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HARNACK TYPE INEQUALITIES FOR THE PARABOLIC LOGARITHMIC P-LAPLACIAN EQUATION

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In this note, we concern with a class of doubly nonlinear operators whose prototype is

$$u_t - \operatorname{div}\big(|u|^{m-1}|Du|^{p-2}Du\big) = 0, \qquad p > 1, \qquad m + p = 2.$$

In the last few years many progresses were made in understanding the right form of the Harnack inequalities for singular parabolic equations. For doubly nonlinear equations the singular case corresponds to the range $m+p<3$. For $3-p/N<m+p<3$, where N denotes the space dimension, intrinsic Harnack estimates hold. In the range $2<m+p\leq 3-p/N$ only a weaker Harnack form survives. In the limiting case $m+p=2$, only the case $p=2$ was studied. In this paper we fill this gap and we study the behaviour of the solutions in the full range $p>1$ and $m=2-p$.

1. Introduction

Harnack estimates for parabolic operators were first established in the fifties by Hadamard [20] and Pini [31] who extended the Harnack theory to the heat equation using representation formulas. A breakthrough was made by Moser

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[30] who extended it to linear parabolic equations with measurable coefficients. Ivanov [23] considered the setting of quasilinear second order parabolic equations, and Serrin [33] and Trudinger [34] worked within a nonlinear setting. The extension of Harnack estimates to the p-Laplacian and to the Porous Medium equations were extremely more complicated. In the singular case, i.e $p < 2$ or $m < 1$ (in such a range the Porous Medium Equations are called Fast Diffusion Equations), the first results were obtained for the prototype equations for the super-critical case, i.e $2N/(N+1) < p < 2$ and $(N-2)_+/N < m < 1$ (N is the space dimension) - DiBenedetto and Kwong [11] proved the so called intrinsic Harnack estimates. The right form of singular intrinsic Harnack estimates and the extension to general operators were obtained by DiBenedetto, Gianazza and Vespri in [8]. In that paper, it was also proved that intrinsic Harnack inequalities cannot hold in the so called sub-critical range, i.e $1 < p \leq 2N/(N+1)$ and $0 < m \leq (N-2)_+/N$. The right form of the weak Harnack inequalities for the sub-critical range is due by Bonforte, Iagar and Vázquez ([1] and [2]) for the prototype equations. The result for the general case was proved in [10] (see also [18]). The singular equation

$$u_t - \operatorname{div}\big(D \ln u\big) = 0,$$

that can be seen as the limit case of the Porous Medium equation when $m \to 0^+$, was studied by Davis, DiBenedetto and Diller [4] and, some years later, DiBenedetto, Gianazza and Liao [6] proved an intrinsic Harnack-type inequality. These same authors worked with a logarithmically singular equation which was treated as the limit of a family of porous medium equations, for $0 < |m| < 1$ (see [7]). This equation, as it is linked to the Ricci flow, was studied by several authors (among them, we quote [3], [12], [13], [35], [36]) that proved interesting and quite surprising properties satisfied by its solutions.

As there is a strong correspondence between Porous Medium Equation and the p-Laplacian equation, it is natural to find a class of equations enjoying the same properties. This class is the one of the doubly nonlinear equations

$$u_t - \operatorname{div}\big(|u|^{m-1}|Du|^{p-2}Du\big) = 0, \qquad p > 1, \quad m > 0. \tag{1}$$

The doubly nonlinear term, appearing in the principal part of the operator, is due to the fact that the diffusion coefficient depends both on the gradient and on the solution itself. Such kind of equations rules several physical phenomena such as the dynamics of the turbulent flow of a non-Newtonian polytropic fluid through a porous medium. These equations were introduced, for the first time, by Lions [28]. Especially in the last years, many are the papers devoted to this topic (for the first contributions on the subject, we refer the reader to the survey by Kalashnikov [25]). But, as already written above, the reason of the interest of these equations consists in being a natural bridge between the more natural generalisations of the heat equation: the p-Laplacian and the Porous Medium equations. For doubly nonlinear equations, the degenerate case is represented by the values m, p verifying $m+p>3$. The case when $m+p=3$ is known as the Trudinger's equation (it was introduced by Trudinger in [34]) . The case $2<m+p<3$ corresponds to the singular case. Lastly, when $m+p=2$, we are in the logarithmic case.

The regularity for the degenerate case was faced by Ivanov [24] and Porzio-Vespri [32]. For the Harnack inequality in the degenerate range (corresponding to $m+p\geq 3$, $p\geq 2$ and $m\geq 1$) see, for instance, Fornaro and Sosio [14]. Nowadays the Trudinger's equation is widely studied by the Scandinavian school (see, for instance, [26]). The first proof of a Harnack inequality can be found in [19].

In the case of singular equations, Vespri [37] obtained a Harnack estimate within the singular setting $3-p/N<m+p<3$. By considering a more general equation, having (1) as a prototype and working for $2<m+p<3$, Fornaro, Sosio and Vespri obtained an integral Harnack estimate and a pointwise Harnack-type estimate, respectively in [15] and [16].

To our knowledge, Harnack estimates for doubly nonlinear logarithmic equations were never proved. The main goal of this work is to fulfill this gap, i.e. to present a weak Harnack estimate for the weak solutions to (1) taken within the wider setting $m+p=2$, $p>1$. Note that it corresponds to see (1) as

$$u_t-\operatorname{div}\left(u^{1-p}|Du|^{p-2}Du\right)=0, \quad p>1 \tag{2}$$

which is equivalent to consider the parabolic p-Laplacian of $\ln u$ (say parabolic logarithmic p-Laplacian)

$$u_t-\Delta_p(\ln u)=u_t-\operatorname{div}\left(|D\ln u|^{p-2}D\ln u\right)=0, \quad p>1. \tag{3}$$

To prove an intrinsic-weak Harnack estimate (for a better understanding of this see Remark 2.4) we follow the regularity approach due to DiBenedetto ([10]) that relies on several estimates, namely energy estimates, L^r_{loc} and $L^r_{loc}-L^\infty_{loc}$ estimates, and results such as a De Giorgi-type lemma and an expansion of positivity. These will be the contents of the sections to come.

2. Intrinsic Harnack estimate

We start this section by defining what we mean by weak solution, which will be done in the usual regularity setting.

Let E be an open set of $\mathbf{R}^N$ and $T>0$ and define $E_T=E\times(0,T]$. We say that a nonnegative function

$$u\in C(0,T;L^2_{\text{loc}}(E)), \qquad \ln u\in L^p_{\text{loc}}(0,T;W^{1,p}_{\text{loc}}(E))$$

is a locally weak sub(super)-solution to (3) if

$$\int_K u\psi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \Big(-u\psi_t + |D\ln u|^{p-2}D\ln u\cdot D\psi\Big) dxdt \leq (\geq)\ 0 \qquad (4)$$

for every compact set $K \subset E$, for every sub-interval $[t_1,t_2] \subset (0,T]$ and for all nonnegative test functions

$$\psi \in W^{1,2}_{\text{loc}}(0,T;L^2(K)) \cap L^p_{\text{loc}}(0,T;W^{1,p}_0(K)).$$

A nonnegative function u is called a locally weak *solution* if it is both a locally weak sub-solution and super-solution.

Remark 2.1. The integrability hypothesis on u ensures that the integrals in (4) are well defined.

It is known that in the singular setting, the notion of boundedness for the weak solutions does not come along with the definition of weak solution; thereby an extra regularity assumption is in force. For a full proof on this subject see [21]-[22].

Following the ideas presented in [4]-[6], let $r > 1$ be any number such that

$$\lambda_r \stackrel{\text{def}}{=} rp - N > 0 \text{ and } u \in L^r_{\text{loc}}(E_T), \qquad (5)$$

which will allow us to transform qualitatively information on the local boundedness of u into quantitatively information (see section 4 for more details). This qualitatively information gives sense to the space derivative Du. In fact, observe that by taking into account the work of Markus and Mizel [29], one has that $u = e^v$, $v = \ln u$, is locally strongly differentiable if and only if its formal derivative is locally integrable, in which case the formal derivative of u is also its strong derivative. The formal derivative of u is $Du = uDv$, being $u \in L^\infty_{loc}(E_T)$ and $Dv \in L^p_{\text{loc}}(E_T)$, therefore Du is locally integrable. Hence we also have for locally bounded u the inequalities

$$\int_K u\psi dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \Big(-u\psi_t + u^{1-p}|Du|^{p-2}Du\cdot D\psi\Big) dxdt \leq (\geq)\ 0 \qquad (6)$$

for every compact set $K \subset E$, for every sub-interval $[t_1,t_2] \subset (0,T]$ and for all nonnegative test functions

$$\psi \in W^{1,2}_{\text{loc}}(0,T;L^2(K)) \cap L^p_{\text{loc}}(0,T;W^{1,p}_0(K)).$$

Before presenting the main result, we introduce some notation.

We denote by $K_\rho(y)$ the cube of $\mathbf{R}^N$ centered at y with edge 2ρ. If $y=0$, we simply write K_ρ instead of $K_\rho(0)$. Let $\theta>0$. We define the cylinders

$$Q_\rho^-(\theta)=K_\rho\times(-\theta\rho^p,0], \qquad Q_\rho^+(\theta)=K_\rho\times(0,\theta\rho^p]$$

and, for $(y,s)\in\mathbf{R}^N\times\mathbf{R}$,

$$(y,s)+Q_\rho^-(\theta)=K_\rho(y)\times(s-\theta\rho^p,s], \quad (y,s)+Q_\rho^+(\theta)=K_\rho(y)\times(s,s+\theta\rho^p].$$

Fix $(x_0,t_0)\in E_T$ and $\rho>0$ such that $K_{8\rho}(x_0)\subset E$, and introduce the quantity

$$\theta_0=\varepsilon\left(\fint_{K_\rho(x_0)}u^q(\cdot,t_0)dx\right)^{\frac{1}{q}}, \tag{7}$$

where $\varepsilon\in(0,1)$ is to be chosen, and $q>1$ is arbitrary. If $\theta_0>0$ assume that

$$(x_0,t_0)+Q_{8\rho}^-(\theta_0)=K_{8\rho}(x_0)\times(t_0-\theta_0(8\rho)^p,t_0]\subset E_T,$$

and set

$$\eta=\left[\frac{\left(\fint_{K_\rho(x_0)}u^q(\cdot,t_0)dx\right)^{\frac{1}{q}}}{\left(\fint_{K_{4\rho}(x_0)}u^r(\cdot,t_0-\theta_0\rho^p)dx\right)^{\frac{1}{r}}}\right]^{\frac{rp}{\lambda_r}}, \tag{8}$$

where r satisfies (5). In addition, assume that u satisfies

$$\ln u\in L^\infty_{\mathrm{loc}}(0,T;L^\alpha_{\mathrm{loc}}(E)), \qquad \text{for } \alpha>N+p. \tag{9}$$

Set

$$\Lambda=\left[\sup_{t_0-\theta_0(8\rho)^p<t<t_0}\fint_{K_{8\rho}(x_0)}\left(\ln\frac{\sup_{(x_0,t_0)+Q_{8\rho}^-(\theta_0)}u}{u}\right)^\alpha dx\right]^{\frac{1}{\alpha}}. \tag{10}$$

The main result can be understood in the following way: if $u(\cdot,t_0)$ is not identically zero in a neighborhood of a certain point x_0, in a measure-theoretical sense, then it is pointwise positive in a neighborhood of (x_0,t_0). This is established by means of an intrinsic-weak Harnack-type inequality, which also determines the size of the neighborhood.

Theorem 2.2. *Let u be a nonnegative, locally bounded local weak solution to the singular equation (2), satisfying (9) in E_T. Introduce θ_0 as in (7) and assume that $\theta_0>0$. There exist constants $\varepsilon\in(0,1)$, and a continuous, increasing* *function $f_\Lambda(\eta)$, defined in $\mathbf{R}^+$ and such that $f_\Lambda(\eta)\to 0$, as $\eta\to 0$, that can be determined a priori only in terms of $\{\alpha,N,p,q,r\}$ and Λ, such that*

$$\inf_{K_{4\rho}(x_0)\times(t_0-\frac{\theta_0}{16}\rho^p,t_0)} u \geq f_\Lambda(\eta) \sup_{K_{2\rho}(x_0)\times(t_0-\frac{\theta_0}{2}\rho^p,t_0)} u. \tag{11}$$

The function $f_\Lambda(\eta)$ behaves has

$$\exp\left\{-\left(\frac{\Lambda^{C_1}}{\eta^{C_2}}\right)^2 e^{2\left(\frac{\Lambda^{C_1}}{\eta^{C_2}}\right)}\ln\Lambda\right\}, \qquad \textit{for} \quad 0<\eta\ll 1 \quad \textit{and} \quad \Lambda\gg 1,$$

being C_1 and C_2 positive constants depending only upon $\{p,N,q,\alpha\}$.

Remark 2.3. Inequality (11) is not a Harnack inequality per se, since η depends upon the solution itself. Therefore it can be regarded as a weak form of a Harnack estimate. Also the size of the cylinder depends on the solution, giving thereby the name intrinsic to the inequality. A similar result has been established in [17], for $2<m+p<3$.

3. Two results on supersolutions

Proposition 3.1. [Energy estimates] *Let u be a nonnegative, locally bounded local weak supersolution to (3) in E_T. Then for every cylinder $(y,s)+Q_\rho^-(\theta)\subset E_T$, $k>0$ and every nonnegative smooth cutoff function ζ vanishing on the boundary of $K_\rho(y)$, it holds*

$$\begin{aligned}
\sup_{s-\theta\rho^p<t\leq s}\int_{K_\rho(y)}&(u-k)_-^2\zeta^p(x,t)dx-2k\int_{K_\rho(y)}(u-k)_-\zeta^p(x,s-\theta\rho^p)dx\\
&+\frac{1}{k^{p-1}}\iint_{(y,s)+Q_\rho^-(\theta)}|D[(u-k)_-\zeta]|^p dxdt\\
&\leq 2kp\iint_{(y,s)+Q_\rho^-(\theta)}(u-k)_-\zeta|\zeta_t|dxdt\\
&+2^p(p-1)^{p-1}k\iint_{(y,s)+Q_\rho^-(\theta)}\chi_{[u<k]}\left(\ln\frac{k}{u}\right)_+^p|D\zeta|^p dxdt\\
&+\frac{1}{k^{p-1}}\iint_{(y,s)+Q_\rho^-(\theta)}(u-k)_-^p|D\zeta|^p dxdt.
\end{aligned} \tag{12}$$

Analogous estimates hold in the cylinder $(y,s)+Q_\rho^+(\theta)\subset E_T$.

Proof. Without loss of generality, we assume $(y,s)=(0,0)$. It is well known that the time derivative u_t has to be avoided in a certain sense since it does not necessarily exist in Sobolev's sense. We use a regularization to overcome this difficulty, more precisely we consider

$$u^\star(x,t) = \frac{1}{\sigma}\int_0^t e^{\frac{s-t}{\sigma}} u(x,s)\, ds, \quad \sigma > 0 \tag{13}$$

used by Kinnunen and Lindqvist [27] when studying several properties for the porous medium equation. This average only needs to consider values of $u(x,t)$ taken in E_T, it is defined at each point, for continuous or bounded and semicontinuous functions u, and verifies

$$\frac{u-u^\star}{\sigma} = (u^\star)_t$$

which implies

$$(\ln u - \ln u^\star)(u^\star)_t \geq 0\ .$$

For regularity results on this average we refer to Lemma 2.1 in [27].

The average inequality for a nonnegative weak supersolution u in E_T to equation (3) is the following. For every compact set $K \subset E$, for every subinterval $[t_1,t_2] \subset (0,T]$ and for all nonnegative test functions ψ belonging to $L^2_{\text{loc}}(0,T;L^2(K)) \cap L^p_{\text{loc}}(0,T;W^{1,p}_0(K))$,

$$\begin{aligned}\int_{t_1}^{t_2}\int_K \Big((u^\star)_t\psi + \big(|D\ln u|^{p-2}D\ln u\big)^\star \cdot D\psi\Big)dxdt \\ \geq \int_K u(x,t_1)\left(\frac{1}{\sigma}\int_{t_1}^{t_2}\psi(x,s)e^{-s/\sigma}\, ds\right)\, dx\end{aligned}$$

and then, since both u and ψ are nonnegative functions, one gets

$$\int_{t_1}^{t_2}\int_K \Big((u^\star)_t\psi + \big(|D\ln u|^{p-2}D\ln u\big)^\star \cdot D\psi\Big)dxdt \geq 0. \tag{14}$$

Now in (14) we consider the integration over $Q_\tau = K_\rho \times (-\theta\rho^p, \tau]$, where $-\theta\rho^p < \tau \leq 0$, and take $\psi = (\ln k - \ln u)_+\zeta^p$ where ζ is a nonnegative smooth cutoff function vanishing on the boundary. The parabolic term verifies

$$
\begin{aligned}
&\iint_{Q_\tau} (u^\star)_t (\ln k - \ln u)_+ \zeta^p dxdt \\
&\qquad = \iint_{Q_\tau} (u^\star)_t (\ln u - \ln u^\star + \ln u^\star - \ln k) \zeta^p \chi_{[u<k]} dxdt \\
&\qquad = \iint_{Q_\tau} \partial_t \left(\int_{u^\star}^k (\ln k - \ln s)\, ds \right) \zeta^p \chi_{[u<k]} dxdt \\
&\qquad\quad + \iint_{Q_\tau} (u^\star)_t (\ln u - \ln u^\star) \zeta^p \chi_{[u<k]} dxdt \\
&\qquad \geq \iint_{Q_\tau} \partial_t \left(\int_{u^\star}^k (\ln k - \ln s)\, ds \right) \zeta^p \chi_{[u<k]} dxdt \\
&\qquad = \int_{K_\rho} \left(\int_{u^\star(x,\tau)}^k (\ln k - \ln s)\, ds \right) \zeta^p(x,\tau) \chi_{[u<k]} dx \\
&\qquad\quad - \int_{K_\rho} \left(\int_{u^\star(x,-\theta\rho^p)}^k (\ln k - \ln s)\, ds \right) \zeta^p(x,-\theta\rho^p) \chi_{[u<k]} dx \\
&\qquad\quad - p \iint_{Q_\tau} \left(\int_{u^\star}^k (\ln k - \ln s)\, ds \right) \zeta^{p-1} \zeta_t \chi_{[u<k]} dxdt.
\end{aligned}
$$

We then pass to the limit as $\sigma \to 0$ to obtain the inferior bound

$$
\begin{aligned}
&\int_{K_\rho \times \{\tau\}} \left(\int_u^k (\ln k - \ln s)\, ds \right) \zeta^p \chi_{[u<k]} dx \\
&\qquad - \int_{K_\rho \times \{-\theta\rho^p\}} \left(\int_u^k (\ln k - \ln s)\, ds \right) \zeta^p \chi_{[u<k]} dx \\
&\qquad - p \iint_{Q_\tau} \int_u^k (\ln k - \ln s)\, ds \zeta^{p-1} \zeta_t \chi_{[u<k]} dxdt.
\end{aligned}
$$

In order to estimate $\displaystyle\int_u^k (\ln k - \ln s)_+ ds$, from below and above, we use two different arguments.

On the one hand, due to the convexity of $f(s) = \ln k - \ln s$, we have $\ln k - \ln s \geq \frac{1}{k}(k-s)$, for every $s \in [u,k]$, thereby

$$
\int_u^k (\ln k - \ln s)_+ ds \geq \frac{1}{k} \int_u^k (k-s) ds = \frac{1}{2k}(u-k)_-^2.
$$

On the other hand, by performing the integration and recalling $u(\ln k - \ln u) \geq 0$, when $0 \leq u \leq k$, we have

$$
\int_u^k (\ln k - \ln s)_+ ds = k - u - u(\ln k - \ln u) \leq (u-k)_-.
$$

Therefore the parabolic term of (3) can be estimated from below by

$$\frac{1}{2k}\int_{K_\rho}(u-k)_-^2\zeta^p(x,\tau)dx-\int_{K_\rho}(u-k)_-\zeta^p(x,-\theta\rho^p)dx \\ -p\iint_{Q_\tau}(u-k)_-\zeta^{p-1}|\zeta_t|dxdt.$$

Concerning the elliptic term, one starts by recalling the regularity assumption $D\ln u\in L^p_{loc}$ and then the properties stated in Lemma 2.1 [27] allow us to pass to the limit as $\sigma\to 0$ and get

$$\iint_{Q_\tau}|D\ln u|^{p-2}D(\ln u)\cdot D[(\ln k-\ln u)_+\zeta^p]dxdt.$$

By means of Young's inequality with $\varepsilon=(2(p-1))^{\frac{p-1}{p}}$, we get

$$\begin{aligned}
&\iint_{Q_\tau}|D\ln u|^{p-2}D(\ln u)\cdot D[(\ln k-\ln u)_+\zeta^p]dxdt\\
&=\iint_{Q_\tau}\frac{|D(u-k)_-|^p}{u^p}\zeta^p dxdt\\
&\quad -p\iint_{Q_\tau}\zeta^{p-1}(\ln k-\ln u)_+|D\ln u|^{p-2}D(\ln u)\cdot D\zeta dxdt\\
&\geq\frac{1}{2}\iint_{Q_\tau}\frac{|D(u-k)_-|^p}{u^p}\zeta^p dxdt\\
&\quad -[2(p-1)]^{p-1}\iint_{Q_\tau}(\ln k-\ln u)_+^p|D\zeta|^p dxdt\\
&\geq\frac{1}{2k^p}\iint_{Q_\tau}|D(u-k)_-|^p\zeta^p dxdt\\
&\quad -[2(p-1)]^{p-1}\iint_{Q_\tau}\chi_{[u<k]}\left(\ln\frac{k}{u}\right)_+^p|D\zeta|^p dxdt\\
&\geq\frac{1}{2^pk^p}\iint_{Q_\tau}|D[(u-k)_-\zeta]|^p dxdt\\
&\quad -[2(p-1)]^{p-1}\iint_{Q_\tau}\chi_{[u<k]}\left(\ln\frac{k}{u}\right)_+^p|D\zeta|^p dxdt\\
&\quad -\frac{1}{2k^p}\iint_{Q_\tau}(u-k)_-^p|D\zeta|^p dxdt
\end{aligned}$$

Combining all the estimates so far and taking the supremum over τ, we obtain (12). □

Let us now consider u to be a nonnegative locally bounded local weak supersolution to (3) in E_T and $\theta>0$. Assume that the cylinder

$$(y,s)+Q^-_{8\rho}(\theta)=K_{8\rho}(y)\times(s-\theta(8\rho)^p,s]$$

is contained in E_T and take $M>0$ such that

$$\operatorname{essinf}_{(y,s)+Q^-_{8\rho}(\theta)} u < M \le \operatorname{essup}_{(y,s)+Q^-_{8\rho}(\theta)} u. \tag{15}$$

Assuming that u satisfies (9), set

$$\Lambda_{8\rho,M} = \max\left\{1, \left[\sup_{s-\theta(8\rho)^p<t<s} \fint_{K_{8\rho}(y)} \left(\ln\frac{M}{u}\right)_+^{\alpha} dx\right]^{\frac{1}{\alpha}}\right\}. \tag{16}$$

Lemma 3.2. [De Giorgi-type lemma] *For every ξ and a in the interval $(0,1)$, there exists a positive number ν, depending on $M,\theta,\xi,a,N,p,\Lambda_{8\rho,M}$ such that if*

$$|[u\le \xi M]\cap (y,s)+Q^-_{2\rho}(\theta)| \le \nu |Q^-_{2\rho}(\theta)|$$

then

$$u \ge a\xi M \qquad \text{a.e. in } (y,s)+Q^-_{\rho}(\theta).$$

Proof. We limit ourselves to the case $(y,s)=(0,0)$, which is admissible via a translation argument.

Introduce the decreasing sequences of numbers

$$\rho_n = \rho + \frac{\rho}{2^n} \in (\rho, 2\rho], \qquad k_n = \xi_n M, \text{ where } \quad \xi_n = a\xi + \frac{1-a}{2^n}\xi \in (a\xi, \xi]$$

and construct the sequences of nested cubes and cylinders

$$K_n = K_{\rho_n}, \qquad Q_n = K_n \times (-\theta\rho_n^p, 0],$$

for $n=0,1,2,\ldots$, over which we define the cutoff function $\zeta(x,t)=\zeta_1(x)\zeta_2(t)$ verifying

$$\zeta_1 = \begin{cases} 1 & \text{in} \quad K_{n+1} \\ 0 & \text{in} \quad \mathbb{R}^N \setminus K_n \end{cases} \qquad |D\zeta_1| \le \frac{1}{\rho_n - \rho_{n+1}} = \frac{2^{n+1}}{\rho}, \tag{17}$$

and

$$\zeta_2 = \begin{cases} 0 & \text{if} \quad t \le -\theta\rho_n^p \\ 1 & \text{if} \quad t \ge -\theta\rho_{n+1}^p \end{cases} \qquad 0 \le (\zeta_2)_t \le \frac{2^{p(n+1)}}{\theta\rho^p}. \tag{18}$$

For the above choices, the energy estimates (12) now read

$$\begin{aligned}
&\sup_{-\theta\rho_n^p<t\le 0} \int_{K_n} (u-k_n)_-^2 \zeta^p(x,t)dx + \frac{1}{k_n^{p-1}} \iint_{Q_n} |D[(u-k_n)_-\zeta]|^p dxd\tau \\
&\quad \le p2^{2p+1}\frac{2^{pn}}{\rho^p} k_n \left(1+\frac{k_n}{\theta}\right) \big|[u<k_n]\cap Q_n\big| \\
&\quad + \frac{2^{pn}}{\rho^p} 2^{2p}(p-1)^{p-1} k_n \left(\iint_{Q_n} \left(\ln\frac{M}{u}\right)_+^{\alpha} dxd\tau\right)^{\frac{p}{\alpha}} \big|[u<k_n]\cap Q_n\big|^{1-\frac{p}{\alpha}} \\
&\quad \le c_p \frac{2^{pn}}{\rho^p} k_n \left\{ \left(1+\frac{k_n}{\theta}\right)|A_n| + \Lambda_{8\rho,M}^p |A_n|^{1-\frac{p}{\alpha}} |Q_n|^{\frac{p}{\alpha}} \right\},
\end{aligned} \tag{19}$$

where $c_p = \max\{2^{2p}(p-1)^{p-1}, 2^{2p+1}p\}$ and $A_n = [u < k_n] \cap Q_n$.

Since $(u-k_n)_- \le k_n \le \xi M$ and $p > 1$, in the previous estimate we can substitute k_n by ξM. To proceed with the estimates we have to treat separately the cases $p \ge 2$ and $1 < p < 2$.

At first consider $p \ge 2$.

Now observe that, by recalling the definition of A_n, applying consecutively Hölder's inequality and then the Sobolev embedding (see Proposition3.1, chapter I, in [5]) and using the previous estimate, we arrive at

$$\begin{aligned}
&\left(\frac{1-a}{2^{n+1}}\right)^p (\xi M)^p |A_{n+1}| \\
&\qquad \le \iint_{Q_{n+1}} (u-k_n)_-^p \, dx d\tau \\
&\qquad \le \left(\iint_{Q_n} [(u-k_n)_- \zeta]^{p\frac{N+p}{N}} dx d\tau \right)^{\frac{N}{N+p}} |A_n|^{\frac{p}{N+p}} \\
&\qquad \le \gamma \left(\iint_{Q_n} |D[(u-k_n)_- \zeta]|^p dx d\tau \right)^{\frac{N}{N+p}} \\
&\qquad\quad \times \left(\sup_{-\theta\rho_n^p < t \le 0} \int_{K_n} |(u-k_n)_- \zeta|^p (x,t) dx \right)^{\frac{p}{N+p}} |A_n|^{\frac{p}{N+p}} \\
&\qquad \le \gamma \left(c_p \frac{2^{pn}}{\rho^p} (\xi M)^p \left\{ \left(1 + \frac{\xi M}{\theta}\right) |A_n| + \Lambda^p_{8\rho,M} |A_n|^{1-\frac{p}{\alpha}} |Q_n|^{\frac{p}{\alpha}} \right\} \right)^{\frac{N}{N+p}} \\
&\qquad\quad \times \left(\sup_{-\theta\rho_n^p < t \le 0} \int_{K_n} |(u-k_n)_- \zeta|^p (x,t) dx \right)^{\frac{p}{N+p}} |A_n|^{\frac{p}{N+p}},
\end{aligned} \tag{20}$$

where γ depends only upon N and p.

Since $p \ge 2$ we have

$$(u-k_n)_-^p \le (\xi M)^{p-2} (u-k_n)_-^2$$

and this allows to estimate the sup-term in (20). Combining (20) and (19) we obtain

$$\begin{aligned}
&\left(\frac{1-a}{2^{n+1}}\right)^p (\xi M)^p |A_{n+1}| \\
&\le \gamma c_p \frac{2^{pn}}{(\xi M)^{\frac{p}{N+p}} \rho^p} (\xi M)^p \left\{ \left(1 + \frac{\xi M}{\theta}\right) |A_n|^{1+\frac{p}{N+p}} + \Lambda^p_{8\rho,M} |A_n|^{1+\frac{p}{N+p}-\frac{p}{\alpha}} |Q_n|^{\frac{p}{\alpha}} \right\}
\end{aligned} \tag{21}$$

Setting $Y_n := \frac{|A_n|}{|Q_n|}$, from (21) and since $Y_n \leq 1$ and $\alpha > N+p$, we get

$$\begin{aligned} Y_{n+1} &\leq \tilde{\gamma}\frac{2^{n(p+1)}}{(1-a)^p}\left(\frac{\theta}{\xi M}\right)^{\frac{p}{p+N}}\left(1+\frac{\xi M}{\theta}\right)\Lambda^p_{8\rho,M}\left(Y_n^{1+\frac{p}{N+p}} + Y_n^{1+\frac{p}{N+p}-\frac{p}{\alpha}}\right) \\ &\leq \tilde{\gamma}\frac{2^{n(p+1)}}{(1-a)^p}\left(\frac{\theta}{\xi M}\right)^{\frac{p}{p+N}}\left(1+\frac{\xi M}{\theta}\right)\Lambda^p_{8\rho,M} Y_n^{1+\beta}, \end{aligned}$$

for $\beta = \frac{p}{N+p} - \frac{p}{\alpha} > 0$ and $\tilde{\gamma} = 2^{N+2p+1+\frac{Np}{N+p}}\gamma c_p$.

If $1 < p < 2$, we start by considering the average level $\tilde{k}_n = \frac{k_n+k_{n+1}}{2}$ and observing that

$$\begin{aligned} \int_{K_n}(u-k_n)^2_-\zeta^p(x,t)dx &\geq \int_{K_n\cap[u<\tilde{k}_n]}(u-k_n)_-^{2-p}(u-k_n)^p_-\zeta^p(x,t)dx \\ &\geq \left(\frac{(1-a)\xi M}{2^{n+2}}\right)^{2-p}\int_{K_n}(u-\tilde{k}_n)^p_-\zeta^p(x,t)dx. \end{aligned}$$

Then

$$\begin{aligned} &\left(\frac{1-a}{2^{n+2}}\right)^p(\xi M)^p|A_{n+1}| \\ &\quad\leq \iint_{Q_{n+1}}(u-\tilde{k}_n)^p_-dxd\tau \\ &\quad\leq \left(\iint_{Q_n}[(u-\tilde{k}_n)_-\zeta]^{p\frac{N+p}{N}}dxd\tau\right)^{\frac{N}{N+p}}|A_n|^{\frac{p}{N+p}} \\ &\quad\leq \gamma\left(\iint_{Q_n}|D[(u-\tilde{k}_n)_-\zeta]|^p dxd\tau\right)^{\frac{N}{N+p}} \\ &\quad\times\left(\sup_{-\theta\rho_n^p<t\leq 0}\int_{K_n}|(u-\tilde{k}_n)_-\zeta|^p(x,t)dx\right)^{\frac{p}{N+p}}|[u<\tilde{k}_n]\cap Q_n|^{\frac{p}{N+p}} \\ &\quad\leq \gamma\left(c_p\frac{2^{pn}}{\rho^p}(\xi M)^p\left\{\left(1+\frac{\xi M}{\theta}\right)|A_n| + \Lambda^p_{8\rho,M}|A_n|^{1-\frac{p}{\alpha}}|Q_n|^{\frac{p}{\alpha}}\right\}\right)^{\frac{N}{N+p}} \\ &\quad\times\left(\sup_{-\theta\rho_n^p<t\leq 0}\int_{K_n}|(u-\tilde{k}_n)_-\zeta|^p(x,t)dx\right)^{\frac{p}{N+p}}|A_n|^{\frac{p}{N+p}}. \end{aligned} \tag{22}$$

Combining the two previous estimates we arrive at

$$\begin{aligned} |A_{n+1}| &\leq \gamma c_p 2^{2p+1}\frac{2^{(2p+1)n}}{(\xi M)^{\frac{p}{N+p}}\rho^p}\left(\frac{1}{1-a}\right)^{\frac{p(N+2)}{N+p}} \\ &\quad\times\left\{\left(1+\frac{\xi M}{\theta}\right)|A_n|^{1+\frac{p}{N+p}} + \Lambda^p_{8\rho,M}|A_n|^{1+\frac{p}{N+p}-\frac{p}{\alpha}}|Q_n|^{\frac{p}{\alpha}}\right\} \end{aligned} \tag{23}$$

and therefore, for $\tilde{\gamma} = \gamma c_p 2^{N+4p+2+\frac{Np}{N+p}}$ and β the same as before,

$$Y_{n+1} \leq \tilde{\gamma} \frac{2^{n(2p+1)}}{(1-a)^{\frac{p(N+2)}{N+p}}} \left(\frac{\theta}{\xi M}\right)^{\frac{p}{p+N}} \left(1 + \frac{\xi M}{\theta}\right) \Lambda^p_{8\rho,M} Y_n^{1+\beta}$$

So, choosing conveniently $C = C(N, p, \Lambda, M, \theta, a)$ and $b = 2^{2p+1}$, we have a recursive algebraic estimate of the type

$$Y_{n+1} \leq C b^n Y_n^{1+\beta},$$

therefore, from a fast geometric convergence result (see Lemma 4.1, chap.1, in [5]), one has $Y_n \to 0$, as $n \to \infty$, if

$$Y_0 \leq C^{-\frac{1}{\beta}} b^{-\frac{1}{\beta^2}} =: \nu$$

More explicitly

$$\nu = \left(\frac{(1-a)^p}{\gamma 2^{2p/\beta}} \frac{\theta}{\theta + \xi M} \frac{1}{\Lambda^p_{8\rho,M}} \left(\frac{\xi M}{\theta}\right)^{\frac{p}{N+p}} \right)^{\frac{1}{\beta}}. \tag{24}$$

□

4. L^r_{loc} and $L^r_{loc} - L^\infty_{loc}$ estimates

In this section we present quantitative information on the local boundedness of u.

Proposition 4.1. [L^r_{loc} estimates backwards in time] *Let u be a nonnegative locally bounded local weak solution to (2) in E_T satisfying $u \in L^r_{\rm loc}(E_T)$, for $r > 1$. Assume that the cylinder $K_{2\rho}(y) \times [s,t]$ is included in E_T. Then there exists a positive constant γ, depending only upon the data $\{p, N\}$ and r, such that*

$$\sup_{s \leq \tau \leq t} \int_{K_\rho(y)} u^r(x,\tau) dx \leq \gamma \left(\int_{K_{2\rho}(y)} u^r(x,s) dx + \frac{(t-s)^r}{\rho^{pr-N}} \right).$$

Proof. At this stage we present the proof in a formal way, however one could argue as in the proof of the energy estimates considering the average function $u^\star$ and working with the average equation. So formally we multiply (2) by $\varphi = u^{r-1} \zeta^p$ and integrate over the cylinder $Q_\tau = K_{(1+\sigma)\rho}(y) \times (s, \tau]$, where $s < \tau \leq t$ and $0 < \sigma < 1$. The function $0 \leq \zeta(x) \leq 1$ is defined in $K_{(1+\sigma)\rho}(y)$ and verifies

$$\zeta = 1 \text{ in } K_\rho(y), \quad \zeta = 0 \text{ in } E \setminus K_{(1+\sigma)\rho}(y), \quad |D\zeta| \leq \frac{1}{\sigma\rho}.$$

We then get, for $s < \tau \leq t$,

$$\iint_{Q_\tau} u_t u^{r-1}\zeta^p dxdt + \iint_{Q_\tau} u^{1-p}|Du|^{p-2}Du \cdot D(u^{r-1}\zeta^p)dxdt = 0.$$

Since the cutoff function is independent of t, the parabolic term is easily given by

$$\frac{1}{r}\int_{K_{(1+\sigma)\rho}(y)\times\{\tau\}} u^r\zeta^p dx - \frac{1}{r}\int_{K_{(1+\sigma)\rho}(y)\times\{s\}} u^r\zeta^p dx.$$

As for the elliptic term, one gets

$$(r-1)\iint_{Q_\tau} u^{r-p-1}|Du|^p\zeta^p dxdt - p\iint_{Q_\tau} u^{r-p}|Du|^{p-2}\zeta^{p-1}Du\cdot D\zeta dxdt$$

and then, by means of Young's inequality with ε, it is bounded from below by

$$(r-1)(1-\frac{p-1}{(r-1)\varepsilon^{p/(p-1)}})\iint_{Q_\tau} u^{r-p-1}|Du|^p\zeta^p dxdt - \varepsilon^p\iint_{Q_\tau} u^{r-1}|D\zeta|^p dxdt.$$

$$= -\left(2\frac{p-1}{r-1}\right)^{p-1}\iint_{Q_\tau} u^{r-1}|D\zeta|^p dxdt,$$

by taking $\varepsilon = \left(2\ \frac{p-1}{r-1}\right)^{\frac{p-1}{p}}$.

Combining the previous estimates, recalling the definition of ζ and applying Hölder's inequality, one gets

$$\begin{aligned}
\sup_{s<\tau\leq t}\int_{K_\rho(y)\times\{\tau\}} u^r dx &\leq \sup_{s<\tau\leq t}\int_{K_{(1+\sigma)\rho}(y)\times\{\tau\}} u^r\zeta^p dx \\
&\leq \int_{K_{(1+\sigma)\rho}(y)\times\{s\}} u^r dx + r\left(2\ \frac{p-1}{r-1}\right)^{p-1}\iint_{Q_\tau} u^{r-1}|D\zeta|^p dxdt \\
&\leq \int_{K_{(1+\sigma)\rho}(y)\times\{s\}} u^r dx \\
&\quad + \frac{r}{\sigma^p\rho^p}\left(2\ \frac{p-1}{r-1}\right)^{p-1}\left(\sup_{s<\tau\leq t}\int_{K_{(1+\sigma)\rho}(y)\times\{\tau\}} u^r dx\right)^{\frac{r-1}{r}}(t-s)^{\frac{r-1}{r}}|Q_t|^{\frac{1}{r}} \\
&\leq \int_{K_{(1+\sigma)\rho}(y)\times\{s\}} u^r dx \\
&\quad + \frac{r2^{\frac{2N}{r}+p-1}}{\sigma^p}\left(\frac{p-1}{r-1}\right)^{p-1}\left(\frac{(t-s)^r}{\rho^{rp-N}}\right)^{\frac{1}{r}}\left(\sup_{s<\tau\leq t}\int_{K_{(1+\sigma)\rho}(y)\times\{\tau\}} u^r dx\right)^{\frac{r-1}{r}}
\end{aligned}$$

The idea now is to obtain an iterative relation regarding the values $\sup \int u^r$. For that purpose, for $n \in \mathbb{N}$, we consider the sequence of radii,

$$\rho_n = \rho \sum_{i=1}^{n} \frac{1}{2^i} = \rho\left(1 - \frac{1}{2^n}\right), \qquad \text{being} \quad \rho_{n+1} = (1+\sigma_n)\rho_n,$$

and define

$$Y_n = \sup_{s<\tau\leq t} \int_{K_{\rho_n}(y)\times\{\tau\}} u^r dx.$$

Applying the previous integral estimate to the sequences above, we arrive at the recursive inequality

$$Y_n \leq \int_{K_{2\rho}(y)\times\{s\}} u^r dx + \gamma 2^{np} \left(\frac{(t-s)^r}{\rho^{rp-N}}\right)^{\frac{1}{r}} Y_{n+1}^{1-\frac{1}{r}},$$

where $\gamma_1 = r2^{\frac{N}{r}+3p-1}\left(\frac{p-1}{r-1}\right)^{p-1}$. Moreover, after applying consecutive Young's inequality with ε, we obtain

$$\begin{aligned} Y_1 \leq & \left(1+\sum_{i=1}^{n-2}\varepsilon^i\right)\int_{K_{2\rho}(y)\times\{s\}} u^r dx \\ & + \left(1+\sum_{i=1}^{n-2}(2^{rp}\varepsilon)^i\right)\frac{2^{rp}}{\varepsilon^r}B + \varepsilon^{n-1}Y_n, \qquad \text{for} \quad n\geq 2, \end{aligned}$$

where $B = \frac{1}{r}\frac{(t-s)^r}{\rho^{rp-N}}\gamma_1^r$. Taking $\varepsilon = 2^{-(rp+1)}$, and since $u \in L^r_{\mathrm{loc}}(E_T)$,

$$Y_1 \leq \gamma\left\{\int_{K_{2\rho}(y)\times\{s\}} u^r dx + \frac{(t-s)^r}{\rho^{rp-N}}\right\},$$

where $\gamma = \max\left\{\frac{2^{rp+1}+1}{2^{rp+1}-1}, \frac{2^{rp+1}}{r^2}\gamma_1^r\right\}$.

$\square$

Remark 4.2. The constant $\gamma = \gamma(N,p,r)$ and it goes to infinity as $r \searrow 1$.

In what follows we present and prove a sup estimate for the weak solutions of (2), known in the literature as $L^r_{loc} - L^\infty_{loc}$ estimate.

Theorem 4.3. **[$L^r_{loc} - L^\infty_{loc}$ estimate]** *Let u be a nonnegative locally bounded local weak solution to the singular equation (2) in E_T satisfying (5), for $r > 1$. Then there exists a positive constant γ_r, depending only upon N, p and r, such that*

$$\sup_{K_{\frac{\rho}{2}}(y)\times[s,t]} u \leq \gamma_r \left(\frac{\rho^p}{t-s}\right)^{\frac{N}{rp-N}} \left(\frac{1}{\rho^N(t-s)}\int_{-t+2s}^{t}\int_{K_\rho(y)} u^r\right)^{\frac{p}{rp-N}} + \gamma_r\frac{t-s}{\rho^p}$$

for all cylinders

$$K_{2\rho}(y)\times[s-(t-s),s+(t-s)]\subset E_T.$$

The constant $\gamma_r\to\infty$ if either $\lambda_r\to 0$ or, when $1<p<2$, $r\to 1$.

Proof. Assume $(y,s)=(0,0)$ and for fixed $\sigma\in(0,1)$ set

$$\rho_n=\sigma\rho+\frac{1-\sigma}{2^{n+1}}\rho,\quad t_n=-\sigma t-\frac{1-\sigma}{2^n}t,\quad n=0,1,2,\dots$$

Consider the sequence of nested and shrinking cylinders $Q_n=K_{\rho_n}\times(t_n,t)$ with common vertex $(0,t)$ and observe that, by construction

$$Q_0=K_\rho\times(-t,t),\quad Q_\infty=K_{\sigma\rho}\times(-\sigma t,t).$$

Set

$$M=\sup_{Q_0}u,\qquad M_\sigma=\sup_{Q_\infty}u.$$

We first prove an estimate of M_σ in terms of M.

Consider cutoff functions $\xi\in C_0^\infty(Q_n)$, verifying $\xi(x,t)=\xi_1(x)\xi_2(t)\in[0,1]$

$$\xi_1=1\text{ in }K_{\rho_{n+1}},\qquad \xi_1=0\text{ in }\mathbb{R}^N\setminus K_{\rho_n},\qquad |D\xi_1|\le\frac{2^{n+1}}{(1-\sigma)\rho}$$

$$\xi_2=1\ ,\ \tau\ge t_{n+1},\qquad \xi_2=0\ ,\ \tau\le t_n,\qquad 0\le(\xi_2)_t\le\frac{2^{n+1}}{(1-\sigma)t}.$$

Finally define the sequence of levels

$$k_n=k\left(1-\frac{1}{2^{n+1}}\right),\quad n=0,1,2,\dots$$

where $k>0$ is to be chosen.

Consider first $1<p<2$. Formally multiply equation (2) by $(u-k_{n+1})_+^{r-1}\xi^p$, where $r>1$ satisfies (5), and integrate over the cylinders $K_{\rho_n}\times(t_n,\tau)$, for $\tau\in(t_n,t]$.

The parabolic term is easily estimated from below by

$$\frac{1}{r}\int_{K_{\rho_n}}(u-k_{n+1})_+^r\xi^p(x,\tau)dx-\frac{p}{r}\frac{2^{n+1}}{(1-\sigma)t}\iint_{Q_n}(u-k_{n+1})_+^r\,dxdt$$

As for the elliptic term, we integrate by parts and then use Young's inequality (with ε) to arrive at the inferior bound

$$\begin{aligned}
&(r-1)\iint_{Q_n} \frac{|D(u-k_{n+1})_+|^p}{u^{p-1}}(u-k_{n+1})_+^{r-2}\xi^p \, dxdt \\
&- p\iint_{Q_n} \frac{|D(u-k_{n+1})_+|^{p-1}}{u^{p-1}}\xi^{p-1}(u-k_{n+1})_+^{r-1}|D\xi| \, dxdt \\
\geq &\frac{r-1}{2}\iint_{Q_n} \frac{|D(u-k_{n+1})_+|^p}{u^{p-1}}(u-k_{n+1})_+^{r-2}\xi^p \, dxdt \\
&-\left(2\frac{p-1}{r-1}\right)^{p-1}\iint_{Q_n} u^{1-p}(u-k_{n+1})_+^{p+r-2}|D\xi|^p \, dxdt \; .
\end{aligned}$$

By observing that

$$\begin{aligned}
\left|D(u-k_{n+1})_+^{\frac{r}{p}}\right|^p &= \left(\frac{r}{p}\right)^p (u-k_{n+1})_+^{r-p}|D(u-k_{n+1})_+|^p \\
&\leq \left(\frac{r}{p}\right)^p (u-k_{n+1})_+^{r-2}u^{2-p}|D(u-k_{n+1})_+|^p \\
&= u\left(\frac{r}{p}\right)^p (u-k_{n+1})_+^{r-2}u^{1-p}|D(u-k_{n+1})_+|^p
\end{aligned}$$

we get, by noticing that $\dfrac{k}{2} < k_{n+1} < u \leq M$,

$$\begin{aligned}
\iint_{Q_n} &\frac{|D(u-k_{n+1})_+|^p}{u^{p-1}}(u-k_{n+1})_+^{r-2}\xi^p \, dxdt \\
&\geq \frac{1}{M}\left(\frac{p}{r}\right)^p \iint_{Q_n} |D(u-k_{n+1})_+^{\frac{r}{p}}|^p\xi^p \, dxdt \\
&\geq \frac{1}{2^{(p-1)}M}\left(\frac{p}{r}\right)^p \iint_{Q_n} \left|D[(u-k_{n+1})_+^{\frac{r}{p}}\xi]\right|^p \, dxdt \\
&\qquad - \frac{2}{k}\left(\frac{p}{r}\right)^p \iint_{Q_n} (u-k_{n+1})_+^{r}|D\xi|^p \, dxdt
\end{aligned}$$

and for $\tilde{k}_n = \frac{k_n+k_{n+1}}{2}$

$$\begin{aligned}
\iint_{Q_n} u^{1-p}(u-k_{n+1})_+^{p+r-2}|D\xi|^p \, dxdt &\leq \iint_{Q_n} (u-k_{n+1})_+^{r-1}|D\xi|^p \, dxdt \\
&\leq \iint_{Q_n} (u-\tilde{k}_n)_+^{r-1}|D\xi|^p \, dxdt, \\
&\leq \frac{2^{n+3}}{k}\iint_{Q_n} (u-k_n)_+^{r}|D\xi|^p \, dxdt \; .
\end{aligned}$$

Combining all the previous estimates, and taking

$$k \geq \frac{t}{\rho^p}$$

we obtain, for all $\tau \in (t_n, t]$

$$\begin{aligned}
&\int_{K_{\rho_n}} (u-k_{n+1})_+^r \xi^p(x,\tau)dx + \frac{r(r-1)}{2^p M}\left(\frac{p}{r}\right)^p \iint_{Q_n} \left|D[(u-k_{n+1})_+^{\frac{r}{p}}\xi]\right|^p dxdt \\
&\quad \leq \frac{2^{n(p+1)}}{(1-\sigma)^p}\left\{\frac{2p}{t} + \frac{r2^{p+3}}{k\rho^p}\left(\frac{p-1}{r-1}\right)^{p-1} + r(r-1)\left(\frac{p}{r}\right)^p \frac{2^{2p}}{k\rho^p}\right\} \\
&\qquad\qquad \times \iint_{Q_n} (u-k_n)_+^r \, dxdt \\
&\quad \leq C_1 \frac{2^{n(p+1)}}{(1-\sigma)^p t} \iint_{Q_n} (u-k_n)_+^r \, dxdt,
\end{aligned}$$

for $C_1 = \max\left\{6p, 3r2^{p+3}\left(\frac{p-1}{r-1}\right)^{p-1}, 3r(r-1)\left(\frac{p}{r}\right)^p 2^{2p}\right\}$. By first applying Hölder's inequality (with exponent $(N+p)/N$), afterwards Sobolev's embedding (with exponent $p(N+p)/N$) and finally using the previous estimate we get

$$\begin{aligned}
X_{n+1} &= \iint_{Q_{n+1}} (u-k_{n+1})_+^r \, dxdt \leq \iint_{Q_n} (u-k_{n+1})_+^r \xi^p \, dxdt \\
&= \iint_{Q_n} \left((u-k_{n+1})_+^{\frac{r}{p}}\xi\right)^p dxdt \\
&\leq \left(\iint_{Q_n} \left((u-k_{n+1})_+^{\frac{r}{p}}\xi\right)^{p(N+p)/N} dxdt\right)^{N/(N+p)} |Q_n \cap [u > k_{n+1}]|^{p/(N+p)} \\
&\leq \gamma(N,p)\left\{\left(\iint_{Q_n} |D[(u-k_{n+1})_+^{\frac{r}{p}}\xi]|^p \, dxdt\right)\right. \\
&\qquad \left. \times \left(\sup_{t_n \leq \tau \leq t} \int_{K_{\rho_n}} (u-k_{n+1})_+^r \xi^p(x,\tau)dx\right)^{p/N}\right\}^{N/(N+p)} |Q_n \cap [u > k_{n+1}]|^{p/(N+p)} \\
&\leq CM^{\frac{N}{N+p}} \frac{2^{n(p+1)}}{(1-\sigma)^p t} X_n \, |Q_n \cap [u > k_{n+1}]|^{p/(N+p)} \\
&\leq CM^{\frac{N}{N+p}} \frac{2^{n(p+1)}}{(1-\sigma)^p t}\left(\frac{2^{(n+1)r}}{k^r}\right)^{\frac{p}{N+p}} X_n^{1+\frac{p}{N+p}},
\end{aligned}$$

where $C = \gamma(N,p)\dfrac{C_1}{C_2}$, for $C_2 = \min\left\{1, \dfrac{r(r-1)}{2^{2p}}\left(\dfrac{p}{r}\right)^p\right\}$. The last inequality was obtained by noticing that

$$X_n \geq \iint_{Q_n \cap [u > k_{n+1}]} (u - k_n)_+^r \, dxdt \geq \left(\frac{k}{2^{n+1}} \right)^r |Q_n \cap [u > k_{n+1}]| \,. \tag{25}$$

From the previous estimate and by defining $Y_n = \dfrac{X_n}{|Q_n|}$, we have, for $\bar{C} = 2^{\frac{rp}{N+p}} C$,

$$Y_{n+1} \leq \bar{C} \frac{M^{\frac{N}{N+p}}}{k^{rp/(N+p)} (1-\sigma)^p} \left(\frac{\rho^p}{t} \right)^{\frac{N}{N+p}} b^n Y_n^{1+\frac{p}{N+p}}, \qquad b = 2^{p+1+rp/(N+p)} > 1.$$

From a geometric convergence lemma, one has $Y_n \to 0$, as $n \to \infty$, if

$$Y_0 \leq \left(\bar{C} \frac{M^{\frac{N}{N+p}}}{k^{rp/(N+p)} (1-\sigma)^p} \left(\frac{\rho^p}{t} \right)^{\frac{N}{N+p}} \right)^{-\frac{N+p}{p}} b^{-(\frac{N+p}{p})^2}.$$

This estimate (and also the previous one $k \geq t/\rho^p$) is verified once we take

$$k = C(N,p,r) \frac{M^{\frac{N}{pr}}}{(1-\sigma)^{\frac{N+p}{r}}} \left(\fint\!\!\fint_Q u^r \right)^{1/r} \left(\frac{\rho^p}{t} \right)^{\frac{N}{pr}} + \frac{t}{\rho^p}.$$

For this choice of k we have

$$M_\sigma = \sup_{Q_\infty} u \leq C(N,p,r) \frac{M^{\frac{N}{pr}}}{(1-\sigma)^{\frac{N+p}{r}}} \left(\fint\!\!\fint_Q u^r \, dxdt \right)^{1/r} \left(\frac{\rho^p}{t} \right)^{\frac{N}{pr}} + \frac{t}{\rho^p}. \tag{26}$$

Now consider the sequences, $n = 0, 1, \cdots$,

$$\tilde{\rho}_n = \sigma\rho + (1-\sigma)\rho \sum_{i=1}^n \frac{1}{2^i} \qquad \text{and} \qquad \tilde{t}_n = -\sigma t - (1-\sigma) t \sum_{i=1}^n \frac{1}{2^i}$$

for which $K_{\tilde{\rho}_n} \times (\tilde{t}_n, t) = \tilde{Q}_n \subset \tilde{Q}_{n+1}$, and define $M_n = \sup_{\tilde{Q}_n} u$. Applying (26) to the cylinders $\tilde{Q}_n$ and $\tilde{Q}_{n+1}$ and then Young's inequality (with $0 < \varepsilon < 1$) we arrive at

$$M_n \leq \varepsilon M_{n+1} + C(N,p,r,\varepsilon) I$$

for

$$I = \frac{1}{(1-\sigma)^{\frac{(N+p)p}{pr-N}}} \left(\frac{\rho^p}{t} \right)^{\frac{N}{rp-N}} \left(\fint\!\!\fint_Q u^r \, dxdt \right)^{\frac{p}{rp-N}} + \frac{t}{\rho^p}.$$

By iteration

$$M_0 \leq \varepsilon^n M_n + C(N,p,r,\varepsilon) I \sum_{i=1}^n \varepsilon^i$$

and then, since $(M_n)_n$ is equibounded, when taking $n \to \infty$,

$$\sup_{Q_\sigma} u \le C(N,p,r,\varepsilon) I$$

and the proof is complete once we take $\sigma = \dfrac{1}{2}$.

Now consider $p > 2$. Proceed in a formal way and multiply equation (2) by $(u-k_{n+1})_+\xi^p$ and then integrate over the cylinders $K_{\rho_n} \times (t_n,\tau)$, for $\tau \in (t_n,t]$.

While there are no substantial changes in the estimation of the parabolic term, the elliptic term is estimated from below as follows

$$\begin{aligned}
&\frac{M^{1-p}}{2^p}\iint_{Q_n} |D(u-k_{n+1})_+\xi|^p \, dxdt \\
&\qquad - \frac{2^p(p-1)^{p-1}}{2}\iint_{Q_n} u^{1-p}(u-k_{n+1})_+^p |D\xi|^p \, dxdt \\
&\ge \frac{M^{1-p}}{2^p}\iint_{Q_n} |D(u-k_{n+1})_+\xi|^p \, dxdt \\
&\qquad - \frac{2^p(p-1)^{p-1}}{2}\iint_{Q_n} u^{1-p+p-2}(u-k_{n+1})_+^2 |D\xi|^p \, dxdt \\
&\ge \frac{M^{1-p}}{2^p}\iint_{Q_n} |D(u-k_{n+1})_+\xi|^p \, dxdt \\
&\qquad - \frac{2^p(p-1)^{p-1}}{2k}\frac{2^{p(n+1)}}{(1-\sigma)^p\rho^p}\iint_{Q_n} (u-k_{n+1})_+^2 \, dxdt \ .
\end{aligned}$$

By considering $k \ge \dfrac{t}{\rho^p}$, one gets, for all $\tau \in (t_n,t]$

$$\begin{aligned}
\int_{K_{\rho_n}} (u-k_{n+1})_+^2 \xi^p(x,\tau)dx + \frac{1}{(2M)^{p-1}}\iint_{Q_n} |D(u-k_{n+1})_+\xi|^p \, dxdt \\
\le C(p)\frac{2^{np}}{(1-\sigma)^p t}\iint_{Q_n} (u-k_n)_+^2 \, dxdt.
\end{aligned} \tag{27}$$

Set $X_n = \iint_{Q_n} (u-k_n)_+^2 \, dxdt$. Arguing as before we have

$$
\begin{aligned}
X_{n+1} &\le \iint_{Q_n} (u-k_{n+1})_+^2 \xi^2 \, dxdt \\
&\le \left(\iint_{Q_n} ((u-k_{n+1})_+\xi)^{\frac{p(N+p)}{N}} \, dxdt \right)^{\frac{2N}{p(N+p)}} |Q_n \cap [u>k_{n+1}]|^{1-\frac{2N}{p(N+p)}} \\
&\le \gamma(N,p) \left\{ \left(\iint_{Q_n} |D[(u-k_{n+1})_+\xi]|^p \, dxdt \right) \right. \\
&\quad \times \left. \left(\sup_{t_n \le \tau \le t} \int_{K_{\rho_n}} (u-k_{n+1})_+^p \xi^p(x,\tau) dx \right)^{\frac{p}{N}} \right\}^{\frac{2N}{p(N+p)}} |Q_n \cap [u>k_{n+1}]|^{1-\frac{2N}{p(N+p)}} \\
&\le \gamma(N,p) \left\{ \left(\iint_{Q_n} |D[(u-k_{n+1})_+\xi]|^p \, dxdt \right) \right. \\
&\quad \times \left. \left(M^{p-2} \sup_{t_n \le \tau \le t} \int_{K_{\rho_n}} (u-k_{n+1})_+^2 \xi^p(x,\tau) dx \right)^{\frac{p}{N}} \right\}^{\frac{2N}{p(N+p)}} |Q_n \cap [u>k_{n+1}]|^{1-\frac{2N}{p(N+p)}} \\
&\le \gamma(N,p) M^{[(p-1)+(p-2)\frac{p}{N}]\frac{2N}{p(N+p)}} \left(\frac{2^{np}}{(1-\sigma)^p t} \right)^{\frac{2}{p}} \left(\frac{2^{(n+1)2}}{k^2} \right)^{1-\frac{2N}{p(N+p)}} X_n^{\frac{2}{p}+1-\frac{2N}{p(N+p)}}
\end{aligned}
$$

Set $Y_n = \dfrac{X_n}{|Q_n|}$. Then

$$
\begin{aligned}
Y_{n+1} &\le C(N,p) M^{[(p-1)+(p-2)\frac{p}{N}]\frac{2N}{p(N+p)}} \\
&\quad \times \frac{b^n}{(1-\sigma)^2} \left(\frac{\rho^p}{t} \right)^{\frac{2N}{p(N+p)}} \left(\frac{1}{k^2} \right)^{1-\frac{2N}{p(N+p)}} Y_n^{1+\frac{2}{N+p}}.
\end{aligned}
$$

The fast geometric convergence lemma says that $Y_n \to 0$, when $n \to \infty$, if

$$
Y_0 \le \left(C(N,p) M^s \frac{1}{(1-\sigma)^2} \left(\frac{\rho^p}{t} \right)^{\frac{2N}{p(N+p)}} k^{-2\left(1-\frac{2N}{p(N+p)}\right)} \right)^{-\frac{N+p}{2}} b^{-(\frac{N+p}{2})^2},
$$

where $s = [(p-1)+(p-2)\frac{p}{N}]\frac{2N}{p(N+p)}$. Observe that if $r \ge 2$ then we estimate

$$
Y_0 = ⨍⨍_Q u^2 \le k^{2-r} ⨍⨍_Q u^r.
$$

Thus we choose k in order to satisfy the inequality

$$
k^{2-r} ⨍⨍_Q u^r \le \left(C(N,p) M^s \frac{1}{(1-\sigma)^2} \left(\frac{\rho^p}{t} \right)^{\frac{2N}{p(N+p)}} k^{-2\left(1-\frac{2N}{p(N+p)}\right)} \right)^{-\frac{N+p}{2}} b^{-(\frac{N+p}{2})^2}.
$$

This is assured once we take

$$k = C(N,p)\left\{ \fint\!\!\fint_Q u^r \right\}^{\frac{p}{p(N+p+r-2)-2N}} M^{\frac{(p-1)N+p(p-2)}{p(N+p+r-2)-2N}}$$
$$\times \left(\frac{\rho^p}{t}\right)^{\frac{N}{p(N+p+r-2)-2N}} \left(\frac{1}{1-\sigma}\right)^{\frac{p(N+p)}{p(N+p+r-2)-2N}} + \frac{t}{\rho^p}.$$

Thus, by Young's inequality with exponent $\frac{[p(N+p+r-2)-2N]}{(p-1)N+p(p-2)}$, we obtain

$$M_\sigma \leq \varepsilon M + C(N,r,p,\varepsilon)\left\{ \frac{1}{(1-\sigma)^{\frac{p(N+p)}{rp-N}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{rp-N}} \left(\fint\!\!\fint_Q u^r \right)^{\frac{p}{rp-N}} + \frac{t}{\rho^p} \right\}.$$

Arguing as in the case $1 < p < 2$, we first apply the estimate above to the sequence M_n and then, by iteration and taking the limit as $n \to \infty$, we arrive at

$$\sup_{K_{\sigma\rho}\times(-\sigma t,t)} u \leq C(N,p,r)\frac{1}{(1-\sigma)^{\frac{p(N+p)}{rp-N}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{rp-N}} \left(\fint\!\!\fint_Q u^r \right)^{\frac{p}{rp-N}} + \frac{t}{\rho^p}.$$

If $r < 2$ then we modify the estimate of Y_0 according to

$$\fint\!\!\fint_Q u^2 \leq M^{2-r} \fint\!\!\fint_Q u^r$$

and consequently

$$M_\sigma \leq C(N,p)\left\{ \fint\!\!\fint_Q u^r \right\}^{\frac{p}{[p(N+p)-2N]}} M^{\frac{(p-1)N+p(p-r)}{p(N+p)-2N}}$$
$$\times \left(\frac{\rho^p}{t}\right)^{\frac{N}{p(N+p)-2N}} \left(\frac{1}{1-\sigma}\right)^{\frac{p(N+p)}{p(N+p)-2N}} + \frac{t}{\rho^p}$$
$$\leq \varepsilon M + C(N,r,p,\varepsilon)\left\{ \frac{1}{(1-\sigma)^{\frac{p(N+p)}{rp-N}}} \left(\frac{\rho^p}{t}\right)^{\frac{N}{rp-N}} \left(\fint\!\!\fint_Q u^r \right)^{\frac{p}{rp-N}} + \frac{t}{\rho^p} \right\}.$$

The conclusion is as above.

□

Remark 4.4. When $1 < p < 2$, the constant $\gamma_r \to \infty$ when $r \to 1$ due to the presence of the factor $r-1$ in the denominator; for $p > 1$, γ_r becomes unbounded when $\lambda_r \to 0$.

Combining Theorem 4.3 and Proposition 4.1 we are led to the following result.

Theorem 4.5. *Let u be a nonnegative locally bounded local weak solution to (2) in E_T satisfying $u \in L^r_{\rm loc}(E_T)$, for $r > \max\left\{1, \frac{N}{p}\right\}$. Then there exists a positive constant γ_r depending on N, r such that, for all cylinders $K_\rho(y) \times [2s-t, t] \subset E_T$,*

$$\sup_{K_{\frac{\rho}{2}}(y)\times[s,t]} u \leq \gamma_r \frac{1}{(t-s)^{\frac{N}{\lambda_r}}} \left(\int_{K_{2\rho}(y)} u^r(x, 2s-t)dx \right)^{\frac{p}{\lambda_r}} + \gamma_r \frac{t-s}{\rho^p}.$$

The constant $\gamma_r \to \infty$ as either $r \to 1$ or $\lambda_r \to 0$.

5. Expansion of positivity and Harnack Inequality

In this section we will be concerned in these two issues: the expansion of positivity and a Harnack-type inequality. Being the first one presented (and proved), since it is the heart of any form of Harnack inequality, the second one follows.

5.1. Estimating the positivity set of the solution

Proposition 5.1. *Assume that u is a nonnegative locally bounded local weak super-solution to (2) satisfying (9). Assume that for some $M > 0$ as in (15) and parameters $a, \delta \in (0,1)$ there holds*

$$|[u(\cdot,\tau) \geq M] \cap K_\rho(y)| \geq a|K_\rho(y)| \tag{28}$$

for all τ such that

$$s - \delta M \rho^p \leq \tau \leq s\ .$$

Then there exist a constant $\sigma \in (0,1)$ that can be determined in terms of a, δ, $\Lambda_{8\rho,M}$ such that

$$u(\cdot,t) \geq \sigma M \qquad \text{in } K_{2\rho}(y)$$

for all times

$$s - \frac{1}{8}\delta M \rho^p \leq t \leq s.$$

To prove the previous result we need the following lemma.

Lemma 5.2. *Under the assumptions of Proposition 5.1, for every $\nu > 0$ there exist $\sigma \in (0,1)$ such that*

$$|[u(\cdot,t) \leq 2\sigma M] \cap K_{4\rho}(y)| \leq \nu |K_{4\rho}|$$

for all $s - \frac{1}{4}\delta M \rho^p < t \leq s$. The number σ depends on a, δ and $\Lambda_{8\rho,M}$.

By performing the change of variables

$$x \to \frac{x-y}{\rho}, \qquad t \to 8^p \frac{t-(s-\delta M\rho^p)}{\delta M\rho^p}, \quad u \to \frac{u}{M}$$

the cylinder $K_{8\rho}(y)\times(s-\delta M\rho^p,s]$ is transformed into $Q_8^+ = K_8\times(0,8^p]$ and the new function (still denoted by u) solves the equation

$$u_t - \delta 8^{-p}\Delta_p(\ln u) = 0, \quad \text{weakly in} \quad Q_8^+. \tag{29}$$

Moreover, the assumption (28) yields

$$|[u(\cdot,t)\geq 1]\cap K_4| \geq \frac{a}{4^N}|K_4| \tag{30}$$

and consequently

$$|[u(\cdot,t)\geq 1]\cap K_8| \geq \frac{a}{8^N}|K_8| \tag{31}$$

for all $t\in(0,8^p]$. Assumption (9) is preserved under the change of variables and the quantities Λ and $\Lambda_{8\rho,M}$ remain unchanged. This procedure allows us to reformulate Lemma 5.2 as follows.

Lemma 5.3. *Let u be a locally bounded, nonnegative local weak super-solution to equation (29) satisfying (31). Then for every $\nu>0$ there exist $\sigma\in(0,1)$ such that*

$$|[u(\cdot,t)\leq \sigma]\cap K_4| \leq \nu|K_4| \tag{32}$$

for all $\frac{3}{4}8^p < t \leq 8^p$. The number σ depends on a,δ and $\Lambda_{8\rho,M}$.

Proof. Assume that $u_t\in C(0,8^p;L^1(K_8))$. Since u is a super-solution, for every nonnegative test function $\phi\in C(Q_8^+)\cap C(0,8^p;W_0^{1,p}(K_8))$ we have

$$\int_{K_8}\frac{\partial}{\partial t}(k-u)_+\phi\,dx + \frac{\delta}{8^p}\int_{K_8}u^{1-p}|D(k-u)_+|^{p-2}D(k-u)_+\cdot D\phi\,dx \leq 0.$$

Take $\phi=\left(\ln\frac{k}{u}\right)_+\zeta^p$, where $k\in(0,1]$ and $\zeta\in C_0^\infty(Q_8^+)$ is such that $\zeta(x,t) = \zeta_1(x)\zeta_2(t)\in[0,1]$

$$\zeta_1 = 1 \text{ in } K_4, \qquad \zeta_1 = 0 \text{ in } \mathbb{R}^N\setminus K_8, \qquad |D\zeta_1|\leq\frac{1}{4}$$

$$\zeta_2 = 1\ ,\ t\geq\frac{3}{4}8^p, \qquad \zeta_2 = 0\ ,\ t\leq 0, \qquad 0\leq(\zeta_2)_t\leq\frac{4}{3\cdot 8^p}.$$

By Young's inequality we get

$$\begin{aligned}&\frac{d}{dt}\int_{K_8}\Phi_k(u)\zeta^p(x,t)dx + \frac{\delta}{2\cdot 8^p}\int_{K_8}\frac{|D(k-u)_+|^p}{u^p}\zeta^p dx\\ &\quad\leq c_p\delta\int_{K_8}\Psi_k^p(u)|D\zeta|^p dx + c_p\int_{K_8}\Phi_k(u)\zeta^{p-1}\zeta_t dx\end{aligned}$$

where

$$\Phi_k(u) = \left(\int_u^k \ln \frac{k}{s} \, ds \right)_+ \qquad \Psi_k(u) = \left(\ln \frac{k}{u} \right)_+ .$$

We estimate $\Psi_k(u)$ taking into account the definition of $\Lambda_{8\rho,M}$ and by means of Hölder's inequality, and $\Phi_k(u)$ by the estimate

$$\Phi_k(u) \le \int_0^1 |\ln s| ds = 1.$$

Hence we infer

$$\frac{d}{dt} \int_{K_8} \Phi_k(u) \zeta^p(x,t) dx + \frac{\delta}{2 \cdot 8^p} \int_{K_8} \frac{|D(k-u)_+|^p}{u^p} \zeta^p dx \le c(1 + \Lambda^p_{8\rho,M})$$

where c depends on N, p and α.
Since $k \in (0,1]$, from the assumption (31) it follows that

$$|[\Psi_k(u) = 0] \cap K_4| \ge \frac{a}{4^N} |K_4| \,, \quad \forall t \in (0, 8^p).$$

Next, we apply the Poincaré's inequality as stated in Proposition 2.1 of Chapter I of [5] and we obtain

$$\frac{d}{dt} \int_{K_8} \Phi_k(u) \zeta^p(x,t) dx + a^p \delta \int_{K_8} \Psi^p_k(u) \zeta^p dx \le c(1 + \Lambda^p_{8\rho,M}). \tag{33}$$

Let us introduce the quantities

$$Y_n = \sup_{0<t<8^p} \int_{K_8} \chi_{[u(\cdot,t)<h^n]} \zeta^p(x,t) dx \tag{34}$$

where $h \in (0,1)$ is to be chosen. We claim that,

given $\nu > 0$, there exist $h, \xi \in (0,1)$, depending on a, δ, ν and $\Lambda^p_{8\rho,M}$, such that for every $n = 0, 1, \dots$

$$\textit{either} \quad Y_n \le \nu \quad \textit{or} \quad Y_{n+1} \le \max\{\nu, \xi Y_n\}. \tag{35}$$

Now (32) is a straightforward consequence of this claim. In fact, by iterating (35) we find $Y_n \le \max\{\nu, \xi^n Y_0\}$ for every $n \ge 1$. Choosing $\bar{n}$ such that $\xi^{\bar{n}} < \nu 2^{-N}$, we have $Y_{\bar{n}} \le \nu |K_4|$, since $Y_0 \le |K_8|$. By the definition of $Y_{\bar{n}}$ we get

$$\sup_{\frac{3}{4} 8^p < t < 8^p} \left| K_4 \cap [u(\cdot,t) < h^{\bar{n}}] \right| \le \sup_{0<t<8^p} \int_{K_8} \chi_{[u(\cdot,t)<h^{\bar{n}}]} \zeta^p(x,t) dx \le \nu |K_4| \,,$$

which yields (32) with $\sigma = h^{\bar{n}}$.
To complete the proof we have to prove the claim. For that purpose, fix $\nu > 0$,

take $n \in \mathbb{N}$ and assume that $Y_n > \nu$, otherwise there is nothing to prove. By the definition of Y_{n+1}, for every $\varepsilon \in (0, \frac{\nu}{2})$ there exists $t_\varepsilon \in (0, 8^p]$ such that

$$\int_{K_8} \chi_{[u(\cdot,t_\varepsilon)<h^{n+1}]} \zeta^p(x,t_\varepsilon)dx \geq Y_{n+1} - \varepsilon.$$

At this point we have two alternatives, either

$$\frac{d}{dt}\int_{K_8} \Phi_{h^n}[u(\cdot,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \geq 0$$

or

$$\frac{d}{dt}\int_{K_8} \Phi_{h^n}[u(\cdot,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx < 0.$$

Assume that the first alternative holds true. Then, by (33) we deduce that

$$\int_{K_8} \Psi^p_{h^n}[u(\cdot,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \leq c\frac{\Lambda^p_{8\rho,M}}{a^p\delta}.$$

On the set $[u(\cdot,t_\varepsilon) < h^{n+1}]$ we have

$$\Psi_{h^n}[u(\cdot,t_\varepsilon)] = \left(\ln\frac{h^n}{u(\cdot,t_\varepsilon)}\right)_+ \geq \ln\frac{h^n}{h^{n+1}} = \ln\frac{1}{h}.$$

Therefore

$$|\ln h|^p \int_{K_8} \chi_{[u(x,t_\varepsilon)<h^{n+1}]}\zeta^p(x,t_\varepsilon)dx \leq \int_{K_8} \Psi^p_{h^n}[u(x,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \leq c\frac{\Lambda^p_{8\rho,M}}{a^p\delta}.$$

and also

$$Y_{n+1} \leq c\frac{\Lambda^p_{8\rho,M}}{a^p\delta|\ln h|^p} + \varepsilon.$$

By considering $\varepsilon \in (0, \frac{\nu}{2})$ and then choosing h sufficiently small in order to have $c\frac{\Lambda^p_{8\rho,M}}{a^p\delta|\ln h|^p} \leq \frac{\nu}{2}$, the thesis follows.
Now assume that the second alternative holds and define

$$t_* = \sup\left\{t \in (0,t_\varepsilon) \mid \frac{d}{dt}\int_{K_8} \Phi_{h^n}[u(x,t)]\zeta^p(x,t)dx \geq 0\right\}.$$

It follows that the function $t \to \int_{K_8} \Phi_{h^n}[u(x,t)]\zeta^p(x,t)dx$ has negative derivative in the interval $(t_*, t_\varepsilon]$ and this yields

$$\int_{K_8} \Phi_{h^n}[u(x,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \leq \int_{K_8} \Phi_{h^n}[u(x,t_*)]\zeta^p(x,t_*)dx. \tag{36}$$

Due to the definition of t_* and arguing as in the first alternative we have

$$\int_{K_8} \Psi^p_{h^n}[u(x,t_*)]\zeta^p(x,t_*)dx \le c\frac{\Lambda^p_{8\rho,M}}{a^p\delta}.$$

For every $s\in(0,1)$, on the set $[u(\cdot,t_*)<h^n(1-s)]$ we have

$$\Psi_{h^n}[u(\cdot,t_*)] = \left(\ln\frac{h^n}{u(\cdot,t_*)}\right)_+ \ge \ln\frac{1}{1-s}$$

and therefore

$$\int_{K_8} \chi_{[u(x,t_*)<h^n(1-s)]}\zeta^p(x,t_*)dx \le c\left(\ln\frac{1}{1-s}\right)^{-p}\frac{\Lambda^p_{8\rho,M}}{a^p\delta}.$$

By the definition of Y_n, since $[u<h^n(1-s)]\subseteq[u<h^n]$, we have

$$\int_{K_8} \chi_{[u(x,t_*)<h^n(1-s)]}\zeta^p(x,t_*)dx \le \min\left\{Y_n, c\left(\ln\frac{1}{1-s}\right)^{-p}\frac{\Lambda^p_{8\rho,M}}{a^p\delta}\right\}.$$

Since the function $s\to\left(\ln\frac{1}{1-s}\right)^{-p}$ is decreasing in $(0,1)$, there exists s_* such that

$$Y_n = c\left(\ln\frac{1}{1-s_*}\right)^{-p}\frac{\Lambda^p_{8\rho,M}}{a^p\delta}.$$

This implies that

$$\int_{K_8} \chi_{[u(x,t_\varepsilon)<h^n(1-s)]}\zeta^p(x,t_*)dx \le \begin{cases} Y_n & \text{if } 0<s<s_* \\ c\left(\ln\frac{1}{1-s}\right)^{-p}\frac{\Lambda^p_{8\rho,M}}{a^p\delta} & \text{if } s_*\le s<1 \end{cases} \tag{37}$$

One can compute

$$s_* = \frac{\exp\left\{\left(\frac{c}{Y_n}\frac{\Lambda^p_{8\rho,M}}{a^p\delta}\right)^{\frac{1}{p}}\right\}-1}{\exp\left\{\left(\frac{c}{Y_n}\frac{\Lambda^p_{8\rho,M}}{a^p\delta}\right)^{\frac{1}{p}}\right\}}$$

and then, since $Y_n>\nu$, we have

$$s_* < \frac{\exp\left\{\left(\frac{c}{\nu}\frac{\Lambda^p_{8\rho,M}}{a^p\delta}\right)^{\frac{1}{p}}\right\}-1}{\exp\left\{\left(\frac{c}{\nu}\frac{\Lambda^p_{8\rho,M}}{a^p\delta}\right)^{\frac{1}{p}}\right\}} = \tilde{\xi}\ .$$

Observe that, due to (36) and (37)

$$
\begin{aligned}
&\int_{K_8} \Phi_{h^n}[u(x,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \\
&\qquad \le \int_{K_8} \Phi_{h^n}[u(x,t_*)]\zeta^p(x,t_*)dx \\
&\qquad = \int_{K_8}\left(\int_0^{(h_n-u(x,t_*))_+} \chi_{[s<(h^n-u)_+]} \ln\frac{h^n}{s+u}\,ds\right)\zeta^p(x,t_*)dx \\
&\qquad \le \int_{K_8}\left(\int_0^{h_n} \chi_{[s<(h^n-u)_+]} \ln\frac{h^n}{s+u}\,ds\right)\zeta^p(x,t_*)dx \\
&\qquad \le \int_0^{h_n} \ln\frac{h^n}{s}\left(\int_{K_8} \chi_{[s<(h^n-u)_+]}\zeta^p(x,t_*)dx\right)ds \\
&\qquad = \int_0^1 h^n \ln\frac{1}{s}\left(\int_{K_8} \chi_{[sh^n<(h^n-u)_+]}\zeta^p(x,t_*)dx\right)ds \\
&\qquad = \int_0^1 h^n \ln\frac{1}{s}\left(\int_{K_8\cap[u<h^n(1-s)]} \zeta^p(x,t_*)dx\right)ds \\
&\qquad \le \int_0^{s_*} h^n \ln\frac{1}{s}\,Y_n\,ds + \int_{s_*}^1 h^n c\frac{\Lambda^p_{8\rho,M}}{a^p\delta}\ln\frac{1}{s}\left(\ln\frac{1}{1-s}\right)^{-p} ds \\
&\qquad = \int_0^1 h^n \ln\frac{1}{s}\,Y_n\,ds - \int_{s_*}^1\left[Y_n - c\frac{\Lambda^p_{8\rho,M}}{a^p\delta|\ln(1-s)|^p}\right] h^n\ln\frac{1}{s}\,ds \\
&\qquad = h^nY_n\left(\int_0^1 \ln\frac{1}{s}\,ds - \int_{s_*}^1\left[1-\frac{c}{Y_n}\frac{\Lambda^p_{8\rho,M}}{a^p\delta|\ln(1-s)|^p}\right]\ln\frac{1}{s}\,ds\right) \\
&\qquad = h^nY_n\left(1 - \int_{s_*}^1\left[1-\left|\frac{\ln(1-s_*)}{\ln(1-s)}\right|^p\right]\ln\frac{1}{s}\,ds\right) \\
&\qquad \le h^nY_n\left(1 - \int_{\tilde\xi}^1\left[1-\left|\frac{\ln(1-s_*)}{\ln(1-s)}\right|^p\right]\ln\frac{1}{s}\,ds\right) \\
&\qquad \le h^nY_n\left(1 - \int_{\tilde\xi}^1\left[1-\left|\frac{\ln(1-\tilde\xi)}{\ln(1-s)}\right|^p\right]\ln\frac{1}{s}\,ds\right) = c_*h^nY_n\,.
\end{aligned}
$$

The last three estimates were obtained by making use of the definitions of s_* and $\tilde\xi$ and of the monotonicity of the function $s \to |\ln(1-s)|^p$ in the interval $(0,1)$. The constant c_* depends only on $\tilde\xi$ and hence on $N, p, a, \delta, \Lambda_{8\rho,M}$.

On the other hand, observe that

$$\begin{aligned}
\int_{K_8} \Phi_{h^n}[u(x,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx &\geq \int_{K_8\cap[u(x,t_\varepsilon)<h^{n+1}]} \Phi_{h^n}[u(x,t_\varepsilon)]\zeta^p(x,t_\varepsilon)dx \\
&= \int_{K_8\cap[u(x,t_\varepsilon)<h^{n+1}]} \left(\int_{u(x,t_\varepsilon)}^{h^n} \ln\frac{h^n}{s}\,ds\right)_+ \zeta^p(x,t_\varepsilon)dx \\
&\geq \int_{K_8\cap[u(x,t_\varepsilon)<h^{n+1}]} \left(\int_{h^{n+1}}^{h^n} \ln\frac{h^n}{s}\,ds\right)_+ \zeta^p(x,t_\varepsilon)dx \\
&= h^n(1-h+h\ln h)\int_{K_8\cap[u(x,t_\varepsilon)<h^{n+1}]} \zeta^p(x,t_\varepsilon)dx \\
&\geq h^n(1-h+h\ln h)(Y_{n+1}-\varepsilon)\ .
\end{aligned}$$

Combining the last two estimates we have

$$Y_{n+1}-\varepsilon \leq \frac{c_*}{1-h+h\ln h}\,Y_n.$$

Taking h sufficiently small and letting $\varepsilon \to 0$ we finally get $Y_{n+1} \leq \xi Y_n$, for a constant $\xi \in (0,1)$ depending only on $N, p, a, \delta, \Lambda_{8\rho,M}$ and our claim is proved. A final remark: the assumption $u_t \in C(0,8^p;L^1(K_8))$ can be removed and one has to argue in a similar way as in [5], chapter IV, section 9. $\square$

Proof of Proposition 5.1. We are now is position to prove Proposition 5.1. For that, we start by considering any cylinder of the form $(y,t)+Q^-_{4\rho}(\theta)$, where

$$\theta = \sigma M \quad \text{and} \quad s-\frac{\delta M}{8}\rho^p \leq t \leq s.$$

Then the inclusion $(t-(4\rho)^p\theta, t] \subset (s-\frac{1}{4}\delta M\rho^p, s]$ holds true for any t as above if and only if

$$s-\frac{\delta M}{8}\rho^p-(4\rho)^p\sigma M \geq s-\frac{1}{4}\delta M\rho^p,$$

which we may assume, without loss of generality. From Lemma 5.2, we know that for every such cylinder

$$|(y,t)+Q^-_{4\rho}(\theta)\cap[u\leq 2\sigma M])| \leq \nu|Q^-_{4\rho}(\theta)|,$$

where σ is fixed once ν is chosen. Then, we fix ν according to Lemma 3.2 with $a=\dfrac{1}{2}$ and ξM replaced by $2\sigma M$; formula (24) yields

$$\nu = \left(\gamma_0 2^{2p/\beta}\Lambda^p_{8\rho,M}\right)^{-\frac{1}{\beta}}, \qquad \beta = \frac{p}{N+p}-\frac{p}{\alpha} > 0$$

ensuring that ν depends on M through $\Lambda_{8\rho,M}$. Thus we arrive at

$$u \geq \sigma M \quad \text{in } K_{2\rho}(y) \times (t - \theta(2\rho)^p, t]$$

and then the Proposition follows once we recall the previous choices on t.
Just a final remark on the dependence of σ: going back to the proof of auxiliar Lemma 5.3, we have $\sigma = h^{\bar{n}}$, where $h \in (0,1)$ has to be chosen such that

$$c\frac{\Lambda_{8\rho,M}^p}{a^p\delta|\ln h|^p} \leq \frac{\nu}{2} \qquad \text{and} \qquad \frac{c_*}{1-h+h\ln h} = \xi \in (0,1)$$

and $\bar{n}$ is such that $\xi^{\bar{n}} \leq \nu 2^{-N}$. □

5.2. Proving Theorem 2.2

Having fixed $(x_0,t_0) \in E_T$, assume it coincides with the origin, write $K_\rho(0) = K_\rho$ and introduce the quantity θ_0 as in (7), which is assumed to be positive. From Proposition 4.1, considered for the cylinder $K_{2\rho} \times (s,0)$, $s \in (-\theta_0\rho^p, 0]$, where $r = q$, and recalling the definition (7) of θ_0, one gets

$$\begin{aligned}
\int_{K_\rho} u^q(x,0)dx &\leq \gamma_q \int_{K_{2\rho}} u^q(x,s)dx + \gamma_q \frac{(\theta_0\rho^p)^q}{\rho^{\lambda_q}} \\
&= \gamma_q \int_{K_{2\rho}} u^q(x,s)dx + \gamma_q\varepsilon^q \int_{K_\rho} u^q(x,0)dx
\end{aligned}$$

and then, by choosing $\gamma_q\varepsilon^q \leq \frac{1}{2}$, one arrives at

$$\int_{K_{2\rho}} u^q(x,s)dx \geq \frac{1}{2\gamma_q}\int_{K_\rho} u^q(x,0)dx \tag{38}$$

for all $s \in (-\theta_0\rho^p, 0]$. Observe that being ε fixed the length θ_0 of the cylinder is completely determined.

Now consider the cylinder $K_{2\rho} \times (-\frac{1}{2}\theta_0\rho^p, 0]$ for which we apply Theorem 4.5. Recalling the definitions (7) and (8), of θ_0 and of η respectively, and assuming $0 < \eta < 1$, one obtains

$$\begin{aligned}
\sup_{K_{2\rho}\times(-\frac{1}{2}\theta_0\rho^p,0]} u &\leq \gamma_r \frac{(4\rho)^{\frac{Np}{\lambda_r}}}{(\theta_0\rho^p)^{\frac{N}{\lambda_r}}} \left(\fint_{K_{4\rho}} u^r(x,-\theta_0\rho^p)dx \right)^{\frac{1}{r}\frac{rp}{\lambda_r}} + \gamma_r\theta_0 \\
&\leq \frac{\gamma_r'}{\varepsilon^{\frac{N}{\lambda_r}}}\frac{1}{\eta}\left(\fint_{K_\rho} u^q(x,0)dx \right)^{\frac{1}{q}} + \gamma_r'\varepsilon \left(\fint_{K_\rho} u^q(x,0)dx \right)^{\frac{1}{q}} \\
&= \gamma_r'\varepsilon\left(1 + \frac{1}{\eta\varepsilon^{\frac{rp}{\lambda_r}}}\right)\left(\fint_{K_\rho} u^q(x,0)dx \right)^{\frac{1}{q}} \\
&\leq \frac{1}{\varepsilon'\eta}\left(\fint_{K_\rho} u^q(x,0)dx \right)^{\frac{1}{q}} = \tilde{M}, \quad \varepsilon' = \frac{\varepsilon^{\frac{N}{\lambda_r}}}{2\gamma_r'}
\end{aligned}$$

for a constant γ_r' depending only upon the data p, N and r. One verifies that $\gamma_r' \to \infty$, as either $\lambda_r \to 0$ or $\lambda_r \to \infty$.

From this

$$\varepsilon' \eta \tilde{M} = \left(\fint_{K_\rho} u^q(x,0)dx \right)^{\frac{1}{q}} = \frac{\theta_0}{\varepsilon}. \tag{39}$$

Let $\mu \in (0,1)$ to be chosen. Using (38) and (39) one arrives at

$$\begin{aligned}
(\varepsilon' \eta \tilde{M})^q &\leq 2^{N+1} \gamma_q \fint_{K_{2\rho}} u^q(x,s)dx \\
&\leq 2^{N+1} \gamma_q \left(\fint_{K_{2\rho} \cap [u < \mu \eta \tilde{M}]} u^q(x,s)dx + \fint_{K_{2\rho} \cap [u \geq \mu \eta \tilde{M}]} u^q(x,s)dx \right) \\
&\leq 2^{N+1} \gamma_q \mu^q (\eta \tilde{M})^q + 2^{N+1} \gamma_q \tilde{M}^q \frac{|[u(\cdot,s) > \mu \eta \tilde{M}] \cap K_{2\rho}|}{|K_{2\rho}|}
\end{aligned}$$

for all $s \in (-\frac{1}{2}\theta_0 \rho^p, 0]$. From this

$$|[u(\cdot,s) > \mu \eta \tilde{M}] \cap K_{2\rho}| \geq \alpha_0 \eta^q |K_{2\rho}|,$$

where

$$\alpha_0 = \frac{\varepsilon'^q - \mu^q 2^{N+1} \gamma_q}{2^{N+1} \gamma_q},$$

for all $s \in (-\frac{1}{2}\theta_0 \rho^p, 0]$. By choosing $\mu \in (0,1)$ sufficiently small, depending only on the data $\{p, N\}$ and on γ_q and γ_r', $\alpha_0 \in (0,1)$ depends only upon the data $\{p, N\}$ and on $\{r, q\}$, and is independent of η. We summarize what we have obtained so far.

Proposition 5.4. *Let u be a nonnegative locally bounded local weak solution to the singular equation (2) satisfying (9). Fix $(x_0, t_0) \in E_T$, let $K_{4\rho}(x_0) \subset E$ and let θ_0 and η be defined by (7), (8) respectively, for some $\varepsilon \in (0,1)$. Suppose $0 < \eta < 1$. For every $r > 1$ satisfying (5) and every $q > 1$, there exist constants $\varepsilon, \mu, \alpha_0 \in (0,1)$, depending only upon the data $\{p, N\}$, q and r, such that*

$$|[u(\cdot,t) > \mu \eta \tilde{M}] \cap K_{2\rho}(x_0)| \geq \alpha_0 \eta^q |K_{2\rho}|$$

for all $t \in (t_0 - \frac{1}{2}\theta_0 \rho^p, t_0]$.

We are one step way from proving the Harnack inequality given by Theorem 2.2. As said previously, one just has to make use of the expansion of positivity - that is the content of the following lines.

From the result above we have the hypothesis of Proposition 5.1 satisfied for $M = \mu\eta\tilde{M}$, $a = \alpha_0\eta^q$ and considering $\delta = \dfrac{\theta_0}{2M} = \dfrac{\varepsilon\varepsilon'}{2\mu}$. In fact, M verifies (15)

$$M = \mu\eta\tilde{M} = \frac{\mu}{\varepsilon'}\left(\fint_{K_\rho} u^q(x,0)dx\right)^{\frac{1}{q}} \le \frac{\mu}{\varepsilon'}\sup_{K_\rho} u(x,0) \le \sup_{Q^-_{8\rho}(\theta_0)} u$$

since, from the previous choice of μ, we have $\mu < \varepsilon'$. Therefore, there exists a constant σ in $(0,1)$, depending upon the data $\{p,N\}$ and α_0, η and δ such that

$$u(x,t) > \sigma\,\mu\eta\tilde{M} \quad x \in K_{4\rho},$$

for all $t \in (-\frac{1}{16}\theta_0\rho^p, 0)$; thereby, recalling the estimate for $\tilde{M}$,

$$\inf_{K_{4\rho}\times(-\frac{\theta_0}{16}\rho^p,0)} u \ge f_\Lambda(\eta) \sup_{K_{2\rho}\times(-\frac{\theta_0}{2}\rho^p,0)} u\,, \qquad f_\Lambda(\eta) = \sigma\,\mu\eta.$$

Remark 5.5. The inequality has been derived assuming that $0 < \eta < 1$. If $\eta \ge 1$ then $f_\Lambda(\eta) \ge f_\Lambda(1)$, thereby establishing a strong form of the Harnack estimate for these solutions, but this fails as pointed out in [6], whose result corresponds to our when $p = 2$.

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