\langle X_i u, X_j \Lambda^{2\varepsilon-2} X_{[I]} u \right\rangle + \left\langle X_i u, g \Lambda^{2\varepsilon-2} X_{[I]} u \right\rangle \\
&\equiv -A_1 + A_2.
\end{aligned}$$

To bound A_1, let us write

$$X_j \Lambda^{2\varepsilon-2} X_{[I]} u = \Lambda^{2\varepsilon-2} X_{[I]} X_j u + \Lambda^{2\varepsilon-2} \left[X_j, X_{[I]}\right] u + \left[X_j, \Lambda^{2\varepsilon-2}\right] X_{[I]} u.$$

Then we have

$$\begin{aligned}
A_1 =& \left\langle X_i u, \Lambda^{2\varepsilon-2} X_{[I]} X_j u \right\rangle + \left\langle X_i u, \Lambda^{2\varepsilon-2} \left[X_j, X_{[I]}\right] u \right\rangle \\
&+ \left\langle X_i u, \left[X_j, \Lambda^{2\varepsilon-2}\right] X_{[I]} u \right\rangle \\
\equiv& A_{1,1} + A_{1,2} + A_{1,3}.
\end{aligned}$$

By Proposition 2.1, $\Lambda^{-1}X_{[I]}$ is an operator of order 0. Then by Lemma 2.2 (b),(a) we have

$$\begin{aligned}|A_{1,1}| &= \left|\left\langle \Lambda^{2\varepsilon-1}X_iu, \Lambda^{-1}X_{[I]}X_ju\right\rangle\right| \leqslant \|X_iu\|_{H^{2\varepsilon-1}} \left\|\Lambda^{-1}X_{[I]}X_ju\right\|_{L^2} \\ &\leqslant c\,\|X_iu\|_{H^{2\varepsilon-1}} \left\|X_ju\right\|_2 \leqslant c\left(\|X_iu\|^2_{H^{2\varepsilon-1}} + \left\|X_ju\right\|_2^2\right).\end{aligned}$$

Similarly $\Lambda^{-1}\left[X_j, X_{[I]}\right]$ is an operator of type 0, hence

$$|A_{1,2}| = \left|\left\langle \Lambda^{2\varepsilon-1}X_iu, \Lambda^{-1}\left[X_j, X_{[I]}\right]u\right\rangle\right| \leqslant c\left(\|X_iu\|^2_{H^{2\varepsilon-1}} + \|u\|_2^2\right).$$

To bound $A_{1,3}$, note that, by Proposition 2.1, $\Lambda^{-2\varepsilon+1}\left[X_j, \Lambda^{2\varepsilon-2}\right]X_{[I]}$ is an operator of type 0, so that by Lemma 2.2 (b)

$$\begin{aligned}|A_{1,3}| &= \left|\left\langle \Lambda^{2\varepsilon-1}X_iu, \Lambda^{-2\varepsilon+1}\left[X_j, \Lambda^{2\varepsilon-2}\right]X_{[I]}u\right\rangle\right| \\ &\leqslant c\left(\|X_iu\|^2_{H^{2\varepsilon-1}} + \|u\|_2^2\right).\end{aligned}$$

To bound A_2, note that, by Proposition 2.1, $\Lambda^{-2\varepsilon+1}g\Lambda^{2\varepsilon-2}X_{[I]}$ is an operator of type 0. Hence, again by Lemma 2.2 (b) we have

$$\begin{aligned}|A_2| &= \left|\left\langle X_iu, g\Lambda^{2\varepsilon-2}X_{[I]}u\right\rangle\right| = \left|\left\langle \Lambda^{2\varepsilon-1}X_iu, \Lambda^{-2\varepsilon+1}g\Lambda^{2\varepsilon-2}X_{[I]}u\right\rangle\right| \\ &\leqslant c\,\|X_iu\|_{H^{2\varepsilon-1}}\|u\|_2 \leqslant c\left(\|X_iu\|^2_{H^{2\varepsilon-1}} + \|u\|_2^2\right).\end{aligned}$$

Hence

$$|A| \leqslant c\left(\left\|X_ju\right\|_2^2 + \|X_iu\|^2_{H^{2\varepsilon-1}} + \|u\|_2^2\right),$$

which is (8) for $|I'| = 1$. As already noted, the general case is not substantially different.

This concludes our discussion of the proof of (4). Starting with this estimate, again by techniques of pseudodifferential operators, the following localized version can be established:

Theorem 2.3. *Let $\varepsilon > 0$ as in the above proof and let $\eta_1, \eta_2 \in C_0^\infty(\Omega)$ such that $\eta_1 \prec \eta_2$. For every $\sigma \in \mathbb{R}$ there exists $c > 0$ such that for every $u \in C_0^\infty(\Omega)$*

$$\|\eta_1 u\|_{H^{\sigma+\varepsilon}} \leqslant C\left(\|\eta_2\mathcal{L}u\|_{H^\sigma} + \|\eta_2 u\|_{H^\sigma}\right). \tag{9}$$

An easy iteration of the last estimate then gives, for every $\sigma, m > 0$, the following:

$$\|\eta_1 u\|_{H^{\sigma+\varepsilon}} \leqslant C\left(\|\eta_2\mathcal{L}u\|_{H^\sigma} + \|\eta_2 u\|_{H^{-m}}\right), \tag{10}$$

which is Theorem 1.2.

The last step in the previous reasoning enlightens the role of ε in the subelliptic estimates. If one just looks at the *final result* (10), letting $\varepsilon = 0$ does not cause a serious loss of information: the inequality just says that every distribution u can be made as regular as we please, provided $\mathcal{L}u$ is assumed regular enough. But, obviously, what makes possible to get (10) from (9) by iteration, is the positivity of ε in (4) and then in (9).

3. Hörmander's proof of the subelliptic estimates

Let us now turn to Hörmander's proof of subelliptic estimates. We stress, once more, that we are dealing with the simplified version of the proof that applies to operators of the kind "sum of squares". (The whole section 5 in [7] becomes superfluous in this special case).

The major difference with Kohn's approach appears in the proof of the basic subelliptic estimate, that is Proposition 1.1, which immediately gives (4). Once this is established, Hörmander derives Theorem 1.3 by a proof which makes use of pseudodifferential operators (although the argument given in [7, pp.153-156] is rather condensed). Kohn's approach, in some sense, consists in applying to the whole proof the techniques that Hörmander applies to a part of it.

In order to prove the estimate

$$\|u\|_{H^{\varepsilon}} \leqslant c\,\|u\|_{W_X^{1,2}}$$

Hörmander introduces a seminorm which weights in L^2 norm the derivatives of fractional order $r > 0$:

$$|u|_r = \sup_{0<|h|<\delta} \|\tau_h u - u\|_{L^2}\,|h|^{-r}$$

(where $\tau_h u(x) = u(x+h)$, and $\delta > 0$ is a fixed small number, which is not important to specify). The relations between these seminorms and the norms of fractional Sobolev spaces defined via the Fourier transform are:

$$\begin{aligned} |u|_r &\leqslant C\,\|u\|_{H^r} \\ \|u\|_{H^t} &\leqslant C\left(|u|_r + \|u\|_{L^2}\right) \text{ for } t < r. \end{aligned} \tag{11}$$

The basic subelliptic estimate that Hörmander proves is the following:

Proposition 3.1. *For every $K \Subset \Omega$ there exists $c > 0$ such that for every $u \in C_0^{\infty}(K)$,*

$$|u|_{1/s} \leqslant c\,\|u\|_{W_X^{1,2}}$$

(where s is the step of the Lie algebra).

By (11), this implies an H^{ε}-subelliptic estimate for every $\varepsilon < 1/s$ (which for $s > 2$ is a better value of ε than the one found in Kohn's proof, $\varepsilon = 2^{1-s}$, while for $s = 2$ is slightly worse).

The use of the seminorm $|\cdot|_r$ instead of the norm $\|\cdot\|_{H^r}$ makes this part of Hörmander's proof more geometric, with no use of pseudodifferential operators and Fourier transform.

Let us define also the seminorms corresponding to fractional derivatives with respect to a vector field X:

$$|u|_{X,r} = \sup_{0<|t|<\delta} \left\| e^{tX}u - u \right\|_{L^2} |t|^{-r}$$

where the exponential $e^{tX}u$ is defined, as usual, letting

$$\left(e^{tX}u\right)(x) = u\left(f\left(x,t\right)\right)$$

when f is the solution to the Cauchy problem

$$\begin{cases} \frac{df}{dt}(x,t) = X\left(f\left(x,t\right)\right) \\ f\left(x,0\right) = x. \end{cases}$$

Also, we write

$$\left(e^{tX}\right)(x) = f\left(x,t\right)$$

with f as above. It can be proved that

$$|u|_{X,r} \leqslant c\,|u|_r\,.$$

The proof of Proposition 3.1 is obtained in two steps, combining the following inequalities

$$|u|_{1/s} \leqslant c\left\{ \sum_{j=1}^{q} |u|_{X_j,1} + \|u\|_{L^2} \right\} \tag{12}$$

$$\leqslant c\left\{ \sum_{j=1}^{q} \left\|X_j u\right\|_{L^2} + \|u\|_{L^2} \right\} \tag{13}$$

where (13) is easy, while (12) contains the relevant piece of information: the fractional derivatives of order $1/s$ *in any direction* can be controlled by first order derivatives with respect to the directions of the vector fields $X_1,...,X_q$ alone.

Exploiting Hörmander's condition, the fractional derivative in any direction can be expressed in terms of fractional derivatives along vector fields which are linear combinations of suitable commutators of $X_1, \dots, X_q$ (with smooth functions as coefficients), therefore the proof of (12) will be achieved showing that

$$|u|_{X,1/s} \leqslant c \left\{ \sum_{j=1}^{q} |u|_{X_j,1} + \|u\|_{L^2} \right\} \tag{14}$$

for every

$$X \in T^s(\Omega) = \left\{ \sum \phi_I X_{[I]} : |I| \leqslant s, \phi_I \in C^\infty(\Omega) \right\}.$$

Proving (14) for $X \in T^s(\Omega)$ requires expressing the exponential of a commutator $X_{[I]}$ in terms of exponentials of the generators $X_1, \dots, X_q$. This can be done exploiting the *Campbell-Hausdorff formula*, a well-known deep result from noncommutative algebra, which lies at the core of Hörmander's proof of (14) and can be stated as follows:

Theorem 3.2. *Let X, Y be two smooth vector fields defined in some domain $\Omega \subset \mathbb{R}^N$. Then, for $x_0 \in \Omega$ and σ, τ small enough we have*

$$e^{\tau X} e^{\sigma Y}(x_0) = e^{Z(\sigma Y, \tau X)}(x_0)$$

where

$$Z(\sigma Y, \tau X) = \sigma Y + \tau X + \frac{1}{2}[\sigma Y, \tau X] + \sum_{k=3}^{\infty} C_k(\sigma Y, \tau X) \tag{15}$$

and each term $C_k(x, y)$ is a suitable linear combination of iterated commutators of length k of x, y.

If X, Y were two commuting vector fields, then the composition of the exponentials $e^{\tau X} e^{\sigma Y}(x_0)$ would simply equal $e^{\sigma Y + \tau X}(x_0)$. The previous theorem specifies how this identity needs to be corrected to keep into account the possible noncommutativity of X and Y. The exact sense of the infinite series appearing in (15) could be made precise in terms of truncated sums with an error term controlled in terms of the smallness of σ, τ, but we do not go into details.

By means of the Campbell-Hausdorff formula, Hörmander proves the following two key technical results:

Lemma 3.3 (see [7, Lemma 4.5.]). *Let $\sigma \in (0,1)$, $n \geq 2$ an integer, X, Y two vector fields and $K \Subset \Omega$. Then there exists $C > 0$ such that for every $t > 0$ small enough and $u \in C_0^\infty(K)$,*

$$\left\| e^{t(X+Y)} u - u \right\|_{L^2} \leqslant C \left(\left\| e^{tX} u - u \right\|_{L^2} + \left\| e^{tY} u - u \right\|_{L^2} + \sum_{j=2}^{n-1} \left\| e^{t^j Z_j} u - u \right\|_{L^2} + t^{\sigma n} |u|_\sigma \right)$$

where Z_j are suitable iterated commutators of X, Y of length j.

Lemma 3.4 (see [7, Lemma 4.6.]). *Let $\sigma \in (0,1)$, I a multiindex and $K \Subset \Omega$. Then for every $t>0$ small enough and $u \in C_0^\infty(K)$,*

$$\left\| e^{t^{|I|}X_{[I]}}u - u \right\|_{L^2} \leqslant Ct \left(\sum_{j=1}^{q} |u|_{X_j,1} + |u|_\sigma \right).$$

Let us now sketch the proof of (14). The result will be achieved showing that for every $\sigma > 0$ there exists $c > 0$ such that

$$|u|_{X,1/s} \leqslant c \left\{ \sum_{j=1}^{q} |u|_{X_j,1} + |u|_\sigma \right\}. \tag{16}$$

Once (16) is established, an interpolation inequality of the kind

$$|u|_\sigma \leqslant \delta \, |u|_\sigma + c_\delta \, \|u\|_{L^2}$$

for every $\delta \in (0,1)$, allows to get (14). So, we are reduced to the proof of (16).

If $\frac{1}{s} < \sigma$ this is trivial because

$$|u|_{X,1/s} \leqslant c_1 \, |u|_{X,\sigma} \leqslant c_2 \, |u|_\sigma \, ,$$

so assume $\sigma \leqslant \frac{1}{s}$.

If X is a commutator $X_{[I]}$ of the generators $X_1,\dots,X_q$ (for $|I| \leqslant s$), then by Lemma 3.4,

$$\begin{aligned} |u|_{X_{[I]},1/s} &= \sup_{0<t<\varepsilon} \frac{\left\| e^{tX_{[I]}}u - u \right\|_{L^2}}{|t|^{1/s}} = \sup_{0<t<\varepsilon} \frac{\left\| e^{t^{|I|}X_{[I]}}u - u \right\|_{L^2}}{|t|^{|I|/s}} \\ &\leqslant \sup_{0<t<\varepsilon} Ct^{1-\frac{|I|}{s}} \left(\sum_{j=1}^{r} |u|_{X_j,1} + |u|_\sigma \right) = C \left(\sum_{j=1}^{r} |u|_{X_j,1} + |u|_\sigma \right) \end{aligned}$$

since $1 - \frac{|I|}{s} \geq 0$

So, let us show that this estimate remains true passing from commutators to linear combinations of commutators. We will concentrate on showing how one can extend this estimate from two vector fields to their sum. In other words, assume that X,Y satisfy (16), and let us show that the same is true for $X+Y$. By Lemma 3.3,

$$\begin{aligned} &|u|_{X+Y,1/s} = \sup_{0<t<\varepsilon} \frac{\left\| e^{t(X+Y)}u - u \right\|_{L^2}}{|t|^{1/s}} \\ &\leqslant \sup_{0<t<\varepsilon} C|t|^{-1/s} \left(\left\| e^{tX}u - u \right\|_{L^2} + \left\| e^{tY}u - u \right\|_{L^2} + \sum_{j=2}^{n-1} \left\| e^{t^j Z_j}u - u \right\|_{L^2} + t^{\sigma n} |u|_\sigma \right) \\ &\leqslant C \left\{ |u|_{X,1/s} + |u|_{Y,1/s} + \sum_{j=2}^{n-1} \sup_{0<t<\varepsilon} |t|^{-1/s} \left\| e^{t^j Z_j}u - u \right\|_{L^2} + \sup_{0<t<\varepsilon} |t|^{\sigma n - 1/s} |u|_\sigma \right\}. \end{aligned}$$

For $\sigma > 0$ and n large enough we can assume $\sigma n - 1/s \geq 0$, so

$$\sup_{0<t<\varepsilon} |t|^{\sigma n - 1/s} |u|_\sigma \leqslant |u|_\sigma$$

and we are left to bound the terms

$$\sum_{j=2}^{n-1} \sup_{0<t<\varepsilon} \frac{\left\| e^{t^j Z_j} u - u \right\|_{L^2}}{t^{1/s}} = \sum_{j=2}^{n-1} |u|_{Z_j, 1/sj}$$

where Z_j are suitable iterated commutators of X, Y of length $j \geq 2$. Recall that the estimate

$$|u|_{Z_j, 1/sj} \leqslant c\, |u|_\sigma \leqslant c \left\{ \sum_{j=1}^{q} |u|_{X_j, 1} + |u|_\sigma \right\}$$

is trivial as soon as $\frac{1}{sj} \leqslant \sigma$. For instance, if $\frac{1}{2s} \leqslant \sigma$ we are done. Otherwise we can proceed iteratively, and reproducing the same procedure arrive in a finite number k of steps to a situation where $\frac{1}{2^k s} \leqslant \sigma$. This concludes the proof.

4. Proof of regularity estimates for sublaplacians on Carnot groups

A special class of operators of the kind "sum of squares of Hörmander's vector fields" is that of sublaplacians on Carnot groups. These operators are important both for themselves and as model operators of more general Hörmander operators, as was first recognized in [6], [5][2].

Let us introduce the framework.

A *homogeneous group* (in $\mathbb{R}^N$) is a Lie group $\left(\mathbb{R}^N, \circ\right)$ (where $\circ$ is thought as "translation") endowed with a family $\{D_\lambda\}_{\lambda>0}$ of group automorphisms ("dilations") given by:

$$D_\lambda (x_1, x_2, ..., x_N) = (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, ..., \lambda^{\alpha_N} x_N) \tag{17}$$

for integers $1 = \alpha_1 \leqslant \alpha_2 \leqslant ... \leqslant \alpha_N$.

We will denote by $\mathbb{G} = \left(\mathbb{R}^N, \circ, D_\lambda\right)$ this structure. The number

$$Q = \sum_{i=1}^{N} \alpha_i$$

is called *homogeneous dimension* of $\mathbb{G}$.

[2]For an introductory explanation of the last statement, see for instance [3, Chap. 3].

On every homogeneous group we can define (in several different ways) a *homogeneous norm* $\|\cdot\|$ such that, for every $x,y\in\mathbb{G}$,

$$\begin{aligned} \|x\| &\geqslant 0 \text{ and } (\|x\|=0 \iff x=0); \\ \left\|x^{-1}\right\| &= \|x\| \\ \|D_\lambda(x)\| &= \lambda\|x\| \text{ for every } \lambda>0 \\ \|x\circ y\| &\leqslant c(\|x\|+\|y\|) \end{aligned}$$

for some constant $c\geq 1$.

Example 4.1 (The Heisenberg group $\mathbb{H}^n$)**.** This is the most famous homogeneous group. In $\mathbb{R}^{n+n+1}\ni(x,y,t)$, let

$$\begin{aligned} (x,y,t)\circ(x',y',t') &= \left(x+x',y+y',t+t'-2\left(x\cdot y'-x'\cdot y\right)\right) \\ D_\lambda(x,y,t) &= \left(\lambda x,\lambda y,\lambda^2 t\right). \end{aligned}$$

A tedious computation shows that $\circ$ is actually a (noncommutative) group operation. It is easy to see that

$$(x,y,t)^{-1}=(-x,-y,-t)$$

and that $\{D_\lambda\}_{\lambda>0}$ is a family of group automorphisms. Note that $N=2n+1$ while $Q=2n+2$. In this case one can define a homogeneous norm letting

$$\|(x,y,t)\|=\sqrt[4]{(x^2+y^2)^2+t^2}.$$

A differential operator P (defined in the whole $\mathbb{R}^N$) is said *left invariant* if for every smooth function u

$$P(\tau_y u)(x)=\tau_y(Pu(x))\ \forall x,y\in\mathbb{R}^N,$$

where

$$\tau_y u(x)=u(y\circ x).$$

Analogously one defines *right invariant operators*.

Also, P is said *β–homogeneous* (for $\beta\in\mathbb{R}$) if for every smooth function u

$$P(u(D_\lambda(x)))=\lambda^\beta(Pu)(D_\lambda(x))\ \forall\lambda>0,x\in\mathbb{R}^N\setminus\{0\}.$$

Let us denote by X_i ($i=1,2,\dots,N$) the only left invariant vector field which agrees with ∂_{x_i} at 0.

Analogously, let X_i^R denote the only right invariant vector field which agrees with ∂_{x_i}, and therefore with X_i, at 0.

Example 4.2. On the Heisenberg group $\mathbb{H}^1$ (see Example 4.1), we can compute

$$\begin{array}{ll} X_1 = \partial_x + 2y\partial_t & X_1^R = \partial_x - 2y\partial_t \\ X_2 = \partial_y - 2x\partial_t & X_2^R = \partial_y + 2x\partial_t \\ X_3 = \partial_t & X_3^R = \partial_t \end{array}$$

Henceforth we will assume that for some $q < N$ the vector fields $X_1, \dots, X_q$ are 1-homogeneous and their iterated commutators up to step s satisfy Hörmander's condition. Then we say that $\mathbb{G}$ is a *Carnot group of step s with generators* $X_1, \dots, X_q$, and that

$$\mathcal{L} = \sum_{i=1}^{q} X_i^2$$

is the *canonical sublaplacian* on $\mathbb{G}$.

Example 4.3 (The Kohn Laplacian on the Heisenberg group)**.** On the Heisenberg group $\mathbb{H}^n$ (see Example 4.1), we have (generalizing the computation in the previous example)

$$X_i = \partial_{x_i} + 2y_i\partial_t; Y_i = \partial_{y_i} - 2x_i\partial_t; [X_i, Y_i] = -4\partial_t.$$

The canonical sublaplacian is

$$\mathcal{L} = \sum_{i=1}^{n} \left(X_i^2 + Y_i^2\right).$$

In this situation $q = 2n < N = 2n+1; s = 2$.

We are now going to describe the proof of Hörmander's theorem for sublaplacians in Carnot groups given in [2].

The starting remark, easy but fundamental, is the following:

Proposition 4.4. *Any two differential operators on $\mathbb{G}$, $\mathcal{L}, \mathcal{R}$ left and right invariant, respectively, commute:*

$$\mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}.$$

Example 4.5. On the Heisenberg group $\mathbb{H}^1$, we have already computed:

$$\begin{array}{ll} X_1 = \partial_x + 2y\partial_t & X_1^R = \partial_x - 2y\partial_t \\ X_2 = \partial_y - 2x\partial_t & X_2^R = \partial_y + 2x\partial_t \\ X_3 = \partial_t & X_3^R = \partial_t \end{array}$$

One can check that, for instance,

$$[X_1, X_2] = -4X_3 \neq 0, \text{ but } \left[X_1, X_2^R\right] = 0.$$

Let us start again with the basic computation leading to (2). In Carnot groups, an easy homogeneity argument implies that the transposed of a generator X_i is just $-X_i$. Then we can write, for every $u \in C_0^\infty(\mathbb{G})$,

$$\sum_{j=1}^{q} \left\| X_j u \right\|_{L^2}^2 = -\sum_{j=1}^{q} \int \left(X_j^2 u \right) u dx \leqslant \left\| \mathcal{L}u \right\|_{L^2} \left\| u \right\|_{L^2}$$

hence

$$\left\| u \right\|_{W_X^{1,2}} \leqslant c \left\{ \left\| \mathcal{L}u \right\|_{L^2} + \left\| u \right\|_{L^2} \right\}$$

with an absolute constant c independent of the support of u (compare with (2)).

Now, assume for a moment that we knew the (apparently similar) estimate

$$\left\| u \right\|_{W_{X^R}^{1,2}} \leqslant c \left(\left\| \mathcal{L}u \right\|_{L^2} + \left\| u \right\|_{L^2} \right) \; \forall u \in C_0^\infty(\mathbb{G})$$

(we are bounding the right invariant derivatives $X_i^R u$, but still in terms of $\mathcal{L}u$). We could then apply this estimate to the functions $X_{i_1}^R X_{i_2}^R ... X_{i_k}^R u$, getting

$$\left\| X_{i_1}^R X_{i_2}^R ... X_{i_k}^R u \right\|_{W_{X^R}^{1,2}} \leqslant c \left(\left\| \mathcal{L} X_{i_1}^R X_{i_2}^R ... X_{i_k}^R u \right\|_{L^2} + \left\| X_{i_1}^R X_{i_2}^R ... X_{i_k}^R u \right\|_{L^2} \right)$$

and since $X_{i_1}^R X_{i_2}^R ... X_{i_k}^R$ and $\mathcal{L}$ commute (because they are a right-invariant and a left-invariant operator)

$$= c \left(\left\| X_{i_1}^R X_{i_2}^R ... X_{i_k}^R \mathcal{L}u \right\|_{L^2} + \left\| X_{i_1}^R X_{i_2}^R ... X_{i_k}^R u \right\|_{L^2} \right)$$

so that

$$\left\| u \right\|_{W_{X^R}^{k+1,2}} \leqslant c \left(\left\| \mathcal{L}u \right\|_{W_{X^R}^{k,2}} + \left\| u \right\|_{W_{X^R}^{k,2}} \right)$$

and, iteratively

$$\left\| u \right\|_{W_{X^R}^{k+1,2}} \leqslant c \left(\left\| \mathcal{L}u \right\|_{W_{X^R}^{k,2}} + \left\| u \right\|_{L^2} \right).$$

So, measuring the degree of regularity of u in terms of $\mathcal{L}u$ (with $\mathcal{L}$ left invariant) using right invariant derivatives, apparently trivializes the problem. The issue is that we actually *do not* have the estimate

$$\left\| u \right\|_{W_{X^R}^{1,2}} \leqslant c \left(\left\| \mathcal{L}u \right\|_{L^2} + \left\| u \right\|_{L^2} \right) \; \forall u \in C_0^\infty(\mathbb{G}).$$

We will see that nevertheless it is possible to control the regularity of u using right invariant vector fields, but this requires asking a higher regularity to $\mathcal{L}u$. The main quantitative estimate proved in [2] is the following:

Theorem 4.6. *Let $u \in W^{1,2}_{X,loc}\left(\mathbb{R}^N\right)$ an let ζ_1, ζ_2 be cutoff functions such that $\zeta_1 \prec \zeta_2$. Let s be the step of the Lie algebra and k be any nonnegative integer.*

(i) If $\mathcal{L}u \in W^{k+s^2-1,2}_{X^R,loc}\left(\mathbb{R}^N\right)$, then $u \in W^{k,2}_{X^R,loc}\left(\mathbb{R}^N\right)$ and

$$\left\|\zeta_1 u\right\|_{W^{k,2}_{X^R}(\mathbb{R}^N)} \leqslant c\left\{\left\|\zeta_2 \mathcal{L}u\right\|_{W^{k+s-1,2}_{X^R}(\mathbb{R}^N)} + \left\|\zeta_2 u\right\|_{L^2(\mathbb{R}^N)}\right\}. \tag{18}$$

(ii) In particular, for every open subset $\Omega \subset \mathbb{R}^N$, $\mathcal{L}u \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega)$.

Assertion (ii) in the above theorem still relies on Hörmander's condition: Euclidean derivatives of any order can be expressed in terms of derivatives (of much higher order) with respect to the vector fields X_i or X_i^R. Therefore, if $\zeta_1 u$ belongs to $W^{k,2}_{X^R}\left(\mathbb{R}^N\right)$ for every k, then it also belongs to $W^{k,2}\left(\mathbb{R}^N\right)$ for every k, and therefore is smooth.

Let us compare this result with Theorem 1.2, containing higher order, localized, subelliptic estimates. A first difference is that Theorem 4.6 applies to functions in $W^{1,2}_{X,loc}\left(\mathbb{R}^N\right)$ and not to general distributions. In this approach, regularization of distributional solutions is a second step in the proof of hypoellipticity, that will not be discussed here. Apart from this fact, the relevant difference is that here the regularity of functions is measured in the scale of anisotropic Sobolev spaces of integral order $W^{k,2}_{X^R}\left(\mathbb{R}^N\right)$, adapted to the right-invariant versions of the vector fields defining the operator, instead of isotropic Sobolev spaces of fractional order $H^\sigma\left(\mathbb{R}^N\right)$.

Let us discuss the line of the proof of Theorem 4.6.

We handle Sobolev norms by means of equivalent norms based on finite difference operators. For a fixed increment $h \in \mathbb{G}$ let

$$\Delta_h u(x) = u(x \circ h) - u(x)$$
$$\widetilde{\Delta}_h u(x) = u(h \circ x) - u(x).$$

and note that Δ_h is left invariant while $\widetilde{\Delta}_h$ is right invariant.

Recall that the exponential $\operatorname{Exp}(tX)$ of a vector field X on a Carnot group is defined, for every $t \in \mathbb{R}$, by:

$$\operatorname{Exp}(tX) = f(t)$$

where

$$\begin{cases} f'(t) = X(f(t)) \\ f(0) = 0. \end{cases}$$

In other words, $\operatorname{Exp}(tX) = e^{tX}(0)$ where $e^{tX}(x)$ is defined as in the previous section.

If $h = \operatorname{Exp}(tX_i)$, then one can easily prove that

$$\|\Delta_h u\|_{L^2} \leqslant |t| \, \|X_i u\|_{L^2} \leqslant \|h\| \, \|X_i u\|_{L^2} \tag{19}$$

where $\|h\|$ is a homogeneous norm on $\mathbb{G}$, and

$$\|X_i u\|_{L^2} \leqslant \sup_{\substack{h=\operatorname{Exp}(tX_i)\\ 0<|t|<1}} \frac{\|\Delta_h u\|_{L^2}}{\|h\|}; \quad \left\|X_i^R u\right\|_{L^2} \leqslant \sup_{\substack{h=\operatorname{Exp}(tX_i)\\ 0<|t|<1}} \frac{\left\|\widetilde{\Delta}_h u\right\|_{L^2}}{\|h\|}.$$

We would like to control the increment of u in every direction, and not just in the directions of the vector fields, in terms of $\|u\|_{W_X^{1,2}}$ (compare with Proposition 1.1 and Proposition 3.1). This is possible exploiting the following global connectivity property which holds for Hörmander vector fields on Carnot groups:

Proposition 4.7. *Let $\mathbb{G} = \left(\mathbb{R}^N, \circ, D_\lambda\right)$ be a Carnot group with generators $X_1, ..., X_q$. Then, there exist constants $M, c > 0$ and for every $x \in \mathbb{R}^N$ there exist numbers $t_1, t_2, ..., t_M$ such that*

$$x = \operatorname{Exp}(t_M X_{k_M}) \circ \cdots \circ \operatorname{Exp}(t_2 X_{k_2}) \circ \operatorname{Exp}(t_1 X_{k_1})$$

with $k_1, ..., k_M \in \{1, 2, ..., q\}$ and

$$|t_i| \leqslant c \, \|x\|.$$

This follows by a result of local approximation of commutators by "quasi-exponential maps", that is suitable compositions of exponential maps of the generators, which can be proved for a general set of vector fields, but in the context of Carnot groups assumes a global and more quantitative form. In particular, we note that the above Proposition can be proved without making use of the Campbell-Hausdorff formula. From (19) and Proposition 4.7 we get, for *every $h \in \mathbb{R}^N$*,

$$\|\Delta_h u\|_{L^2} \leqslant \|h\| \, \|\nabla_X u\|_{L^2} \leqslant c \, \|h\| \left(\|\mathcal{L}u\|_{L^2} + \|u\|_{L^2}\right). \tag{20}$$

Since our goal is to bound some norm of $X_i^R u$, which is related to the finite difference $\widetilde{\Delta}_h$, we need instead a control over $\left\|\widetilde{\Delta}_h u\right\|_{L^2}$. This is possible in view of a technical lemma:

Lemma 4.8. *Let $u \in L^2(\mathbb{G}), U \Subset \mathbb{G}$, $\operatorname{sprt} u \subset U$. There exists $c > 0$, depending on U, such that*

$$\sup_{0<\|h\|\leqslant 1} \frac{\left\|\widetilde{\Delta}_h u\right\|_2}{\|h\|^{1/s}} \leqslant c \sup_{0<\|h\|\leqslant 1} \frac{\|\Delta_h u\|_2}{\|h\|}$$

whenever the right hand side is finite. (Here s is the step of the Lie algebra of $\mathbb{G}$).

Then (20) rewrites as:

$$\sup_{0<\|h\|\leqslant 1}\frac{\left\|\widetilde{\Delta}_h u\right\|_2}{\|h\|^{1/s}}\leqslant c\left(\|\mathcal{L}u\|_{L^2}+\|u\|_{L^2}\right), \tag{21}$$

which closely resembles the basic subelliptic estimate proved by Hörmander (see Proposition 3.1), with the Euclidean translations τ_h replaced by left translations with respect to the group law, and $|h|$ replaced by the homogeneous norm $\|h\|$.

Looking at (21), we note that, in order to get a control on $\left\|X_i^R u\right\|_2$, we would need $\|h\|$ instead of $\|h\|^{1/s}$ on the quotient at the left hand side. Then, for fixed $h=\operatorname{Exp}(tX_i)$, $i=1,\dots,q$, we consider the iterated finite difference

$$\widetilde{\Delta}_h^m u=\widetilde{\Delta}_h\widetilde{\Delta}_h^{m-1}u.$$

From (21) we get

$$\left\|\widetilde{\Delta}_h^m u\right\|_{L^2}\leqslant c\,\|h\|^{1/s}\left(\left\|\mathcal{L}\widetilde{\Delta}_h^{m-1}u\right\|_{L^2}+\left\|\widetilde{\Delta}_h^{m-1}u\right\|_{L^2}\right)$$

Since $\widetilde{\Delta}_h$ and $\mathcal{L}$ commute (because $\widetilde{\Delta}_h$ is right invariant and $\mathcal{L}$ is left invariant), by iteration we get

$$\left\|\widetilde{\Delta}_h^m u\right\|_{L^2}\leqslant c\,\|h\|^{m/s}\left(\|\mathcal{L}u\|_{W^{m-1,2}_{X^R}(\mathbb{G})}+\|u\|_{L^2}\right).$$

In particular, for $m=s+1$,

$$\left\|\widetilde{\Delta}_h^{s+1} u\right\|_{L^2}\leqslant c\,\|h\|^{1+1/s}\left(\|\mathcal{L}u\|_{W^{s,2}_{X^R}(\mathbb{G})}+\|u\|_{L^2}\right) \tag{22}$$

where the power $\|h\|^{1+1/s}$, with an exponent greater than 1, is useful in view of the following fact:

Lemma 4.9 (Marchaud inequality for Carnot groups)**.** *If, for some numbers $\alpha\in(1,2)$, $A>0$, positive integer m and $u\in L^2(\mathbb{G})$ we have*

$$\left\|\widetilde{\Delta}_h^m u\right\|_{L^2}\leqslant A\,\|h\|^{\alpha}\text{ for every }h\in\mathbb{G},$$

then for some $c>0$ independent of u,

$$\left\|\widetilde{\Delta}_h u\right\|_{L^2}\leqslant c\,(A+\|u\|_{L^2})\,\|h\|\text{ for every }h\in\mathbb{G}.$$

Hence from (22) we get

$$\left\| \widetilde{\Delta}_h u \right\|_{L^2} \leqslant c \|h\| \left(\|\mathcal{L}u\|_{W^{s,2}_{X^R}} + \|u\|_{L^2} \right)$$

and therefore

$$\|u\|_{W^{1,2}_{X^R}} \leqslant c \left(\|\mathcal{L}u\|_{W^{s,2}_{X^R}} + \|u\|_{L^2} \right).$$

More precisely, the estimate that we get is the localized version

$$\|\zeta_1 u\|_{W^{1,2}_{X^R}} \leqslant c \left\{ \|\zeta_2 \mathcal{L}u\|_{W^{s,2}_{X^R}} + \|\zeta_2 u\|_{L^2} \right\}$$

for $\zeta_1 \prec \zeta_2$, and this is the first step to prove, iteratively, the desired estimate. But iteration is now very easy because the vector fields X^R commute with $\mathcal{L}$, as already noted. Therefore the previous inequality immediately implies:

$$\|\zeta u\|_{W^{k,2}_{X^R}(\mathbb{R}^N)} \leqslant c \left\{ \|\zeta_1 \mathcal{L}u\|_{W^{k+s-1,2}_{X^R}(\mathbb{R}^N)} + \|\zeta_1 u\|_{L^2(\mathbb{R}^N)} \right\}.$$

5. Final remarks

Let us now make a comparison between the three proofs that we have discussed so far.

Proving the hypoellipticity of Hörmander's operators poses several problems. Two of these can be stated, in very general terms, as follows.

1. Starting with some information about the regularity of $\mathcal{L}u$, that is about the regularity of u in the directions of the vector fields $X_1, \ldots, X_q$ alone, we want to get some information about the regularity of u in *every* direction.

This issue is responsible of the appearance of norms or seminorms weighting derivatives *of fractional order*, which happens in all the three proofs. It is also responsible of the use, within the same proof, of several different norms or seminorms, which happens both in Hörmander's proof and in the proof on Carnot groups: transferring the regularity information from some directions to every direction is a delicate task which requires subsequent steps, involving different ways of measuring the regularity of a function. Under this respect, Kohn's proof has the advantage of using a single scale of spaces.

2. One needs to interchange the order of operators which actually do not commute, controlling the error term which is introduced.

This fact happens both in Hörmander's proof and in Kohn's proof. In the first one, the use of Campbell-Hausdorff formula is the basic ingredient to handle compositions of exponential maps and get a control on the error term. In the second one, the theory of pseudodifferential operators ensures that the commutator of two operators of types m_1, m_2 is an operator of type $m_1 + m_2 - 1$, which is the key fact to control the error term. The proof on Carnot groups, instead, avoids almost completely this problem, exploiting the fact that a left invariant operator and a right invariant operator actually commute. Lemma 4.8, which is not a difficult result, connects difference quotients constructed by right and left translations. This is the key point which makes this proof much easier than the other two.

Both Kohn's proof and the proof on Carnot groups, obviously, take advantage of the ideas contained in the original proof by Hörmander. Kohn's argument, as already noted, applies to the whole proof the techniques of pseudodifferential operators, which Hörmander uses in a portion of the proof. The proof on Carnot groups imitates Hörmander's technique of fractional differentiation along vector fields, but this is implemented exploiting the underlying structure of Carnot groups, which makes the arguments much easier. As a result, both these proofs are technically more homogeneous than the original one, which relies on several tools of different kinds.

REFERENCES

[1] M. Bramanti: An invitation to hypoelliptic operators and Hörmander's vector fields. Springer Briefs in Mathematics. 2014.

[2] M. Bramanti, L. Brandolini: A proof of Hörmander' theorem for sublaplacians on Carnot groups. Nonlinear Analysis Series A: Theory, Methods and Applications. 126 (2015), 170–200.

[3] M. Bramanti, L. Brandolini: Hörmander operators, book, to appear.

[4] S.C. Chen, M.C. Shaw: Partial differential equations in several complex variables. AMS/IP Studies in Advanced Mathematics, 19. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001.

[5] G. B. Folland: Subelliptic estimates and function spaces on nilpotent Lie groups. Ark. Mat. 13 (1975), no. 2, 161–207.

[6] G. B. Folland, E. M. Stein: Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group. Comm. Pure Appl. Math. 27 (1974), 429–522.

[7] L. Hörmander: Hypoelliptic second order differential equations. Acta Math. 119 (1967), 147-171.

[8] J. J. Kohn: Pseudo-differential operators and hypoellipticity. Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), pp. 61–69. Amer. Math. Soc., Providence, R.I., 1973.

[9] O. A. Oleĭnik, E. V. Radkevič: Second order equations with nonnegative characteristic form. Plenum Press, New York-London, 1973. (Translated from the original version in Russian, 1969).

[10] L. P. Rothschild, E. M. Stein: Hypoelliptic differential operators and nilpotent groups. Acta Math. 137 (1976), no. 3-4, 247–320.

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