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MEAN-CONVEX SETS AND MINIMAL BARRIERS

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A mean-convex set is locally a barrier for minimal surfaces but can fail to be a *global* barrier. In this note we suggest how to extend to general dimensions the results of a previous unpublished manuscript [25] on the characterization of the global barriers for minimal surfaces.

0. Introduction

In this note we continue the analysis on the relation between mean-convexity and the maximum principle for minimal surfaces started in an unpublished manuscript [25]. In particular, we show how to extend to arbitrary dimensions the results proven therein and, for the sake of completeness, we also report some of the results contained in the previous note [25].

Let $\Omega \subset M^n$ be an open subset with smooth boundary $\partial\Omega$. We say that Ω has mean-convex boundary, and we shortly call Ω *mean-convex*, if the mean-curvature vector $\vec{H}_{\partial\Omega}$ is a nonpositive multiple of the outward-pointing unit normal $\nu_{\partial\Omega}$ to Ω at every point of $\partial\Omega$:

$$\vec{H}_{\partial\Omega} \cdot \nu_{\partial\Omega} \leq 0.$$

Mean-convex sets are locally *barriers* for minimal surfaces, because they are locally parametrized by supersolutions of the minimal surface equation, and

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can be used as barriers for the existence of minimal surfaces (see, e.g., the work by Meeks and Yau [19]).

However, a mean-convex set Ω may fail to be a *global barrier*. There are simple examples for this phenomenon due to topological obstructions, but there are also instances of the failure of this global barrier principle also under the most restrictive topological assumptions (see the counterexample in § 1).

This arises the question: how can we characterize *global barrier* for minimal hypersurfaces? We say that a set $\Theta \subset \mathbb{R}^n$ is *global barrier* if:

$$\Sigma \text{ minimal hypersurface, } \partial\Sigma \subset \Theta \quad \Longrightarrow \quad \Sigma \subset \Theta,$$

where the notion of "minimal hypersurface" has to been suitably specified (stationary submanifolds with small singular sets, see § 1). In this paper we address this issue by looking at the minimal barrier containing a set Ω, here called the *mean-convex hull* of Ω:

$$\Omega^{mc} := \bigcap_{\Omega \subset \Theta \in \mathcal{A}} \Theta, \tag{0.1}$$

where $\mathcal{A}$ denotes the family of global barriers in $\mathbb{R}^n$. Clearly, by the convex hull property for minimal surfaces, the closed convex hull Ω^{co} is a global barrier containing Ω, hence the intersection in (0.1) is non-trivial. Nevertheless, Ω^{co} may not be the smallest one (see the examples in § 1). On the contrary, a mean-convex set does not need to coincide with its mean-convex hull.

Similar notions of mean-convex hull have been introduced for minimal hypersurfaces spanning a fixed extreme boundary, see e.g. [7]. The main result of the paper is to prove a partial regularity result for Ω^{mc}.

Theorem 0.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded closed set with $\partial\Omega \in C^{1,1}$. Then, $\partial\Omega^{mc}$ is $C^{1,1}$ regular up to a $(n-8)$-dimensional singular set Σ. Moreover, $\partial\Omega^{mc} \setminus \Omega$ is a stable current with boundary in Ω which is regular in a neighborhood of its boundary.*

A straightforward consequence of the theorem is the following.

Corollary 0.2. *If $n \leq 7$, then Ω^{mc} is $C^{1,1}$ regular; if $n = 3$ and Ω is connected, Ω^{mc} is a homology ball.*

Heuristics of the proof: mean-curvature flow with obstacles

The main idea in the proof of Theorem 0.1introduced in [25] is to use an evolution approach trying to characterize the mean-convex hull in terms of the asymptotic evolution of a mean-curvature flow (MCF) with an obstacle. The rough idea is to show that

$$\Omega^{mc} = \lim_{t \to +\infty} F_t,$$

where F_t are sets containing Ω evolving according to the following equation for the normal velocity $\vec{v}_{F_t}$:

$$\vec{v}_{F_t}(x) = \begin{cases} \vec{H}_{\partial F_t}(x) & \text{if } x \in \partial F_t \setminus \Omega, \\ \max\left\{\vec{H}_{\partial F_t} \cdot \vec{\nu}_{F_t}, 0\right\} \vec{\nu}_{F_t} & \text{if } x \in \partial F_t \cap \Omega, \end{cases} \tag{0.2}$$

where $\vec{\nu}_{F_t}$ denotes the outward-pointing unit normal to ∂F_t.

In this paper, we suggest how to extend the results of [25] to higher dimensions, by making a different use of the stability estimate by Schoen and Simon [23] in combination with the regularity results for the classical obstacle problem for the area functional (see, e.g., [6]).

Since the appearance of the first version of this note, there has been various contributions to the definition of MCF with obstacles (see, e.g., [2, 17, 18, 22]), although a detailed analysis of the free boundary regularity has not been yet addressed. Moreover, in the same years several papers have appeared on the notion of minimal hull for minimal surfaces (e.g., [1, 12] and the reference therein) which seem to be closely related to the notion of mean-convex hull used here. Further investigations about this relation are of great interest.

1. Mean-convex sets and barriers

Throughout this section, Ω denotes a bounded closed set in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$. We let ν be the external unit normal to $\partial\Omega$ and $\vec{H}_{\partial\Omega}$ the mean curvature vector of $\partial\Omega$. In contrast with the examples we are going to discuss, we recall the following result (here the term disk refers to a smooth 2-dimensional surface with boundary, having the topology of the planar disk $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$).

Theorem 1.1 (Meeks & Yau [19]). *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded mean-convex set and $\Gamma \subseteq \partial\Omega$ a closed curve, null-homotopic in Ω. Then, there exists an embedded minimal disk $\Sigma \subseteq \Omega$ such that $\partial\Sigma = \Gamma$.*

1.1. Mean-convex $\neq$ global barrier

In general, a mean-convex set does not need to be a global barrier. We describe here a counterexample in the most restrictive hypotheses of Ω a mean-convex set homeomorphic to the 3-dimensional ball and Γ a simple Jordan curve in $\partial\Omega$. Our starting point is the well-known example of a Jordan curve bounding at least two different minimal disks (see, for example, [21, § 389]).

Let us fix cylindrical coordinates in $\mathbb{R}^3$:

$$(x,y,z) = (\rho\cos\theta, \rho\sin\theta, z) \quad \text{with} \quad (\theta,\rho,z) \in [0,2\pi) \times [0,+\infty) \times \mathbb{R}.$$

Figure 1: Curve bounding at least two minimal disks (approximate solutions drawn).

For $\theta_0 > 0$ a parameter to be fixed momentarily, let Ω_{θ_0} be the following closed set (see Figure 2 for two views of this domain):

$$\Omega_{\theta_0} := \Big\{(\theta,\rho,z) : \theta_0 \le \theta \le 2\pi,\ |z| \le L,\ a\cosh(z/a) \le \rho \le 1\Big\},$$

where $L > h := 0.6$ and $0 < a < 1$ are fixed in such a way that $a\cosh(L/a) < 1$. Note that such a choice of parameters is possible, for example $L = 0.62$ and $a = 0.5$. Let $\Gamma \subseteq \partial\Omega$ be the curve given by (see Figure 1 left):

$$\Gamma_{\theta_0} := \Big\{(\theta,1,z) : (\theta,z) \in \partial\big([2\theta_0, 2\pi] \times [-h,h]\big)\Big\}.$$

Figure 2: Two views of the mean-convex domain Ω_{θ_0} (on the left part the transparency shows the interior section).

It is well-known that the area minimizing surface with boundary two axial unitary circles on parallel planes distant $2h$ is the union of the two disks

$$D_+ := \big\{(\theta,\rho,h)\ :\ \theta \in [0,2\pi),\ \rho \in [0,1]\big\}, \tag{1.1}$$
$$D_- := \big\{(\theta,\rho,-h)\ :\ \theta \in [0,2\pi),\ \rho \in [0,1]\big\} \tag{1.2}$$

(see, for example, [21, § 389] and [24]).

By compactness of integral currents, the minimizers Σ_{θ_0} of the area with boundary Γ_{θ_0} converge as $\theta_0 \to 0$ to a current Σ with boundary the two circles ∂D_+, ∂D_-, and

$$\mathbf{M}(\Sigma) \le \liminf_{\theta_0 \to 0} \mathbf{M}(\Sigma_\theta), \tag{1.3}$$

(here $\mathbf{M}$ stands for the *mass* of a current, that is the analog of the volume measure in Geometric Measure Theory). It is a consequence of the Bridge Principle for minimal surfaces [27, Theorem 2.2] that $\Sigma = D_+ \cup D_-$. Indeed, if this is not the case, then being the two disks the absolute minimizers,

$$\mathbf{M}(D_+ \cup D_-) < \mathbf{M}(\Sigma) \overset{(1.3)}{\leq} \liminf_{\theta_0 \to 0} \mathbf{M}(\Sigma_\theta). \tag{1.4}$$

By the Bridge Principle, for every $\varepsilon > 0$ there exists $\theta_\varepsilon > 0$ and an integer rectifiable current T_ε such that $\partial T_\varepsilon = \Gamma_{\theta_\varepsilon}$ and

$$\mathbf{M}(T_\varepsilon) \leq \mathbf{M}(D_+ \cup D_-) + \varepsilon,$$

which together with (1.4) contrasts the minimizing property of $\Sigma_{\theta_\varepsilon}$ for ε sufficiently small.

By a simple consequence of the regularity theory for minimal surfaces, this convergence is smooth away from the points

$$(\theta, \rho, z) = (0, 1, \pm h),$$

and Σ_{θ_0} is contained in a neighborhood of

$$D_+ \cup D_- \cup \left\{(0,1,z) : |z| \leq h\right\},$$

for θ_0 sufficiently small. In particular, for θ_0 small enough, the minimizing disk with boundary Γ_{θ_0} resembles the surface in Figure 1 on the right, and therefore is not contained in Ω_{θ_0}. Both Ω_{θ_0} and Γ_{θ_0} are not smooth, but piecewise smooth. Nevertheless, since all the angles between the faces of Ω_{θ_0} are less than π, it is not difficult (though boring) to modify the above example and reduce to a smooth mean-convex domain and a smooth Jordan curve.

1.2. Mean-convex hull $\neq$ convex hull

It follows directly from the definition that in the plane the mean-convex hull (0.1) coincides with the convex hull. Nevertheless, a simple example shows that the two notions do not need to coincide in dimension $n \geq 3$. Consider the set contained between a vertical catenoid and two horizontal parallel planes, i.e.

$$\Omega = \left\{(x,y,z) \,:\, |z| \leq 1,\, x^2 + y^2 \leq \cosh(z)^2\right\} \subset \mathbb{R}^3,$$

(the fact that Ω is not smooth is not essential, for the example can be modified accordingly). Clearly,

$$\Omega^{co} = \{x^2 + y^2 \leq \cosh(1)^2\}.$$

Nevertheless, it is not difficult to show that $\Omega = \Omega^{mc}$. To see this, let Σ be a minimal hypersurface with $\partial\Sigma \subset \Omega$. By the convex hull property, every minimal surface with boundary in Ω is contained in Ω^{co}, hence, in particular, $\Sigma \subseteq \{|z| \leq 1\}$. On the other hand, consider the foliation by rescaled catenoids:

$$\{|z| \leq 1\} \setminus \Omega = \bigcup_{\lambda \geq 1} \mathrm{Cat}_\lambda,$$

where

$$\mathrm{Cat}_\lambda := \Big\{(x,y,z) \,:\, |z| \leq 1,\, x^2 + y^2 = \lambda^2 \cosh(z/\lambda)^2\Big\}.$$

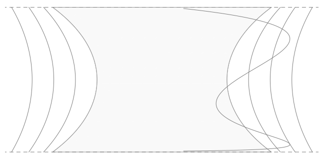

Figure 3: Catenoids' foliation.

Let $\lambda_{\max}$ the maximum λ such that $\Sigma \cap \mathrm{Cat}_\lambda \neq \emptyset$ and assume $\lambda_{\max} > 1$, i.e. Σ is not contained in Ω. By the strong maximum principle, it follows that $\Sigma \equiv \mathrm{Cat}_{\lambda_{\max}}$, thus contradicting $\partial\Sigma \subset \Omega$ and implying that $\Sigma \subset \Omega$, i.e. $\Omega = \Omega^{mc}$.

2. Mean curvature flow with obstacle

For the proof of Theorem 0.1, we need to develop a theory of weak mean curvature flow of Caccioppoli sets with obstacle, following closely the approach of Almgren, Taylor and Wang [3] and Luckhaus and Sturzenhecker [16].

We start recalling the few notions of Geometric Measure Theory which are needed in the sequel (more details on Caccioppoli sets can be found in the monograph [11]).

2.1. Caccioppoli sets

A measurable set $E \subset \mathbb{R}^n$ is said to be a Caccioppoli set if there exist sets $E_j \subset \mathbb{R}^n$ with smooth boundary $\partial E_j \in C^1$ such that $\chi_{E_j} \to \chi_E$ in $L^1(\mathbb{R}^n)$ and

$$\liminf_{j \to +\infty} \mathcal{H}^{n-1}(\partial E_j) < +\infty.$$

Here, as usual, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure and χ_E the characteristic function of the set E, namely

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Note that, according to the above definition, a Caccioppoli set is defined up to a set of Lebesgue measure zero, for $\chi_E \in L^1$ identifies an equivalent class of measurable functions. Nevertheless, we will always assume to have fixed a pointwise representative of E which satisfies the following condition:

$$x \in \partial E \quad \Longleftrightarrow \quad 0 < |B_r(x) \cap E| < \omega_n r^n \quad \forall\, r > 0,$$

where $|A|$ denotes the Lebesgue measure of a measurable set $A \subseteq \mathbb{R}^n$.

The measure of the boundary of E in a open set $\mathcal{O} \subset \mathbb{R}^n$, also called the perimeter of E in $\mathcal{O}$, is then given by the minimum limit of the measure in $\mathcal{O}$ of the boundaries of the approximating sets, i.e.

$$\mathrm{Per}\,(E,\mathcal{O}) := \inf \left\{ \liminf_{j\to+\infty} \mathcal{H}^{n-1}(\partial E_j \cap \mathcal{O}) \,:\, \chi_{E_j} \to \chi_E \ \text{ in } \ L^1(\mathcal{O}),\ \partial E_j \in C^1 \right\}.$$

We will often write $\mathrm{Per}\,(E)$ for $\mathrm{Per}\,(E,\mathbb{R}^n)$. Moreover, it turns out that, in case $\partial E \in C^1$, then $\mathrm{Per}\,(E,\mathcal{O}) = \mathcal{H}^{n-1}(\partial E \cap \mathcal{O})$, thus justifying the term "perimeter". An easy consequence of the definition is the inequality:

$$\mathrm{Per}\,(E \cup F,\mathcal{O}) + \mathrm{Per}\,(E \cap F,\mathcal{O}) \leq \mathrm{Per}\,(E,\mathcal{O}) + \mathrm{Per}\,(F,\mathcal{O}). \tag{2.1}$$

Finally, we will use often the following two properties of Caccioppoli sets.

1. *Lower semicontinuity*:

$$\mathrm{Per}\,(E,\mathcal{O}) \leq \liminf_{j\to+\infty} \mathrm{Per}\,(E_j,\mathcal{O}), \quad \forall\, \chi_{E_j} \to \chi_E \ \text{ in } \ L^1(\mathcal{O}).$$

2. *Compactness*: given $E_j \subseteq B_R \subset \mathbb{R}^n$ with $\sup_j \mathrm{Per}\,(E_j) < +\infty$, there exists $E \subset \mathbb{R}^n$ and a subsequence $(E_{j_k})_{k\in\mathbb{N}}$ such that

$$\chi_{E_{j_k}} \to \chi_E \quad \text{in} \quad L^1(\mathbb{R}^n), \quad \text{as } \ k \to +\infty.$$

2.2. Discrete in time approximate flow with obstacle

In what follows $\Omega \subset \mathbb{R}^n$ is a closed bounded set with $C^{1,1}$ boundary and $E_0 \subseteq \mathbb{R}^n$ is the initial bounded closed set of the evolution such that

$$|\partial E_0| = 0 \quad \text{and} \quad \Omega \subset E_0.$$

We define the approximate flow of time step $h > 0$ in the following way: we set $E_0^{(h)} := E_0$ and, given $E_i^{(h)}$ for some $i \in \mathbb{N}$, we let $E_{i+1}^{(h)}$ be a minimizer of the functional $\mathcal{F}(\cdot,h,E_i^{(h)})$ given by

$$\mathcal{F}(E,h,E_i^{(h)}) := \mathrm{Per}\,(E) + \int_{E \triangle E_i^{(h)}} \frac{\mathrm{dist}\big(x,\partial E_i^{(h)}\big)}{h}\, \mathrm{d}x,$$

where the minimum is taken among all the sets E containing Ω a.e.,

$$\mathcal{F}(E_{i+i}^{(h)}, h, E_i^{(h)}) = \min\left\{\mathcal{F}_{E_i^{(h)}}(E) \,:\, E \supset \Omega \text{ a.e.}\right\}.$$

It is clear that, thanks to the compactness and the semicontinuity properties (1) and (2) in § 2.1, this minimum problem is well-posed in the class of sets of finite perimeter and admits minimizers – note that the L^1 convergence implies the convergence almost everywhere for subsequences, thus preserving the constraint $E \supset \Omega$ in the limit. Notice, however, that uniqueness in general fails. The approximate flow is, hence, defined as:

$$E_t^{(h)} := E_{\lfloor t \rfloor}^{(h)} \quad \forall\, t \geq 0,$$

where $\lfloor t \rfloor \in \mathbb{N}$ is the integer part of t, namely $\lfloor t \rfloor \leq t < \lfloor t \rfloor + 1$.

2.3. Regularity of approximate flows

It follows from the regularity theory in geometric measure theory that the sets $E_t^{(h)}$ have $C^{1,1}$ boundaries up to a singular set of dimension at most $n-8$. To see this, first we note that the functional $\mathcal{F}(\cdot, h, E_i^{(h)})$ can be written in the following way:

$$\mathcal{F}(E, h, E_i^{(h)}) = \operatorname{Per}(E) + \int_{\mathbb{R}^n} u_{i,h}(x)\, \chi_E(x)\, \mathrm{d}x + \int_{\mathbb{R}^n} u_{i,h}(x)\, \chi_{E_i^{(h)}}(x)\, \mathrm{d}x, \quad (2.2)$$

where we set $u_{i,h} := h^{-1} d_i$ and d_i the signed distance from $\partial E_i^{(h)}$:

$$d_i(x) := \begin{cases} \operatorname{dist}\big(x, \partial E_i^{(h)}\big) & \text{if } x \notin E_i^{(h)}, \\ -\operatorname{dist}\big(x, \partial E_i^{(h)}\big) & \text{if } x \in E_i^{(h)}. \end{cases} \quad (2.3)$$

The last term in (2.2) is a constant not depending on E. Therefore, it turns out that $E_{i+1}^{(h)}$ is also a minimizer of the functional $\mathcal{G}(\cdot, h, E_i^{(h)})$:

$$\mathcal{G}(E, h, E_i^{(h)}) := \operatorname{Per}(E) + \int_{\mathbb{R}^n} u_{i,h}(x)\, \chi_E(x)\, \mathrm{d}x. \quad (2.4)$$

Note that

$$\mathcal{G}\Big(E \cap (E_i^{(h)})^{co}, h, E_i^{(h)}\Big) \leq \mathcal{G}\big(E, h, E_i^{(h)}\big),$$

with equality only if $E \subseteq (E_i^{(h)})^{co}$. Hence, it follows by a simple induction argument that

$$E_t^{(h)} \subset (E_0)^{co} \quad \forall\, t \geq 0. \quad (2.5)$$

In turns, this implies that the sets $E_i^{(h)}$ are uniform Λ-minimizers of the perimeter for $\Lambda = \sigma h^{-1}$, where $\sigma > 0$ is a given constant independent of h. Namely, there exists $R > 0$ such that for all $i \in \mathbb{N}$, $x \in \mathbb{R}^n$ and $0 < r < R$, it holds

$$\operatorname{Per}(E_i^{(h)}, B_r(x)) \le \operatorname{Per}(F, B_r(x)) + \sigma h^{-1} r^n \quad \forall\, F \triangle E_i^{(h)} \Subset B_r(x). \tag{2.6}$$

From the regularity theory of Λ-minimizers (see, e.g., [26]), it follows that $\partial E_i^{(h)} \in C^{1,1}$ up to a closed set of dimension at most $n-8$ and the following density estimates hold (see [26, Proposition 3.4]):

$$\frac{\omega_{n-1}}{n} - \sigma h^{-1} r \le \frac{\min\left\{|E_i^{(h)} \cap B_r(x)|, |E_i^{(h)} \setminus B_r(x)|\right\}}{r^n} \qquad \forall\, x \in \mathbb{R}^n, \tag{2.7}$$

$$\omega_{n-1} - (n-1)\,\sigma h^{-1} r \le \frac{\operatorname{Per}(E_i^{(h)}, B_r(x))}{r^{n-1}} \qquad \forall\, x \in \partial E_i^{(h)}. \tag{2.8}$$

2.4. Uniform distance estimate

The main analytical estimate exploited in the proof of Theorem 0.1 is the following on the distance between two successive boundaries of the approximate flow.

Proposition 2.1. *There exists a dimensional constant $\gamma(n) > 0$, such that*

$$\operatorname{dist}\big(\partial E_{i+1}^{(h)}, \partial E_i^{(h)}\big) \le \gamma(n)\, \sqrt{h} \quad \forall\, i \in \mathbb{N},\ \forall\, h > 0. \tag{2.9}$$

The proof of Proposition 2.1 follows by a simple adaptation of the arguments in [16]. For readers' convenience, we give here a detailed proof.

We premise the following density estimate for one-sided minimizers of the perimeter. The estimate can be easily deduce from the original arguments by De Giorgi exploited for minimizers [8] (see also [11]).

Lemma 2.2. *There exists a dimensional constant $\theta = \theta(n) > 0$ with this property. Let $E \subset B_R \subset \mathbb{R}^n$ be a Caccioppoli set such that $0 \in \partial E$ and*

$$\operatorname{Per}(E, B_R) \le \operatorname{Per}(F, B_R) \quad \forall\, E \subseteq F,\ F \setminus E \Subset B_R. \tag{2.10}$$

Then,

$$\theta\, r^n \le |B_r \setminus E| \quad \forall\, 0 < r < R. \tag{2.11}$$

Proof. For $r < R$, set $F_r := E \cup B_r$. Note that, for almost every $r > 0$, it holds

$$\operatorname{Per}(F_r) = \mathcal{H}^{n-1}(\partial B_r \setminus E) + \operatorname{Per}(E, \mathbb{R}^n \setminus B_r(x)),$$
$$\operatorname{Per}(B_r \setminus E) = \mathcal{H}^{n-1}(\partial B_r \setminus E) + \operatorname{Per}(E, B_r),$$
$$\operatorname{Per}(E) = \operatorname{Per}(E, B_r) + \operatorname{Per}(E, \mathbb{R}^n \setminus B_r).$$

Indeed, if E were smooth, these formulas follow for all the r such that B_r and E have transversal intersections. Otherwise one can argue by approximation.

Using now (2.10), we deduce that, for almost every $r > 0$,

$$\begin{aligned}
\operatorname{Per}(F_r) &= \mathcal{H}^{n-1}(\partial B_r \setminus E) + \operatorname{Per}(E, \mathbb{R}^n \setminus B_r(x)) \\
&\geq \operatorname{Per}(E) \\
&= \operatorname{Per}(E, B_r) + \operatorname{Per}(E, \mathbb{R}^n \setminus B_r) \\
&= \operatorname{Per}(B_r \setminus E) - \mathcal{H}^{n-1}(\partial B_r \setminus E) + \operatorname{Per}(E, \mathbb{R}^n \setminus B_r(x)).
\end{aligned} \tag{2.12}$$

By the isoperimetric inequality [11, Corollary 1.29], there exists a dimensional constant $C > 0$, such that

$$C\,|B_r \setminus E|^{\frac{n-1}{n}} \leq \operatorname{Per}(B_r \setminus E) \overset{(2.12)}{\leq} 2\,\mathcal{H}^{n-1}(\partial B_r \setminus E). \tag{2.13}$$

Setting $f(r) := |B_r \setminus E|$, by the coarea formula it holds

$$\mathcal{H}^{n-1}(\partial B_r \setminus E) = f'(r) \quad \text{for a.e. } r > 0.$$

Hence, (2.13) reads as

$$f(r)^{\frac{n-1}{n}} \leq 2C^{-1} f'(r).$$

Integrating (2.13) we get the desired (2.11) for a dimensional constant $\theta > 0$. ☐

Using Lemma 2.2, we can give a proof of the uniform bound in Proposition (2.1).

Proof of Proposition 2.1. We claim that (2.9) holds for

$$\gamma := 2\sqrt{\frac{n\,\omega_n}{\theta}} + 1, \tag{2.14}$$

where θ is the constant in (2.11).

Set for simplicity of notation $L_1 := E^{(h)}_{i+1}$ and $L_0 := E^{(h)}_i$ and assume by contradiction that there exists a point $x \in \partial L_1 \setminus L_0$ such that

$$\operatorname{dist}(x, L_0) > \gamma\sqrt{h}.$$

Let $r := \gamma\sqrt{h}/2$ and note that, since $B_r(x) \cap L_0 = \emptyset$, L_1 satisfies a one-sided minimizing property in $B_r(x)$. Indeed, let F be such that

$$F \subset L_1 \quad \text{and} \quad |L_1 \setminus F| \subset\subset B_r(x).$$

From $\mathcal{G}(L_1,h,L_0) \le \mathcal{G}(F,h,L_0)$ and $u_{i,h}|_{B_r(x)} > 0$ (notation as in (2.2)), it follows that

$$\mathrm{Per}\,(L_1,B_r(x)) \le \mathrm{Per}\,(F,B_r(x)).$$

This implies that we can apply Lemma 2.2 to $B_r(x) \setminus L_1$ and, hence, the density estimate (2.11) gives:

$$|L_1 \cap B_r(x)| \ge \theta \left(\frac{\gamma\sqrt{h}}{2}\right)^n. \tag{2.15}$$

On the other hand, set $L_3 := L_1 \setminus B_r(x)$. By the minimizing property

$$\mathcal{G}(L_1,h,L_0) \le \mathcal{G}(L_3,h,L_0)$$

and $u_{i,h}|_{B_r(x)} \ge \gamma/(2\sqrt{h})$, we get easily the following reversed bound:

$$\frac{\gamma|L_1 \cap B_r(x)|}{2\sqrt{h}} \le n\,\omega_n \left(\frac{\gamma\sqrt{h}}{2}\right)^{n-1}. \tag{2.16}$$

Clearly, (2.15) and (2.16) imply $\gamma \le 2\sqrt{n\,\omega_n/\theta}$, which contradicts (2.14).

Similarly, in the case there exists $x \in \partial L_1 \cap L_0$ with $\mathrm{dist}(x,\partial L_0) > \gamma\sqrt{h}$, we argue in the same way, noticing that L_1 turns out to be one-sided minimizing in a neighborhood of x. $\square$

2.5. Weak flow with obstacle

Though it is not needed to the proof of Theorem 0.1, we note that Proposition 2.1 also leads to the existence of a limit flow with obstacle. Indeed, from the very definition of discrete flow, it follows easily that

$$\mathrm{Per}\,(E_t^{(h)}) \le \mathrm{Per}\,(E_0) \quad \text{for every} \quad h,t \ge 0.$$

Hence, recalling (2.5) and the compactness (2) § 2.1, by a diagonal argument we find a subsequence h (not relabelled) and sets E_t such that

$$E_t^{(h)} \to E_t \quad \text{as } h \to 0 \quad \forall\, 0 \le t \in \mathbb{Q}.$$

Moreover, using Proposition 2.1, one can show that for the whole discrete flow a uniform Hölder continuity in time in the L^1 topology holds (see [25]).

Proposition 2.3. *There exists a constant $C > 0$ such that*

$$|E_t^{(h)} \triangle E_s^{(h)}| \le C\,|s-t|^{\frac{1}{2}} \quad \forall\, h > 0,\ \forall\, t,s \ge h > 0. \tag{2.17}$$

Clearly, this allows us to pass into the limit for every $t \ge 0$ and find a limit flow E_t satisfying the continuity estimate:

$$|E_t \triangle E_s| \le C\,|s-t|^{\frac{1}{2}} \quad \forall\, t,s > 0.$$

3. Monotone flows with obstacle

Since we are interested in the asymptotics of the evolution with obstacle, we can restrict ourself to the case of "nested" flows, i.e. flows satisfying $E_t \subseteq E_s$ for every $0 \leq s \leq t$. For the smooth flow the right condition to look at is the mean-convexity of the initial set. In the context of Caccioppoli sets there are different ways to generalize this notion, such as the local pseudo-convexity introduced in [20] or the minimizing hulls (also called subsolutions) considered in [4, 5, 13]. For our purposes, the latter suffices.

Definition 3.1. A set $E \subseteq \mathbb{R}^n$ is a *Minimizing Hull* in $\mathcal{O} \subseteq \mathbb{R}^n$ open if

$$\operatorname{Per}(E,\mathcal{O}) \leq \operatorname{Per}(F,\mathcal{O}) \quad \forall\, E \subseteq F \quad \text{such that} \quad F \setminus E \Subset \mathcal{O}. \tag{3.1}$$

We often do not specify the open set when $\mathcal{O} = \mathbb{R}^n$. It is easy to verify that a minimizing hull E with smooth boundary is mean-convex, while the reverse implication is in general false. Simple consequences of Definition 3.1 are the following two properties.

(1) *If $E \subseteq \mathbb{R}^n$ is a minimizing hull and $F \subseteq \mathbb{R}^n$, then*

$$\operatorname{Per}(E \cap F) \leq \operatorname{Per}(F). \tag{3.2}$$

Indeed, from the minimizing hull property $\operatorname{Per}(E) \leq \operatorname{Per}(E \cup F)$ and from (2.1), we have

$$\operatorname{Per}(E \cap F) \leq \operatorname{Per}(E) + \operatorname{Per}(F) - \operatorname{Per}(E \cup F) \leq \operatorname{Per}(F).$$

(2) *If $\{E_k\}_{k\in\mathbb{N}}$ is a sequence of minimizing hulls and $\chi_{E_k} \to \chi_E$ in L^1, then E is a minimizing hull.* Indeed, given $E \subset F$ such that $F \setminus E \Subset \mathbb{R}^n$, by the minimizing hull property of E_k we have

$$\operatorname{Per}(E_k) \leq \operatorname{Per}(E_k \cup F). \tag{3.3}$$

On the other hand, $E_k \cap F \to E \cap F = E$ and, by semicontinuity (1) § 2.1,

$$\begin{aligned}
\operatorname{Per}(E) &\leq \liminf_{k\to+\infty} \operatorname{Per}(E_k \cap F) \\
&\overset{(2.1)}{\leq} \liminf_{k\to+\infty} \left[\operatorname{Per}(E_k) + \operatorname{Per}(F) - \operatorname{Per}(E_k \cup F)\right] \\
&\overset{(3.3)}{\leq} \operatorname{Per}(F).
\end{aligned}$$

3.1. Maximal solutions

Given a minimizing hull as initial set E_0, it is possible to define uniquely a maximal approximate flow. The main observation in this regard is contained in the following lemma.

Lemma 3.2. *Let $E_0 \subset \mathbb{R}^n$ be a bounded closed minimizing hull such that*

$$\Omega \subseteq E_0 \quad \text{and} \quad |\partial E_0| = 0.$$

Then, the following holds:

(i) any minimizer $E \supset \Omega$ of $\mathcal{G}(\cdot,h,E_0)$ is a minimizing hull and $E \subseteq E_0$;

(ii) if E' is any other minimizer, then $E \cup E'$ and $E \cap E'$ are minimizers of $\mathcal{G}(\cdot,h,E_0)$ as well.

Proof. Let $u_{0,h} = h^{-1}d_0 \in L^\infty(\mathbb{R}^n)$, with d_0 the rescaled signed distance from ∂E_0 in (2.3), and for simplicity let us write $\mathcal{G}(\cdot)$ for $\mathcal{G}(\cdot,h,E_0)$. We start proving that $E \subseteq E_0$. Indeed, note that

$$\begin{aligned}
\mathcal{G}(E) \le \mathcal{G}(E\cap E_0) &= \mathrm{Per}\,(E\cap E_0) + \int_{\mathbb{R}^n} u_{0,h}\,\chi_{E\cap E_0} \\
&\overset{(3.2)}{\le} \mathrm{Per}\,(E) + \int_{\mathbb{R}^n} u_{0,h}\,\chi_E - \int_{\mathbb{R}^n} u_{0,h}\,\chi_{E\setminus E_0} \\
&= \mathcal{G}(E) - \int_{\mathbb{R}^n} u_{0,h}\,\chi_{E\setminus E_0}.
\end{aligned}$$

Since $u_{0,h} > 0$ in $\mathbb{R}^n \setminus E_0$, this implies $E \subseteq E_0$ a.e.

Next we show that E is a minimizing hull. Let $E \subseteq F$ and $F \setminus E \Subset \mathbb{R}^n$. From the minimizing property of E we infer the following:

$$\begin{aligned}
\mathcal{G}(E) &= \mathrm{Per}\,(E) + \int_{\mathbb{R}^n} u_{0,h}\,\chi_E \\
&\le \mathcal{G}(F\cap E_0) \\
&= \mathrm{Per}\,(F\cap E_0) + \int_{\mathbb{R}^n} u_{0,h}\,\chi_{F\cap E_0} \\
&\overset{(3.2)}{\le} \mathrm{Per}\,(F) + \int_{\mathbb{R}^n} u_{0,h}\,\chi_{F\cap E_0}.
\end{aligned} \tag{3.4}$$

From $E \subseteq E_0$ and (3.4) we have that

$$\mathrm{Per}\,(E) \le \mathrm{Per}\,(F) + \int_{\mathbb{R}^n} u_{0,h}\,\chi_{(F\setminus E)\cap E_0} \le \mathrm{Per}\,(F),$$

where we used $u_{0,h}|_{E_0} \le 0$. This shows that E is a minimizing hull.

Finally, let E' be another minimizer of $\mathcal{G}$. From the minimizing property of E, we get

$$\mathcal{G}(E) \le \mathcal{G}(E \cap E') = \operatorname{Per}(E \cap E') + \int_{\mathbb{R}^n} u_h \, \chi_{E \cap E'}, \tag{3.5}$$

$$\mathcal{G}(E) \le \mathcal{G}(E \cup E') = \operatorname{Per}(E \cup E') + \int_{\mathbb{R}^n} u_h \, \chi_{E \cup E'}. \tag{3.6}$$

Summing the two inequalities, we get

$$\begin{aligned} 2\mathcal{G}(E) &\le \operatorname{Per}(E \cap E') + \operatorname{Per}(E \cup E') + \int_{\mathbb{R}^n} u_h \, \chi_{E \cap E'} + \int_{\mathbb{R}^n} u_h \, \chi_{E \cup E'} \\ &\overset{(2.1)}{\le} \operatorname{Per}(E) + \operatorname{Per}(E') + \int_{\mathbb{R}^n} u_h \, \chi_E + \int_{\mathbb{R}^n} u_h \, \chi_{E'} \\ &= \mathcal{G}(E) + \mathcal{G}(E'). \end{aligned}$$

Since E' is a minimizer, i.e. $\mathcal{G}(E) = \mathcal{G}(E')$, we deduce that (3.5) and (3.6) are equalities, thus concluding that $E \cap E'$ and $E \cup E'$ are both minimizers of $\mathcal{G}$. $\square$

A simple first corollary of Lemma 3.2 is the existence of a maximal minimizer for $\mathcal{G}$.

Corollary 3.3. *Let E_0 be a bounded closed minimizing hull such that*

$$\Omega \subseteq E_0 \quad \textit{and} \quad |\partial E_0| = 0.$$

Then, there exist a maximal minimizer $E_{\max}$ of $\mathcal{G}$ in the following sense: if E is any other minimizer of $\mathcal{G}$, then $E \subseteq E_{\max}$.

Proof. We define $E_{\max}$ as a minimizer which maximize the volume, i.e.

$$|E_{\max}| = \max\big\{|E| \;:\; E \text{ minimizer of } \mathcal{G}\big\}, \tag{3.7}$$

If E is any other minimizer of $\mathcal{G}$, from Lemma 3.2 we deduce that $E \cup E_{\max}$ is also a minimizer. Hence, since

$$|E_{\max}| \le |E \cup E_{\max}|,$$

from (3.7) we infer that $E \subseteq E_{\max}$. $\square$

From now on, we will call the flow constructed from these special solutions the *maximal* approximate flows. Similarly, we deduce the following proposition from Lemma 3.2.

Proposition 3.4. *Let $E_0 \subseteq \mathbb{R}^n$ be a minimizing hull with*

$$\Omega \subset E_0 \quad \text{and} \quad |\partial E_0| = 0,$$

and, for every $h > 0$, let $E^{(h)}_{\max,t}$ denote the maximal flows. Then, the following holds:

(i) $E^{(h)}_{\max,t} \subseteq E^{(h)}_{\max,s}$ for every $0 \le s \le t$;

(ii) $E^{(h)}_{\max,t}$ is a minimizing hull for every $t \ge 0$.

Proof. The proof follows readily from the previous Lemma 3.2, noticing that, by the regularity of the minimizers, it holds $|\partial E^{(h)}_i| = 0$. □

3.2. Monotonicity

In the proof of Theorem 0.1 we need also the following refined monotonicity property. The proof exploits the same arguments used above.

Lemma 3.5. *Let E_0 and F_0 be two closed bounded minimizing hulls such that*

$$\Omega \subseteq E_0 \subseteq F_0 \quad \text{and} \quad |\partial E_0| = |\partial F_0| = 0.$$

Then, the maximal minimizers $E_{\max}$ of $\mathcal{G}(\cdot, h, E_0)$ and $F_{\max}$ of $\mathcal{G}(\cdot, h, F_0)$ satisfy

$$E_{\max} \subseteq F_{\max} \tag{3.8}$$

Proof. For simplicity, set $u_0 := h^{-1} d_{\partial E_0}$ and $u_1 := h^{-1} d_{\partial F_0}$, where $d_{\partial E_0}$ and $d_{\partial F_0}$ are the signed distances from ∂E_0 and ∂F_0 respectively, as defined in (2.3). Using the minimizing properties, we get:

$$\operatorname{Per}(E_{\max}) + \int_{\mathbb{R}^n} u_0\, \chi_{E_{\max}} \le \operatorname{Per}(E_{\max} \cap F_{\max}) + \int_{\mathbb{R}^n} u_0\, \chi_{E_{\max} \cap F_{\max}}, \tag{3.9}$$

$$\operatorname{Per}(F_{\max}) + \int_{\mathbb{R}^n} u_1\, \chi_{F_{\max}} \le \operatorname{Per}(E_{\max} \cup F_{\max}) + \int_{\mathbb{R}^n} u_1\, \chi_{E_{\max} \cup F_{\max}}. \tag{3.10}$$

Summing these two inequalities, and using (2.1), we get

$$\int_{\mathbb{R}^n} u_0\, \chi_{E_{\max}} + \int_{\mathbb{R}^n} u_1\, \chi_{F_{\max}} \le \int_{\mathbb{R}^n} u_0\, \chi_{E_{\max} \cap F_{\max}} + \int_{\mathbb{R}^n} u_1\, \chi_{E_{\max} \cup F_{\max}},$$

which in turn implies

$$\int_{\mathbb{R}^n} (u_0 - u_1)\, \chi_{E_{\max} \setminus F_{\max}} \le 0. \tag{3.11}$$

Since $u_0 \ge u_1$ in E_0 and $E_{\max} \subseteq E_0$ by Lemma 3.2, we infer that (3.11) is an inequality. This implies that also (3.9) and (3.10) are equalities, i.e. $E_{\max} \cup F_{\max}$ is a minimizer of $\mathcal{G}(\cdot, h_1, F_0)$. By maximality of the solution, we conclude (3.8). □

4. Asymptotic evolutions

In this section we study the properties of the approximate asymptotic evolutions (well defined thanks to Proposition 3.4 (i)):

$$E^{(h)}_{\max,\infty} := \bigcap_{t\geq 0} E^{(h)}_{\max,t}.$$

We prove that every $E^{(h)}_{\max,\infty}$ is stationary under the approximate mean-curvature flow with obstacle.

Proposition 4.1. *For every $0 < h' \leq h$, $E^{(h)}_{\max,\infty}$ is the maximal minimizer of $\mathcal{G}(\cdot, h', E^{(h)}_{\max,\infty})$. In particular, $E^{(h)}_{\max,\infty} \subseteq E^{(h')}_{\max,\infty}$.*

Proof. We start proving that $E^{(h)}_{\max,\infty}$ is a minimizer of $\mathcal{G}(\cdot, h, E^{(h)}_{\max,\infty})$. We proceed by contradiction. Assume there exists $F \subset E^{(h)}_{\max,\infty}$ such that

$$\mathcal{G}\big(F, h, E^{(h)}_{\max,\infty}\big) < \mathcal{G}\big(E^{(h)}_{\max,\infty}, h, E^{(h)}_{\max,\infty}\big). \tag{4.1}$$

Note that, by the semicontinuity of the perimeter (1) § 2.1 and the locally uniform convergence $d_i \to d_\infty$, where d_∞ is the signed distance to $\partial E^{(h)}_{\max,\infty}$ as in (2.3), we have

$$\mathcal{G}\big(F, h, E^{(h)}_{\max,\infty}\big) = \lim_{i\to+\infty} \mathcal{G}\big(F, h, E^{(h)}_{\max,i}\big), \tag{4.2}$$

$$\mathcal{G}\big(E^{(h)}_{\max,\infty}, h, E^{(h)}_{\max,\infty}\big) \leq \liminf_{i\to\infty} \mathcal{G}\big(E^{(h)}_{\max,i+1}, h, E^{(h)}_{\max,i}\big). \tag{4.3}$$

From (4.1), (4.2) and (4.3), we infer that, for i big enough,

$$\mathcal{G}\big(F, h, E^{(h)}_{\max,i}\big) < \mathcal{G}\big(E^{(h)}_{\max,i+1}, h, E^{(h)}_{\max,i}\big),$$

thus contrasting with the minimizer property of $E^{(h)}_{\max,i+1}$.

Now, note that

$$\mathcal{G}\big(E^{(h)}_{\max,\infty}, h, E^{(h)}_{\max,\infty}\big) \leq \mathcal{G}\big(F, h, E^{(h)}_{\max,\infty}\big), \quad \forall\, F \subseteq E^{(h)}_{\max,\infty}$$

implies that, for all $h' \leq h$, (recall that $d(\cdot, \partial E^{(h)}_{\max,\infty}) \leq 0$ on $\partial E^{(h)}_{\max,\infty}$)

$$\begin{aligned}\operatorname{Per}\big(E^{(h)}_{\max,\infty}\big) &\leq \operatorname{Per}(F) - \int_{E^{(h)}_{\max,\infty}\setminus F} h^{-1} d(x, \partial E^{(h)}_{\max,\infty})\\ &\leq \operatorname{Per}(F) - \int_{E^{(h)}_{\max,\infty}\setminus F} h'^{-1} d(x, \partial E^{(h)}_{\max,\infty}),\end{aligned}$$

which, in turns, leads to the minimizing property for $\mathcal{G}(\cdot,h',E^{(h)}_{\max,\infty})$:

$$\mathcal{G}\big(E^{(h)}_{\max,\infty},h',E^{(h)}_{\max,\infty}\big) \le \mathcal{G}\big(F,h',E^{(h)}_{\max,\infty}\big).$$

Finally, since $E^{(h)}_{\max,\infty} \subseteq E_0$, the last assertion follows by induction from Lemma 3.5. □

In particular, recalling the regularity theory for almost minimizers of the perimeter (see the details in [25]), it follows from Proposition 4.1 that $E^{(h)}_{\max,\infty}$ is $C^{1,1}$ regular up to a singular set Σ_h of dimension at most $n-8$. Moreover, by the monotonicity property proven in Proposition 4.1, the asymptotic approximate evolutions $E^{(h)}_{\max,\infty}$ have a L^1-limit as $h \to 0$,

$$E^{(h)}_{\max,\infty} \uparrow E_{\max,\infty} := \bigcup_{h>0} E^{(h)}_{\max,\infty}.$$

The regularity of $E_{\max,\infty}$ is shown in the following proposition.

Proposition 4.2. *Let $E_{\max,\infty}$ be the asymptotic evolution defined above. Then, $\partial E_{\max,\infty} = \overline{M}$, where M is a $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ of class $C^{1,1}$ and $\overline{M}\setminus M$ has Hausdorff dimension at most $n-8$. Moreover, there exists a neighborhood U of Ω where $\partial E_{\max,\infty}$ is regular, i.e. $\partial E_{\max,\infty} \cap U = M \cap U$.*

Proof. We start noticing that

(i) since each $E^{(h)}_{\max,\infty}$ is a minimizing hull, $E_{\max,\infty}$ is also a minimizing hull by (2) of § 3;

(ii) by the minimizing hull property, for all $h,r>0$ and for every $p \in \mathbb{R}^n$, we have

$$\max\Big\{\mathrm{Per}\,(E_{\max,\infty},B_r(p)),\mathrm{Per}\,(E^{(h)}_{\max,\infty},B_r(p))\Big\} \le n\,\omega_n\,r^{n-1}; \qquad (4.4)$$

(iii) $M_h := \partial E^{(h)}_{\infty} \setminus (\Sigma_h \cup \Omega)$ is a stable minimal surface in $\mathbb{R}^n$: this follows from the Euler–Lagrange equation for $\mathcal{G}(\cdot,h,E^{(h)}_{\max,\infty})$, i.e.

$$H_{M_h}(\cdot) = h^{-1} d(\cdot,\partial E^{(h)}_{\max,\infty}),$$

and the one-sided area minimizing property of M_h.

By the Schoen-Simon compactness result [23, Theorem 2], we have that, up to passing to a subsequence (not relabelled), the varifold $|M_h|$ naturally associated to M_h converges to a varifold V supported on a stable minimal hypersurface M

with $\mathcal{H}^\alpha(\overline{M}\setminus(M\cup\Omega))=0$ for all $\alpha>n-8$. Therefore, since $E^{(h)}_{\max,\infty}$ converges in L^1 to $E_{\max,\infty}$, we also conclude that

$$\partial E_{\max,\infty}\setminus\Omega\subset \operatorname{spt}(|V|)=\bar{M}\setminus\Omega,$$

where $|V|$ is the weight of the varifold V. In other words, in the complement of Ω the boundary of $E_{\max,\infty}$ is contained in the union of a stable minimal hypersurface and a closed singular set $\Sigma=\overline{M}\setminus(M\cup\Omega)$ of Hausdorff dimension at most $n-8$.

Finally, let $x_0\in\partial\Omega\cap\partial E_{\max,\infty}$. Then, by the uniform bound on the perimeters (4.4), the translated and rescaled sets

$$F_r := r^{-1}\left(E_{\max,\infty}-x_0\right).$$

locally converge up to extracting a subsequence (here and in the sequel not relabelled) to a minimizing hull F (recall (2) § 3). By the $C^{1,1}$-regularity of $\partial\Omega$, the rescaled obstacles

$$\Omega_r := r^{-1}\big(\Omega-x_0\big)$$

converge locally to a closed half space H. Since F is a minimizing hull, $H\subseteq F$ and $0\in\partial F\cap\partial\Omega$, by a simple maximum principle we infer that $F=H$. Indeed, if this is not the case, there exist $R>0$ and a smooth, not constantly zero function $f:\partial B_R\subset H\to H^\perp$ such that $\operatorname{graph}(f)\subset F$. By the classical maximum principle for minimal surfaces, the solution u to the minimal surface equation with boundary f is positive in the whole B_R and $\operatorname{graph}(u)\subset F$ by the minimizing hull property. Deforming the boundary value f continuously to zero, we find a family of solutions to the minimal surface equation foliating a neighborhood of 0 in $\mathbb{R}^n\setminus H$, thus contradicting $0\in\partial F$.

This implies that every point $x_0\in\partial\Omega\cap\partial E_{\max,\infty}$ is a point of the reduced boundary. Moreover, by a simple variant of this argument, considering blowup centers x_0 in a neighborhood of $\partial\Omega$ we also infer that every point in a neighborhood of $\partial\Omega$ is a regular point, uniformly if we assume an upper bound for the norm of the second fundamental form of $\partial\Omega$. Therefore, invoking the regularity of the classical obstacle problem for minimal surfaces (see, e.g., the account given in [9, 10]), we conclude that $\partial E_{\max,\infty}$ is of class $C^{1,1}$ in a neighborhood of $\partial\Omega$. □

As a straightforward consequence of the above proposition (alternatively, see [25] for the proof) is the following.

Corollary 4.3. *Let $\Omega\subset\mathbb{R}^n$, $n\le 7$, be a closed $C^{1,1}$ set and $E_0\supset\Omega$ a closed minimizing hull with $|\partial E_0|=0$. Then, there exists a dimensional constant $c_0>0$ such that $\partial E_{\max,\infty}\in C^{1,1}$ with uniform estimate*

$$\|A_{\partial E_{\max,\infty}}\|_{L^\infty}\le c_0\,\|A_{\partial\Omega}\|_{L^\infty}.$$

5. Mean-convex hull

Now we are ready for the proof of Theorem 0.1. Here we make precise the notion of minimal hypersurface Σ used in the definition of global barriers in the Introduction, that are locally boundary of stationary Caccioppoli sets E with $\mathcal{H}^{n-2}(\mathrm{Sing}(E)) = 0$, where $\mathrm{Sing}(E)$ are the points in the measure theoretic boundary of E which are not in the reduced boundary.

We divide the proof of Theorem 0.1 in several steps.

5.1. Step 1

Consider the closed ε-neighbourhood of the obstacle Ω:

$$\Omega_\varepsilon := \big\{x : \mathrm{dist}(x,\Omega) \le \varepsilon\big\}.$$

Note that $\Omega_\varepsilon \downarrow \Omega$, i.e.

$$\Omega_{\varepsilon_1} \subseteq \Omega_{\varepsilon_2} \quad \forall\, 0 \le \varepsilon_1 \le \varepsilon_2 \quad \text{and} \quad \bigcap_{\varepsilon>0} \Omega_\varepsilon = \Omega_0.$$

Moreover, by the $C^{1,1}$ regularity of $\partial\Omega$ there exists $\varepsilon_0 > 0$ such that $\partial\Omega_\varepsilon \in C^{1,1}$. Let now E_0 be a closed convex set such that $\Omega_\varepsilon \Subset \mathrm{int}(E_0)$ for every $\varepsilon < \varepsilon_0$ and let $E^\varepsilon_{\max,\infty}$ be the asymptotic limit of the maximal flows starting at E_0 with respect to the obstacle Ω_ε. Set

$$\mathcal{E}(\Omega) := \bigcap_{\varepsilon>0} E^\varepsilon_{\max,\infty}.$$

We will show that $\Omega^{mc} = \mathcal{E}(\Omega)$.

5.2. Step 2

The main ingredient for the proof of Theorem 0.1 is contained in the following proposition.

Proposition 5.1. *Let Ω and E_0 be as in Step 1. Then, every minimal hypersurface Σ with $\partial\Sigma \subseteq \Omega$ is contained in $\mathcal{E}(\Omega)$.*

Proof. Let $E^{(h),\varepsilon}_{\max,t}$ denote the approximate maximal flows starting at E_0 with respect to the obstacle Ω_ε. We show that, for every minimal hypersurface Σ with $\partial\Sigma \subset \Omega$, it holds $\Sigma \subset E^{(h),\varepsilon}_{\max,\infty}$ for every $\varepsilon > 0$ and $h < \varepsilon^2/(4\gamma^2)$, where γ is the constant in Proposition 2.1. This implies that

$$\Sigma \subset \bigcup_{h>0} E^{(h),\varepsilon}_{\max,\infty} = E^\varepsilon_{\max,\infty} \quad \forall\, \varepsilon > 0,$$

thus proving the proposition.

The proof of the claim is by contradiction. Assume there exists $i \in \mathbb{N}$ such that

$$\Sigma \subset E^{(h)}_{\max,i} \quad \text{and} \quad \Sigma \setminus E^{(h)}_{\max,i+1} \neq \emptyset. \tag{5.1}$$

Note that here we used the convex hull property for minimal surfaces which implies $\Sigma \subset E_0$. Set for simplicity of notation $L := E^{(h)}_{\max,i}$ and consider the closed set of points of minimum distance between ∂L and $\bar{\Sigma}$:

$$W := \big\{x \in \bar{\Sigma} : \operatorname{dist}(x, \partial L) = \operatorname{dist}\big(\Sigma, \partial L\big)\big\}.$$

From Proposition 2.1 and (5.1), we deduce that

$$\operatorname{dist}(\Sigma, \partial L) \leq \gamma\sqrt{h}.$$

Hence, since $2\gamma\sqrt{h} < \varepsilon$ and $\partial\Sigma \subset \Omega$ is distant at least ε from Ω_ε, the minimum distance is reached in the interior of Σ, i.e. $W \subset \Sigma$. Let $x_0 \in W$ be a boundary point of $W \subset \Sigma$ for the induced topology, i.e.

$$B_r(x_0) \cap (\Sigma \setminus W) \neq \emptyset \quad \forall\, r > 0, \tag{5.2}$$

and let $y_0 \in \partial L$ be such that $\operatorname{dist}(\Sigma, W) = |x_0 - y_0|$. Consider

$$\Sigma' = \Sigma + y_0 - x_0.$$

We have that $\Sigma' \subset L$ and $\Sigma' \cap \partial L \neq \emptyset$. We can apply the strict maximum principle (see, e.g., [14]) and conclude that $\Sigma' \equiv \partial L$ in a neighborhood of x_0, against (5.2). □

5.3. Step 3

$\mathcal{E}(\Omega)$ satisfies the regularity conclusion of Theorem 0.1, thanks to Proposition 4.1. Moreover, we show that $\mathcal{E}(\Omega)$ is actually a global barrier.

Proposition 5.2. *Let Ω and E_0 be as in Step 1. Then, $\mathcal{E}(\Omega)$ is a global barrier,i.e.*

$$\Sigma \text{ minimal hypersurface, } \partial\Sigma \subset \mathcal{E}(\Omega) \quad \Longrightarrow \quad \Sigma \subset \mathcal{E}(\Omega).$$

Proof. By Proposition 5.1, it is enough to show that

$$\mathcal{E}(\mathcal{E}(\Omega)) = \mathcal{E}(\Omega). \tag{5.3}$$

To this aim, set for simplicity $\mathcal{E}_1 := \mathcal{E}(\Omega)$, $\mathcal{E}_2 := \mathcal{E}(\mathcal{E}(\Omega))$ and $M := \partial\mathcal{E}_2 \setminus \mathcal{E}_1$. We claim that

$$\partial M \subset \Omega. \tag{5.4}$$

Assume, indeed, there exists $x_0 \in \partial M \setminus \Omega$. Then, in particular, since $\partial M \subset \partial\mathcal{E}_1$, we have that $x \in \partial\mathcal{E}_1 \setminus \Omega$. Then, in a neighborhood of x we have two stable currents $\Sigma_1 := B_r(x_0) \cap \partial\mathcal{E}_2$ and $\Sigma_2 := B_r(x_0) \cap \partial\mathcal{E}_1$ which disconnect the ball and one lies on one side of the other. Therefore, we can apply the strong maximum principle by Ilmanen [14, Theorem A (iii)] to infer that

$$B_r(x_0) \cap \partial\mathcal{E}_2 = B_r(x_0) \cap \partial\mathcal{E}_1.$$

This contradicts $x_0 \in \partial M = \partial\mathcal{E}_2 \setminus \mathcal{E}_1$.

The conclusion of the proof is now straightforward. Since by Proposition 5.1 $\mathcal{E}(\Omega)$ is a barrier for minimal hypersurfaces with boundary in Ω, from (5.4) it follows that $\partial\mathcal{E}(\mathcal{E}(\Omega)) \subset \mathcal{E}(\Omega)$, which together with the obvious inclusion $\mathcal{E}(\Omega) \subseteq \mathcal{E}(\mathcal{E}(\Omega))$ gives (5.3). □

5.4. **Step** 4

The proof of Theorem 0.1 now follows straightforwardly. By the previous steps, we deduce that $\mathcal{E}(\Omega)$ is a global barrier containing Ω and satisfying the regularity conclusion of the theorem.

We need only to show that $\mathcal{E}(\Omega)$ is the least possible barrier. To this aim, note that, since $\partial\mathcal{E}(\Omega) \setminus \Omega$ is a minimal surface with boundary in Ω, then necessarily

$$\partial\mathcal{E}(\Omega) \subset \Omega^{mc}. \tag{5.5}$$

The conclusion then follows noting that (5.5) implies $\mathcal{E}(\Omega) \subset \Omega^{mc}$, because $\mathcal{E}(\Omega)$ can be realized as the union of minimal hypersurfaces with boundary on $\partial\mathcal{E}(\Omega)$ (which then necessarily are contained in Ω^{mc}), e.g.

$$\mathcal{E}(\Omega)\setminus = \bigcup_{t\in\mathbb{R}} \Big(\mathcal{E}(\Omega) \cap \{x : x_n = t\}\Big).$$

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