LE MATEMATICHE
Vol. LVII (2002) – Fasc. I, pp. 39–48

ON THE CONVOLUTION PRODUCT OF THE DISTRIBUTIONAL FAMILIES RELATED TO THE DIAMOND OPERATOR

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In this paper, we introduce a distributional family $K_{\alpha,\beta}$ which is related to the Diamond operator $\diamond^k$ iterated k-times. At first we study the properties of $K_{\alpha,\beta}$ and then we give a sense to the convolution product of $K_{\alpha,\beta*}K_{\alpha',\beta'}$.

1. Introduction.

A. Kananthai [4] has first introduced the Diamond operator $\diamond^k$ iterated k-times which is defined by

$$\diamond^k = \left((\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2})^2 - (\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2})^2 \right)^k \tag{1}$$

where $p+q=n$ is the dimension of the n-dimensional Euclidean space $\mathbb{R}^n$ and k is a nonnegative integer. Actually (1) can be rewrite in the following form

$$\diamond^k = \Box^k \Delta^k = \Delta^k \Box^k \tag{2}$$

where the operators $\Box^k$ and Δ^k are defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k \tag{3}$$

Entrato in redazione il 7 dicembre 2001

and

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^k, \, p+q = n \tag{4}$$

In this paper, the family $K_{\alpha,\beta}$ is defined by $K_{\alpha,\beta}(x) = R^e_\alpha * R^H_\beta$ where R^e_α is elliptic kernel defined by (5) and R^H_β is hyperbolic kernel defined by (8) and the symbol $*$ designates as the convolution and $x \in \mathbb{R}^n$. By A. Kananthai ([4] , p. 33 , Theorem 3.1) $(-1)^k K_{\alpha,\beta}(x)$ is an elementary solution of the Diamond operator $\diamondsuit^k$ defined by (1) for $\alpha = \beta = 2k$.

We found the following properties $K_{0,0}(x) = \delta(x)$ where δ is the Dirac-delta distribution, $K_{-2k,-2k}(x) = (-1)^k \diamondsuit^k \delta(x)$, $\diamondsuit^k(K_{\alpha,\beta}(x)) = (-1)^k K_{\alpha-2k,\beta-2k}$ and $\diamondsuit^k(K_{2k,2k}(x)) = (-1)^k \delta(x)$.

Moreover, we found the convolutions product $K_{\alpha,\beta} * K_{\alpha',\beta'} = B_{\beta,\beta'} R^H_{\beta+\beta'} * R^e_{\alpha+\alpha'}$ if p is even, and $K_{\alpha,\beta} * K_{\alpha',\beta'} = \left(R^H_{\beta+\beta'} + T_{\beta+\beta'}\right) * R^e_{\alpha+\alpha'}$ if p is odd, where

$$B_{\beta,\beta'} = \frac{\cos(\frac{\beta}{2}\pi) cos(\frac{\beta'}{2}\pi)}{\cos(\frac{\beta+\beta'}{2})\pi}$$

and

$$T_{\beta,\beta'} = \frac{C(-\beta - \beta' 2) 4^{-1}}{C(-\frac{\beta}{2}) C(-\frac{\beta'}{2})(2\pi i)^{-1}} \left[H^+_{\beta+\beta'} - H^-_{\beta+\beta'}\right],$$

$C(r) = \Gamma(r)\Gamma(1-r)$ and $H^\pm_r = H_r(u \pm io, n) = e^{\mp \frac{r\pi}{2} i} e^{\pm \frac{q\pi}{2} i} a(\frac{r}{2})(u \pm io)^{\frac{r-n}{2}}$ and $a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$.

2. Preliminaries.

Definition 2.1. Let the function $R^e_\alpha(x)$ be defined by

$$R^e_\alpha(x) = \frac{|x|^{\alpha-n}}{W_n(\alpha)} \tag{5}$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, α is a complex parameter, n is the dimension of $\mathbb{R}^n$ and $|x| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}}$ and $W_n(\alpha)$ is defined by the formula

$$W_n(\alpha) = \frac{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

The function $R^e_\alpha(x)$ is precisely the definition of elliptic kernel of Marcel Riesz [2] and the following formula is valid

$$R^e_\alpha(x) * R^e_\beta(x) = R^e_{\alpha+\beta}(x) \tag{6}$$

which hold for $\alpha > 0$, $\beta > 0$ and $\alpha + \beta \leq n$ see([2], p. 20).

Definition 2.2. Let $x = (x_1, x_2, ..., x_n)$ be a point of $\mathbb{R}^n$ and write

$$u = u(x) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \ldots - x_{p+q}^2 \tag{7}$$

where $p + q = n$.

Denote by Γ_+ the interior of the forward cone defined by $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ and by $\bar{\Gamma}_+$ designates its closure.

Similarly, define $\Gamma_- = \{x \in \mathbb{R}^n : x_1 < 0 \text{ and } u > 0\}$ and $\bar{\Gamma}_-$ designates its closure. For any complex number α, define

$$R^H_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{K_n(\alpha)} & if \quad x \in \Gamma_+ \\ 0 & if \quad x \notin \Gamma_+ \end{cases} \tag{8}$$

where $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}} \Gamma(\frac{2+\alpha-n}{2}) \Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)}{\Gamma(\frac{2+\alpha-p}{2}) \Gamma(\frac{p-\alpha}{2})} \tag{9}$$

The function R^H_α was introduced by Y. Nozaki ([3], p. 72). R^H_α, which is an ordinary function if $R_e(\alpha) \geq n$, is a distribution of α and is a distribution of α if $R_e(\alpha) < n$. Let supp $r^H_\alpha(u)$. Suppose

$$supp R^H_\alpha(u) \subset \bar{\Gamma}_+ \tag{10}$$

We shall call R^H_α the Marcel Riesz's ultra-hyperbolic kernel. By putting $p = 1$ in (8) and (9) and remembering the Legendre's duplication formula of $\Gamma(z)$,

$$\Gamma(2z) = 2^{2z-1} \pi^{-\frac{1}{2}} \Gamma(z) \Gamma(z + \frac{1}{2}) \tag{11}$$

see([5], Vol I, p. 5) the formula (8) reduces to

$$M_\alpha = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{H_n(\alpha)} & if \quad x \in \Gamma_+ \\ 0 & if \quad x \notin \Gamma_+ \end{cases} \tag{12}$$

Here $u = u(x) = x_1^2 - x_2^2 - \ldots - x_n^2$ and

$$H_n(\alpha) = 2^{\alpha-1} \pi^{\frac{n-2}{2}} \Gamma(\frac{\alpha}{2}) \Gamma(\frac{\alpha - n + 2}{2}) \tag{13}$$

M_α is precisely the hyperbolic kernel of Marcel Riesz ([2], p. 31).

Lemma 2.1. *The function $R^e_\alpha(x)$ has the following properties*

(i) $R^e_0(x) = \delta(x)$
(ii) $R^e_{-2k}(x) = (-1)^k \Delta^k \delta(x)$
(iii) $\Delta^k R^e_\alpha(x) = (-1)^k R^e_{\alpha-2k}(x)$

where Δ^k is the Laplace operator iterated k-times defined by (4).

The proofs of Lemma 2.3 is given by S.E Trione [5].

Lemma 2.2. (The convolutions of $R^H_\alpha(u)$)

(i) $R^H_\alpha * R^H_\beta = \frac{\cos\alpha\frac{\pi}{2}\cos\beta\frac{\pi}{2}}{\cos(\frac{\alpha+\beta}{2})\pi} R^H_{\alpha+\beta}$ *where R^H_α is defined by (8) and (9) with p is an even.*
(ii) $R^H_\alpha * R^H_\beta = R^H_{\alpha+\beta} + T_{\alpha,\beta}$ *for p is an odd, where*

$$T_{\alpha,\beta} = T_{\alpha,\beta}(u \pm io, n) = \frac{\frac{2\pi i}{4} C(-\frac{\alpha-\beta}{2})}{C(-\frac{\alpha}{2})C(-\frac{\beta}{2})}[H^+_{\alpha+\beta} - H^-_{\alpha+\beta}] \tag{14}$$

$C(r) = \Gamma(r)\Gamma(1-r)$
$H^\pm_r = H_r(u \pm io, n) = e^{\mp r\frac{\pi}{2}i} e^{\pm q\frac{\pi}{2}i} a(\frac{r}{2})(u \pm io)^{\frac{r-n}{2}}$
$a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$
$(u \pm io)^\lambda = \lim_{\epsilon\to 0}(u + i\epsilon|x|^2)^\lambda$ *see([6], p. 275) $u = u(x)$ is defined by (7) and* $|x| = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$.

In particular $R^H_\alpha * R^H_{-2k} = R^H_{\alpha-2k}$ and $R^H_\alpha * R^H_{2k} = R^H_{\alpha+2k}$.
The proofs of this Lemma is given by M. Aguirre Tellez ([1], p. 121–123).

Lemma 2.3.

(i) $R^H_{-2k} = \Box^k \delta$
(ii) $\Box^k R^H_\alpha = R^H_{\alpha-2k}$
(iii) $\Box^k R^H_{2k} = R^H_0 = \delta$

where $\Box^k$ is defined by (3).

Proof. See ([1], p. 123).

3. The family of distributions $K_{\alpha,\beta}(x)$.

Let $K_{\alpha,\beta}(x)$ be a distributional family defined by

$$K_{\alpha,\beta}(x) = R^e_\alpha * R^H_\beta \tag{15}$$

where the functions R^e_α and R^H_β are defined by (5) and (8) respectively. We now show that $K_{\alpha,\beta}$ exists an is in the space O'_c of rapidly decreasing distributions. We know from [1], p. 119, formulae (I,2,2) that the Fourier's transform of $R^H_\alpha(u)$ is given by the following formulae

$$\{R^H_\alpha(u)\}^\wedge = \frac{1}{2}[f_\alpha(Q + i0) + f_\alpha(Q - i0)] \tag{16}$$

if p is odd and

$$\{R^H_\alpha(u)\}^\wedge = \frac{1}{2i}\frac{\cos\frac{\alpha\pi}{2}}{\sin\frac{\alpha\pi}{2}}[f_\alpha(Q + i0) + f_\alpha(Q - i0)] \tag{17}$$

if p is even. Where

$$f_\alpha(Q \pm i0) = e^{\pm\frac{\alpha\pi i}{2}}(Q \pm i0)^{-\frac{\alpha}{2}} \tag{18}$$

and from [7] page 44 and [6], page 194, the Fourier transform of $R^e_\alpha(x)$ is given by the following formula

$$\{R^e_\alpha(x)\}^\wedge = |y|^{-\alpha} = (|y|^2)^{\frac{-\alpha}{2}} \tag{19}$$

Now using the properties

$$(Q \pm i0)^\lambda = Q^\lambda_+ + e^{\pm\lambda\pi i}Q^\lambda_- \tag{20}$$

([6], page 276), where

$$Q^\lambda_+ = \begin{cases} Q^\lambda & \text{if} \quad Q \geq 0 \\ 0 & \text{if} \quad Q < 0 \end{cases} \tag{21}$$

and

$$Q^\lambda_- = \begin{cases} (-Q)^\lambda & \text{if} \quad Q \leq 0 \\ 0 & \text{if} \quad Q > 0 \end{cases} \tag{22}$$

and

$$Q = Q(y) = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_{p+q}^2. \tag{23}$$

From [1] and [2], we have

$$[f_\alpha(Q + i0) + f_\alpha(Q - i0)] = 2\cos\frac{\alpha\pi}{2} Q^{-\frac{\alpha}{2}} + 2Q_-^{-\frac{\alpha}{2}} \tag{24}$$

if p is odd and

$$[f_\alpha(Q - i0) - f_\alpha(Q + i0)] = 2i\ \sin\frac{\alpha\pi}{2} Q_-^{-\frac{\alpha}{2}} \tag{25}$$

if p is even . Therefore

$$\{R_\alpha^H(u)\}^\wedge = \cos\frac{\alpha\pi}{2} Q^{-\frac{\alpha}{2}} + Q_-^{-\frac{\alpha}{2}} \tag{26}$$

if p is odd and

$$\{R_\alpha^H(u)\}^\wedge = \cos\frac{\alpha\pi}{2} Q_-^{-\frac{\alpha}{2}} \tag{27}$$

if p is even.
The formulae (26) and (27) using (21) and (22) can be rewrite

$$\{R_\alpha^H(u)\}^\wedge = \cos\frac{\alpha\pi}{2}(|y|_p^2)^{-\frac{\alpha}{2}}(1-\rho^2)^{-\frac{\alpha}{2}} + (-1)^{-\frac{\alpha}{2}}(|y|_q^2)^{-\frac{\alpha}{2}}(1-s^2)^{-\frac{\alpha}{2}} \tag{28}$$

if p is odd and

$$\{R_\alpha^H(u)\}^\wedge = -\cos\frac{\alpha\pi}{2}(-1)^{-\frac{\alpha}{2}}(|y|_q^2)^{-\frac{\alpha}{2}}(1-s^2)^{-\frac{\alpha}{2}} \tag{29}$$

if p is even, where

$$|y|_p^2 = y_1^2 + \ldots + y_p^2 \tag{30}$$

$$|y|_q^2 = y_{p+1}^2 + \ldots + y_{p+q}^2 \tag{31}$$

$$\rho^2 = \frac{|y|_q^2}{|y|_p^2} < 1 \tag{32}$$

(33) $$s^2 = \frac{|y|_p^2}{|y|_q^2} < 1$$

Now using that

$$(1 + r^2) \in O_M$$

([8], page 271) where

$$r^2 = x_1^2 + \ldots + x_p^2 + x_{p+1}^2 + \ldots + x_{p+q}^2$$

from (28) and (29) we have

(34) $$\{R_\alpha^H(u)\}^\wedge \in O_M$$

where O_M is the space of functions slow growth (slowly increasing, c.f. [8], page 243). Similary from [4], we have

(35) $$\{R_\alpha^e(u)\}^\wedge \in O_M.$$

On the other hand, from [8] theorem XV, page 268 the Fourier's transforms F and F are reciprocal isomorphisms form O_M and O_c' respectively. In addition if

(36) $$T \in O_M \Rightarrow \bar{F}\{T\} \in O_c'$$

and if

(37) $$T \in O_c' \Rightarrow \bar{F}\{T\} \in O_M$$

where O_c' is the space of rapidly decreasing distributions and if

$$g = F\{f\} \Rightarrow f = \bar{F}\{g\} = F^{-1}\{g\}$$

Now putting

(38) $$H_{\alpha,\beta} = \{R_\alpha^H(u)\}^\wedge \{R_\alpha^e(u)\}^\wedge$$

and considering (34) and (35) we have

(39) $$H_{\alpha,\beta} \in O_M$$

Therefore considerring (36), (37), (38) and (39) we have

(40) $$\bar{F}\{H_{\alpha,\beta}\} = F^{-1}\{H_{\alpha,\beta}\} \in O_c'$$

Taking into account (38) and (40) we can define the distribution families $K_{\alpha,\beta}$ in the following from

(41) $$K_{\alpha,\beta} = K_{\alpha,\beta}(x) = R_\alpha^H(u) * R_\alpha^e(x) = F^{-1}\{\{R_\alpha^H(u)\}^\wedge.\{R_\alpha^e(x)\}^\wedge\}$$

From (40) the families $K_{\alpha,\beta}$ exists an is in O_c'.

Lemma 3.1. *The following formulae are valid*

(i) $K_{0,0}(x) = \delta(x)$
(ii) $K_{-2k,-2k}(x) = (-1)^k \Diamond^k \delta(x)$
(iii) $\Diamond^k (K_{\alpha,\beta}(x)) = (-1)^k K_{\alpha-2k,\beta-2k}(x)$
(iv) $\Diamond^k (K_{2k,2k}(x)) = (-1)^k \delta(x)$.

Proof.

(i) By (14) $K_{0,0}(x) = R_0^e * R_0^H$, and by Lemma 2.3(i) and Lemma 2.5(i) we obtain $K_{0,0}(x) = \delta * \delta = \delta$

(ii) We have

$$\begin{aligned}\Diamond^k K_{\alpha,\beta}(x) &= \Diamond^k (R_\alpha^e * R_\beta^H)\\ &= \Box^k \Delta^k (R_\alpha^e * R_\beta^H)\\ &= \Delta^k R_\alpha^e * \Box^k R_\beta^H\\ &= (-1)^k R_{\alpha-2k}^e * R_{\beta-2k}^H \text{ by Lemma 2.3(iii) and Lemma 2.5(ii)}\\ &= (-1)^k K_{\alpha-2k,\beta-2k}(x)\end{aligned}$$

putting $\alpha = \beta = 0$ and (i) we obtain $K_{-2k,-2k}(x) = (-1)^k \Diamond^k \delta(x)$.

(iii) Similarly as (ii)

(iv) Putting $\alpha = \beta = 2k$ in (iii) we obtain

$$\Diamond^k (K_{2k,2k}(x)) = (-1)^k K_{0,0}(x) = (-1)^k \delta(x) \qquad \Box$$

4. Main results.

Theorem 4.1. *Let the families $K_{\alpha,\beta}(x)$ and $K_{\alpha',\beta'}(x)$ be defined by (14) then the convolution product $K_{\alpha,\beta}(x) * K_{\alpha,\beta'}(x)$ can be obtained by the following formulae*

(i) $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = B_{\beta,\beta'} R_{\beta+\beta'}^H * R_{\alpha+\alpha'}^e$ *where* R_β^H *and* R_α^e *are defined by (8) and (5) respectively which* p *is an even and*

$$B_{\beta,\beta'} \frac{\cos(\frac{\beta}{2}\pi)\cos(\frac{\beta'}{2}\pi)}{\cos(\frac{\beta+\beta'}{2}\pi)}$$

(ii) $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R_{\beta+\beta'}^H + T_{\beta,\beta'}) * R_{\alpha+\alpha'}^e$ *if* p *is an odd and* $T_{\beta,\beta'}$ *is defined by (13)*

(iii) $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (-1)^k \Diamond^k K_{\alpha,\beta}(x)$.

Proof.

(i) We have

$$K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R^e_\alpha * R^H_\beta) * (R^e_{\alpha'} * R^H_{\beta'})$$
$$= (R^e_\alpha * R^e_{\alpha'}) * (R^H_\beta * R^H_{\beta'})$$
$$= R^e_{\alpha+\alpha'} * (R^H_\beta * R^H_{\beta'}) \text{by (6)}$$
$$= (R^H_\beta * R^H_{\beta'}) * R^e_{\alpha+\alpha'}$$
$$= B_{\beta,\beta'} R^H_{\beta+\beta'} * R^e_{\alpha+\alpha'} \text{by Lemma 2.4(i) for } p \text{ is even,}$$

where $B_{\beta,\beta'} = \frac{\cos(\frac{\beta}{2}\pi)\cos(\frac{\beta'}{2}\pi)}{\cos(\frac{\beta+\beta'}{2}\pi)}$.

(ii) from(i), $K_{\alpha,\beta}(x) * K_{\alpha',\beta'}(x) = (R^H_\beta * R^H_{\beta'}) * R^e_{\alpha+\alpha'} = (R^H_{\beta+\beta'} + T_{\beta,\beta'}) * R^e_{\alpha+\alpha'}$ by Lemma 2.2(ii) for p is odd and $T_{\beta,\beta'}$ is defined by (14)

(iii) we have $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = B_{\beta,-2k} R^H_{\beta-2k} * R^e_{\alpha-2k}$ for p is even. Since

$$B_{\beta,-2k} = \frac{\cos(\frac{\beta}{2}\pi)\cos(-2k)\frac{\pi}{2}}{\cos(\frac{\beta-2k}{2}\pi)} = 1,$$

we have $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = R^H_{\beta-2k} * R^e_{\alpha-2k} = K_{\alpha-2k,\beta-2k}(x)$. Now for p is odd, we have $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (R^H_{\beta-2k} + T_{\beta,-2k}) * R^e_{\alpha-2k}$. By (14) $T_{\beta,-2k} = \frac{\frac{2\pi i}{4} C(-\frac{\beta+2k}{2})}{C(-\frac{\beta}{2})C(\frac{2k}{2})}[H^+_{\beta-2k} - H^-_{\beta-2k}]$ where $C(r) = \Gamma(r)\Gamma(1-r)$, $H^\pm_r = e^{\mp\frac{r\pi}{2}i} e^{\pm\frac{q\pi}{2}i} a(\frac{r}{2})(u \pm io)^{\frac{r-n}{2}}$ and $a(\frac{r}{2}) = \Gamma(\frac{n-r}{2})[2^r \pi^{\frac{n}{2}} \Gamma(\frac{r}{2})]^{-1}$. Applying the formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin 2\pi}$ to $C(-\frac{\beta+2k}{2})$, $C(-\frac{\beta}{2})$ and $C(k)$ and also the formulae $H^\pm_{\beta-2k}$ and $a(\frac{\beta-2k}{2})$ we obtain $T_{\beta,-2k} = 0$ and $T_{-2k,\beta} = 0$. It follows that $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = R^H_{\beta-2k} * R^e_{\alpha-2k} = K_{\alpha-2k,\beta-2k}(x)$ for p is odd. Now $\diamond^k K_{\alpha,\beta}(x) = (-1)^k K_{\alpha-2k,\beta-2k}(x)$ by Lemma 3.1(iii).

Thus $K_{\alpha,\beta}(x) * K_{-2k,-2k}(x) = (-1)^k \diamond^k K_{\alpha,\beta}(x)$. That completes the proofs. □

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