LE MATEMATICHE
Vol. LXXV (2020) – Issue I, pp. 331–352
doi: 10.4418/2020.75.1.15

ON THE HARMONIC CHARACTERIZATION OF DOMAINS VIA MEAN VALUE FORMULAS

GIOVANNI CUPINI, ERMANNO LANCONELLI

The Euclidean ball have the following harmonic characterization, via Gauss-mean value property: Let D be an open set with finite Lebesgue measure and let x_0 be a point of D. If

$$u(x_0) = \fint_D u(y)dy$$

for every nonnegative harmonic function u in D, then D is a Euclidean ball centered at x_0. On the other hand, on every sufficiently smooth domain D and for every point x_0 in D there exist Radon measures μ such that

$$u(x_0) = \int_D u(y)d\mu(y)$$

for every nonnegative harmonic function u in D. In this paper we give sufficient conditions so that this last mean value property characterizes the domain D.

1. Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 3$, and let $B_r(x_0)$ be the open Euclidean ball with center x_0 and radius $r > 0$. If $\overline{B_r(x_0)} \subseteq \Omega$, by the Gauss mean value

Received on October 30, 2019
AMS 2010 Subject Classification: 35B05, 31B05
Keywords: Mean value formula, rigidity, harmonic functions
G. Cupini is member of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM)

Theorem
$$u(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(y)\,dy, \qquad \forall u \in \mathcal{H}(\Omega),$$
where $\mathcal{H}(\Omega)$ denotes the linear space of the harmonic functions in Ω and $|B_r(x_0)|$ stands for the Lebesgue measure of $B_r(x_0)$.

From this result, by using the dominated and the monotone convergence theorems, one gets:
$$u(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(y)\,dy, \qquad \forall u \in \mathcal{H}(B_r(x_0)) \cap L^1(B_r(x_0)), \tag{1}$$
and
$$u(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(y)\,dy, \qquad \forall u \in \mathcal{H}(B_r(x_0)),\ u \geq 0, \tag{2}$$
respectively.

In particular, if we consider the family of integrable and nonnegative harmonic functions
$$B_r(x_0) \ni y \mapsto \Gamma(y-x), \qquad x \in \mathbb{R}^n \setminus B_r(x_0),$$
where Γ denotes the fundamental solution of the Laplacian, both (1) and (2) imply
$$\Gamma(x_0 - x) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} \Gamma(y-x)\,dy \qquad \forall x \in \mathbb{R}^n \setminus B_r(x_0). \tag{3}$$

The Euclidean balls are the only sets satisfying identity (1), identity (2) or identity (3). The proofs of these rigidity results, giving harmonic characterizations of the Euclidean balls, has a long history, starting with the pioneering papers [8], [9], continued in [12] and [1], and recently resumed in [6], [7]. In these papers the following rigidity theorems were proved.

Theorem K (Kuran [12]). *Let $D \subset \mathbb{R}^n$, $n \geq 3$, be an open set with finite Lebesgue measure and let $x_0 \in D$ be such that*
$$u(x_0) = \fint_D u(x)\,dx \qquad \forall u \in \mathcal{H}(D) \cap L^1(D).$$
Then D is a Euclidean ball centered at x_0.

Theorem CL (Cupini-Lanconelli [6], [7]). Let $D \subset \mathbb{R}^n$, $n \geq 3$, be an open set such that $|\overline{D}| < \infty$, and let $x_0 \in D$. Assume that
$$u(x_0) = \fint_D u(x)\,dx \qquad \forall u \in \mathcal{H}(D),\ u \geq 0.$$
Then D is a Euclidean ball centered at x_0.

Theorem ASZ (Aharonov-Schiffer-Zalcman [1]). *Let $D \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set such that $D = \operatorname{int}\overline{D}$ and $|\partial D| = 0$.*

For a fixed $x_0 \in D$ assume

$$\Gamma(x_0 - x) = \fint_D \Gamma(y - x)\, dy \qquad \forall x \in \mathbb{R}^n \setminus D. \tag{4}$$

Then D is a Euclidean ball centered at x_0.

We would like to stress that Theorem ASZ gives an affirmative answer to the following question:

Let D be a homogeneous body whose Newtonian potential, outside D, is proportional to the Newtonian potential of a mass concentrated at a point $x_0 \in D$. Then: is it true that D is a ball centered at x_0?

As a by-product of the general results presented in this paper, we obtain that Theorem ASZ and Theorem CL hold true merely assuming $|D| < \infty$. Therefore, given this improvement for granted, the following implications hold:

$$\text{Theorem ASZ} \quad \Rightarrow \quad \text{Theorem CL} \quad \Rightarrow \quad \text{Theorem K.}$$

The starting point of our investigations is the paper [10] - by Hansen and Netuka - together with the papers [2] and [3] - by Aikawa. In these works the authors showed that, for any sufficiently smooth open set Ω, and for every point $x_0 \in \Omega$, there exist not trivial Radon measures μ in $\mathbb{R}^n$ such that the following Gauss-type Theorem holds:

$$\int_\Omega u(x)\, d\mu(y) = u(x_0) \qquad \forall x \in \mathcal{H}(\Omega), u \geq 0. \tag{5}$$

Then, a question naturally arises: is it possible to trace the path described above by replacing the Euclidean balls and the Lebesgue measure with Ω and μ, respectively? Or, roughly speaking: does the mean value formula (5) characterize the open set Ω? Or even, stronger question: does Theorem ASZ extend to nonhomogeneous bodies?

To put this last question in a precise form we introduce the notion of rigidity triples (Ω, μ, x_0), that, in the case of the identity (3), will be the triple

$$\left(B(x_0, r), \frac{1}{|B(x_0, r)|} m \llcorner B(x_0, r), x_0 \right), \tag{6}$$

where m is the n-dimensional Lebesgue measure.

Precisely, we give the following definition,

Definition 1.1 (Rigidity triple)**.** Let Ω be an open subset of $\mathbb{R}^n$, $\Omega \ni x_0$, μ be a non-negative Radon measure, $\mu(\mathbb{R}^n \setminus \Omega) = 0$. We say that (Ω, μ, x_0) is a *rigidity-triple* if

- Ω is solid (i.e., $\Omega = \operatorname{int}\overline{\Omega}$ and $\mathbb{R}^n \setminus \overline{\Omega}$ is connected)
- the *ASZ-property* holds, i.e.,

$$\Gamma(x_0 - x) = \int_\Omega \Gamma(y-x)\, d\mu(y), \qquad \forall\, x \in \mathbb{R}^n \setminus \Omega \tag{7}$$

- the *interior potential property* holds, i.e.,

$$\Gamma(x_0 - x) > \int_\Omega \Gamma(y-x)\, d\mu(y) \qquad \forall\, x \in \Omega \setminus \{x_0\}. \tag{8}$$

By (3) and by the Poisson-Jensen formula (see e.g. [4, Theorem 9.5.2]) it turns out that the triple in (6) is a prototype of rigidity triple.

Our question can now be formalized as follows:

Let (Ω, μ, x_0) be a rigidity triple. Let D be an open subset of $\mathbb{R}^n$, $D \ni x_0$, and let ν be a non-negative Radon measure satisfying the ASZ-property:

$$\Gamma(x_0 - x) = \int_D \Gamma(y-x)\, d\nu(y) \qquad \forall x \in \mathbb{R}^n \setminus D. \tag{9}$$

If $\mu \llcorner (\Omega \cap D) = \nu \llcorner (\Omega \cap D)$, is it true that $D = \Omega$ and $\nu = \mu$?

Of course, if this question has an affirmative answer, we will have also the following CL-type result:

$$u(x_0) = \int_D u(y)\, d\nu(y) \quad \forall u \in \mathcal{H}(D), u \geq 0 \qquad \Rightarrow \qquad D = \Omega. \tag{10}$$

In Section 2 we exhibit explicit examples of rigidity triples. In particular, by improving some results by Aikawa and by Hansen-Netuka, we show that every solid Δ-regular open set (i.e. bounded open sets for which the Dirichlet problem is solvable for every continuous boundary data) supports a rigidity triple. The results we present in that section are basically contained in the papers [6] and [7]. We provide a complete proof to put into light the crucial role played by a deep result: the Poisson-Jensen formula for superharmonic functions.

In Section 3 we state several rigidity results. Our main rigidity result is Theorem 3.1. This theorem improves, as for the Laplacian operator is concerned, the rigidity results [6, Theorem 1.1] and [7, Theorem 3.4].

In Section 4 we prove Theorem 3.1. We also show how the same strategy for proving Theorem 3.1 allows to obtain a proof of the original rigidity result for compact sets proved by Aharonov, Schiffer and Zalcman in [1] that avoids Kuran's result, which plays a crucial role in the proof given in [1].

All the main results in this paper can be extended to the L-harmonic functions, i.e. to the solutions to $Lu = 0$, for a wide class of linear second order PDE's L with nonnegative characteristic form. However we decided to confine our presentation here to classical harmonic functions, to put into evidence the main ideas and reduce the technicalities.

We remark that a research problem on the Gauss mean value formula, other than the rigidity one, is that of its *stability*. Roughly speaking, this problem can be stated as follows:

Let $D \subseteq \mathbb{R}^n$ be an open set with finite Lebesgue measure and let $x_0 \in D$. If $u(x_0)$ is close to $\fint_D u\,dx$ for every $u \in \mathcal{H}(D) \cap L^1(D)$, is it true that D is close to a Euclidean ball centered at x_0?

The answer is yes. This has been proved, together with other related results on this subject, in a joint paper with N. Fusco and X. Zhong, see [5].

2. Examples of rigidity triples

As we announced in the Introduction, aim of this section is to exhibit rigidity triples. We first show that they can be defined on every solid and Δ-regular open sets. Then, with an ad hoc procedure, we construct rigidity triples on every bounded strongly star-shaped domain.

As far as the Δ-regular open sets are concerned, we state here an improvement of Aikawa's Theorem in [2].

Proposition 2.1. *Let Ω be a bounded, connected and Δ-regular open set and let $x_0 \in \Omega$.*

Let $\varphi :]0,\infty[\to]0,\infty[$ be a measurable function, such that $\int_0^\infty \varphi(t)\,dt = 1$. Define

$$w(x) := \varphi(G(x_0,x))|\nabla G(x_0,x)|^2, \qquad x \in \Omega \setminus \{x_0\},$$

where $G(x_0,\cdot)$ is the Green function of the Laplace operator in Ω with pole at x_0.

If we denote $\mu := w\, m \llcorner \Omega$, then

$$u(x_0) = \int_\Omega u(x)w(x)\,dx, \qquad \forall u \in \mathcal{H}(\Omega), u \geq 0, \tag{11}$$

and (8) *hold. If moreover Ω is solid then (Ω, μ, x_0) is a rigidity triple.*

Before giving the proof of this result, we exhibit other rigidity triples.

Let $d : \mathbb{R}^n \to [0,\infty[$ be a smooth homogeneous norm in $\mathbb{R}^n$; i.e. $d \in C^\infty(\mathbb{R}^n \setminus \{0\})$,

$$d(x) \geq 0, = 0 \text{ iff } x = 0, \qquad d(\lambda x) = \lambda d(x) \text{ if } \lambda > 0,$$

and denote $B^d_r(x_0)$ the d-balls of radius r centered at x_0; i.e.,

$$B^d_r(x_0) := \{y \in \mathbb{R}^n : d(y - x_0) < r\}.$$

Notice that every d-ball is smooth, solid, bounded, star-shaped and open set with respect to x_0.

Define

$$P_d : \partial B^d_1(0) \to \mathbb{R}, \qquad P_d = -\frac{\partial G}{\partial \nu},$$

where G is the Green function of $B^d_1(0)$ with pole at the origin and ν is the outward normal to $B^d_1(0)$; moreover we let

$$m_d(y) := |\nabla d(y)| P_d\left(\frac{y}{d(y)}\right), \qquad y \neq 0.$$

The following result is a refinement of Theorem 1.2 in [6].

Proposition 2.2. *Let $B^d_r(x_0)$ be a d-ball in $\mathbb{R}^n$ and for every $\alpha > 0$ define*

$$w_\alpha(y) := \frac{\alpha}{r^\alpha} \frac{m_d(y - x_0)}{(d(y - x_0))^{n-\alpha}}, \qquad y \in B^d_r(x_0) \setminus \{x_0\}.$$

Let μ_α be the measure

$$\mu_\alpha := w_\alpha \, m \llcorner B^d_r(x_0). \tag{12}$$

Then

$$u(x_0) = \int_\Omega u(x) w_\alpha(x)\, dx, \qquad \forall u \in \mathcal{H}(\Omega), u \geq 0, \tag{13}$$

and $(B^d_r(x_0), \mu_\alpha, x_0)$ is a rigidity triple.

This proposition slightly improves Theorem 1.2 in [6], where α was assumed greater than $n-2$.

Notice that if d is the Euclidean norm then

$$m_d(y) = \frac{1}{n\omega_n} \qquad \text{and} \qquad w_\alpha(y) = \frac{\alpha}{n\omega_n r^\alpha} \frac{1}{|y - x_0|^{n-\alpha}};$$

thus, if $\alpha = n$, $\mu_n = \frac{1}{|B_r(x_0)|} m \llcorner B_r(x_0)$. Therefore, formula (2) for non-negative harmonic functions is a particular case of Proposition 2.2.

The following lemma, proved in [6], will be used to prove Propositions 2.1 and 2.2. We provide the proof also of this result, to put into light the role played by a deep result: the Poisson-Jensen formula for superharmonic functions.

Lemma 2.3 (Lemma 3.1 in [6])**.** *Let Ω be a connected bounded open subset of $\mathbb{R}^n$ and assume that there exists a family of open sets $(\Omega_t)_{0<t<T}$, $0 < T \leq \infty$, such that* *(i)* $\Omega = \cup_{0<t<T}\Omega_t$,

(ii) $\overline{\Omega_t} \subseteq \Omega_\tau$ *if* $0 < t < \tau < T$,

(iii) Ω_t *is connected and* Δ*-regular for a.e.* $t \in]0,T[$.

Fixed $x_0 \in \Omega$*, for every non-negative and superharmonic function* u *in* Ω *we define,*

$$m_t(u)(x_0) := \int_{\partial\Omega_t} u(y)\, d\mu_{x_0}^{\Omega_t}(y),$$

where $\mu_{x_0}^{\Omega_t}$ *denotes the* harmonic measure *of* Ω_t *at* x_0 *and*

$$M(u)(x_0) := \int_0^T \varphi(t) m_t(u)(x_0)\, dt,$$

where $\varphi :]0,T[\to]0,\infty[$ *is measurable and such that*

$$\int_0^T \varphi(t)\, dt = 1 \tag{14}$$

holds.

Then

(a) $u(x_0) \geq M(u)(x_0)$,

(b) $u(x_0) = M(u)(x_0)$ *if* u *is harmonic in* Ω,

(c) $u(x_0) > M(u)(x_0)$ *if* $u(x_0) < \infty$ *and* $\Delta u \not\equiv 0$ *in* Ω.

Proof. If Ω_t is Δ-regular, we set

$$n_t(u)(x_0) := \int_{\Omega_t} G_{\Omega_t}(x_0, y)\, d\nu_u(y). \tag{15}$$

where $G_{\Omega_t}(x_0,\cdot)$ stands for the Green function of Ω_t with pole at x_0, and ν_u is the *Riesz measure* of u; i.e.,

$$\nu_u := -\Delta u \text{ in the weak sense of distributions.}$$

By Poisson-Jensen formula (see e.g. [11, Theorem 5.27], see also [4, Theorem 9.5.1]) and the assumptions on $(\Omega_t)_{0<t<T}$, we have

$$u(x_0) = m_t(u)(x_0) + n_t(u)(x_0) \quad \text{for a.e. } t \in]0,T[. \tag{16}$$

Since u is non-negative, then $m_t(u)(x_0) \geq 0$. Moreover, since $\Omega_t \subseteq \Omega_\tau$ if $t \leq \tau$, and $\nu_u \geq 0$, the function $t \mapsto n_t(u)$ is increasing and non-negative. By (16) and (14) we get

$$u(x_0) = \int_0^T \varphi(t) m_t(u)(x_0)\,dt + \int_0^T \varphi(t) n_t(u)(x_0)\,dt =: M(u)(x_0) + N(u)(x_0). \tag{17}$$

Since $N(u)(x_0) \geq 0$ and (17) hold, then (a) follows.

If u is harmonic in Ω then $N(u)(x_0) = 0$ and, by (17), (b) follows. Moreover, if $\Delta u \not\equiv 0$ in Ω then $\nu_u \neq 0$ in Ω. Therefore, there exists $t_0 > 0$ such that $\nu_u(\Omega_{t_0}) > 0$. On the other hand $\nu_u(\Omega_t) \geq \nu_u(\Omega_{t_0})$ if $t \geq t_0$, and $G_{\Omega_t}(x_0, \cdot) > 0$ since Ω_t is connected. Then

$$n_t(u)(x_0) := \int_{\Omega_t} G_{\Omega_t}(x_0, y)\,d\nu_u(y) > 0 \quad \forall t \geq t_0,$$

so that

$$N(u)(x_0) := \int_0^T \varphi(t) n_t(u)(x_0)\,dt > 0.$$

Using this information in (17) together with the assumption $u(x_0) \in \mathbb{R}$, we immediately get (c). $\square$

Let us now prove the Propositions above.

Proof of Proposition 2.1. For every $t \in]0, \infty[$ we let

$$\Omega_t := \left\{ x \in \Omega \,:\, G(x_0, x) > \frac{1}{t} \right\}.$$

Then $(\Omega_t)_{0<t<\infty}$ satisfies conditions (i)-(iii) in Lemma 2.3. Since

$$G(x_0, \cdot) - \frac{1}{t}$$

is the Green function of Ω_t, the Green representation formula for harmonic functions implies that

$$\begin{aligned} d\mu_{x_0}^{\Omega_t}(y) &= -\frac{\partial}{\partial \nu}\left(G(x_0, y) - \frac{1}{t} \right) d\sigma(y) = -\frac{\partial}{\partial \nu} G(x_0, y)\,d\sigma(y) \\ &= \langle \nabla G(x_0, y), \nabla G(x_0, y) \rangle \frac{d\sigma(y)}{|\nabla G(x_0, y)|} = |\nabla G(x_0, y)|\,d\sigma(y), \end{aligned}$$

where ν is the outward normal. Then given a measurable function $\psi :]0, \infty[\to]0, \infty[$, with $\int_0^\infty \psi(t)\,dt = 1$, by Lemma 2.3-(b) every non-negative harmonic function u in Ω_t satisfies

$$u(x_0) = \int_0^\infty \left(\psi(t) \int_{G(x_0, y) = \frac{1}{t}} u(y) |\nabla G(x_0, y)|\,d\sigma(y) \right) dt.$$

The change of variable $t = \frac{1}{s}$ gives

$$u(x_0) = \int_0^\infty \left(\psi\left(\frac{1}{s}\right) \int_{G(x_0,y)=s} u(y)|\nabla G(x_0,y)|\, d\sigma(y) \right) \frac{ds}{s^2}$$

and, by the coarea formula, setting

$$\varphi(s) := s^{-2}\psi\left(\frac{1}{s}\right), \qquad w(y) := \varphi(G(x_0,y))|\nabla G(x_0,y)|^2$$

we get

$$u(x_0) = \int_\Omega u(y)\varphi(G(x_0,y))|\nabla G(x_0,y)|^2\, dy = \int_\Omega u(y)w(y)\, dy. \tag{18}$$

This proves (11), since

$$\int_0^\infty \varphi(s)\, ds = \int_0^\infty \psi\left(\frac{1}{s}\right) \frac{ds}{s^2} = \int_0^\infty \psi(t)\, dt = 1.$$

Let us now prove property (8). If in (18) we use the family of non-negative harmonic functions

$$\Omega \ni y \mapsto \Gamma(y-x) =: u_x(y), \qquad x \in \mathbb{R}^n \setminus \Omega$$

we obtain

$$\Gamma(x_0 - x) = \int_\Omega \Gamma(y-x)\, d\mu(y) \qquad \forall x \in \mathbb{R}^n \setminus \Omega, \tag{19}$$

where $\mu := w\, m \llcorner \Omega$. Moreover, since u_x is superharmonic in $\mathbb{R}^n$ and $\Delta u_x = -\delta_x$, the Dirac measure at x, by Lemma 2.3-(c) we have

$$\Gamma(x_0 - x) > \int_\Omega \Gamma(y-x)\, d\mu(y) \qquad \forall x \in \Omega \setminus \{x_0\}.$$

This proves (8).

Keeping in mind (19), and noting that (11) implies (7), if Ω is solid, then (Ω, μ, x_0) is a rigidity triple. □

We now provide a proof of Proposition 2.2.

Proof of Proposition 2.2. The set $B_r^d(x_0)$ is solid, bounded and open subset of $\mathbb{R}^n$. The following properties hold:

(i) $B_r^d(x_0) = \cup_{0<t<r} B_t^d(x_0)$,

(ii) $\overline{B_t^d(x_0)} \subseteq B_\tau^d(x_0)$ if $0 < t < \tau < r$,

(iii) $B_t^d(x_0)$ is connected and Δ-regular for a.e. $t \in]0,r[$.

For every non-negative and superharmonic function u in $B_r^d(x_0)$ we define,

$$m_t(u)(x_0) := \int_{\partial B_t^d(x_0)} u(y)\, d\mu_{x_0}^{B_t^d(x_0)}(y),$$

where $\mu_{x_0}^{B_t^d(x_0)}$ denotes the *harmonic measure* of $B_t^d(x_0)$ at x_0.

It is a standard fact that the measure $\mu_0^{B_1^d(0)}$ is such that

$$d\mu_0^{B_1^d(0)}(y) := P_d(y)\, d\sigma(y),$$

where

$$P_d : \partial B_1^d(0) \to \mathbb{R}, \quad P_d(y) := -\frac{\partial G}{\partial \nu}(0,y)$$

with $G(0,\cdot)$ the Green function of $B_1^d(0)$ with pole at 0, and ν is the outward normal.

Then, since Δ is left translation invariant and homogeneous of degree two w.r.t. the dilation $y \mapsto \lambda y$, one has

$$d\mu_{x_0}^{B_t^d(x_0)}(y) := \frac{1}{t^{n-1}} P_d\left(\frac{y-x_0}{t}\right) d\sigma(y).$$

For $\alpha > 0$ the function

$$\varphi_\alpha :]0,r[\to]0,\infty[, \qquad \varphi_\alpha(t) := \frac{\alpha}{r^\alpha} t^{\alpha-1}$$

is non-negative and measurable, and

$$\int_0^r \varphi_\alpha(t)\, dt = 1.$$

Define

$$\begin{aligned} M(u)(x_0) &:= \int_0^r \varphi_\alpha(t) m_t(u)(x_0)\, dt \\ &= \frac{\alpha}{r^\alpha} \int_0^r \left(\frac{1}{t^{n-\alpha}} \int_{d(y-x_0)=t} u(y) P_d\left(\frac{y-x_0}{t}\right) d\sigma(y) \right) dt \end{aligned}$$

By the coarea formula, the right hand side is equal to

$$\begin{aligned} &\frac{\alpha}{r^\alpha} \int_{B_r^d(x_0)} u(y) \frac{|\nabla d(y-x_0)|}{(d(y-x_0))^{n-\alpha}} P_d\left(\frac{y-x_0}{d(y-x_0)}\right) dy \\ &= \frac{\alpha}{r^\alpha} \int_{B_r^d(x_0)} u(y) \frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}}\, dy. \end{aligned}$$

Thus, keeping in mind the definition of μ_α, see (12), we have

$$M(u)(x_0) = \frac{\alpha}{r^\alpha}\int_{B_r^d(x_0)} u(y)\frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}}\,dy = \int_{B_r^d(x_0)} u(y)\,d\mu_\alpha(y).$$

By Lemma 2.3 (b), we obtain

$$u(x_0) = Mu(x_0) = \int_{B_r^d(x_0)} u(y)\,d\mu_\alpha(y) \tag{20}$$

for every $u \in \mathcal{H}(B_r^d(x_0))$, $u \geq 0$. We have so proved (7).

Using in (20) the family of functions

$$y \mapsto u_x(y) := \Gamma(y-x), \qquad x \notin B_r^d(x_0),$$

which are non-negative and harmonic in $B_r^d(x_0)$, we get

$$\Gamma(x_0-x) = \int_{B_r^d(x_0)} \Gamma(y-x)\,d\mu_\alpha(y), \qquad \forall x \notin B_r^d(x_0).$$

On the other hand, if $x \in B_r^d(x_0)\setminus\{x_0\}$ then

$$u_x \text{ is superharmonic in } B_r^d(x_0), \quad u_x(x_0) = \Gamma(x_0-x) < \infty$$

and

$$\Delta u_x = -\delta_x,$$

where δ_x is the Dirac measure at $\{x\}$. Then, by Lemma 2.3 (c),

$$\Gamma(x_0-x) > \int_{B_r^d(x_0)} \Gamma(y-x)\,d\mu_\alpha(y) \qquad \forall x \in B_r^d(x_0),\ x \neq x_0.$$

The conclusion follows. □

3. Rigidity results

In this section we state our main result, Theorem 3.1 below, its corollaries and other related results.

Theorem 3.1 is a general rigidity result of Aharonov-Schiffer-Zalcman's-type for open sets and general Radon measures. Roughly, it says that any time that a rigidity triple is given, then a rigidity result can be established. For the classical Laplacian, Theorem 3.1 improves [7, Theorem 3.4], since the assumption (i) below is weaker than $\mathbb{R}^n \setminus (\overline{\Omega}\cup\overline{D}) \neq \emptyset$, previously assumed. This apparently irrevelant modification, allows to establish Corollary 3.3, and the Kuran's-type rigidity result for non-negative harmonic functions, see Corollary 3.4.

Before stating the main result, we recall that the support of a measure μ in $\mathbb{R}^n$ can be defined as follows:

$$\operatorname{supp}\mu := \{x \in \mathbb{R}^n : (A \text{ open set}, x \in A) \Rightarrow \mu(A) > 0\}.$$

Theorem 3.1. *Let D be an open subset of $\mathbb{R}^n$, $x_0 \in D$, and let ν be a non-negative Radon measure, $\nu(\mathbb{R}^n \setminus D) = 0$, such that $\partial D \subseteq \operatorname{supp}\nu$ and*

$$\Gamma(x_0 - x) = \int_D \Gamma(y - x)\, d\nu(y) \qquad \forall x \in \mathbb{R}^n \setminus D. \tag{21}$$

If there exists a rigidity triple (Ω, μ, x_0) such that

(i) $\mathbb{R}^n \setminus (\overline{\Omega} \cup D) \neq \emptyset$,

(ii) $\mu \llcorner (\Omega \cap D) = \nu \llcorner (\Omega \cap D)$,

then $D = \Omega$ and $\nu = \mu$.

Before discussing the sharpness of the assumptions, we list some consequences of this result.

Corollary 3.2. *Let D be an open subset of $\mathbb{R}^n$, $x_0 \in D$, and let ν be a non-negative Radon measure, $\nu(\mathbb{R}^n \setminus D) = 0$, such that $\partial D \subseteq \operatorname{supp}\nu$ and*

$$u(x_0) = \int_D u(x)\, d\nu(x) \qquad \forall u \in \mathcal{H}(D),\ u \geq 0.$$

If there exists a rigidity triple (Ω, μ, x_0) such that

(i) $\mathbb{R}^n \setminus (\overline{\Omega} \cup D) \neq \emptyset$,

(ii) $\mu \llcorner (\Omega \cap D) = \nu \llcorner (\Omega \cap D)$,

then $D = \Omega$ and $\nu = \mu$.

Proof. Since for every $x \in \mathbb{R}^n \setminus D$

$$y \mapsto \Gamma(y - x) \text{ is a non-negative harmonic function in } D$$

the conclusion immediately follows by Theorem 3.1. □

As a consequence of this result we obtain an improvement of Theorem ASZ already announced in Introduction.

Corollary 3.3. *Let D be an open subset of $\mathbb{R}^n$ with finite Lebesgue measure and $x_0 \in D$.*

Assume that

$$\fint_D \Gamma(y-x)\,dy = \Gamma(x_0 - x) \qquad \forall x \in \mathbb{R}^n \setminus D.$$

Then D is a Euclidean ball centered at x_0.

Proof. Let r be the positive real number such that $|B(x_0,r)| = |D|$.

Since $\overline{B(x_0,r)} \cup D$ has a finite Lebesgue measure, then

$$|\mathbb{R}^n \setminus (\overline{B(x_0,r)} \cup D)| = +\infty,$$

so the set $\mathbb{R}^n \setminus (\overline{B(x_0,r)} \cup D)$ is not empty.

Define μ and ν the Radon measures

$$\mu := \frac{1}{|B(x_0,r)|} m \llcorner B(x_0,r), \qquad \nu := \frac{1}{|D|} m \llcorner D.$$

Trivially,

$$\mu \llcorner (B(x_0,r) \cap D) = \nu \llcorner (B(x_0,r) \cap D), \qquad \partial D \subseteq \operatorname{supp} \nu.$$

Since $(B(x_0,r), \mu, x_0)$ is a rigidity triple, we conclude by using Theorem 3.1. $\square$

Since for every $x \notin D$ the function $D \ni y \mapsto \Gamma(y-x)$ is harmonic and non-negative in D, a straightforward consequence of the above result is the following.

Corollary 3.4. *Let D be an open subset of $\mathbb{R}^n$ with finite Lebesgue measure and $x_0 \in D$.*

Assume that

$$\fint_D u(x)\,dx = u(x_0) \qquad \forall u \in \mathcal{H}(D),\, u \geq 0. \tag{22}$$

Then D is a Euclidean ball centered at x_0.

Notice that this result is more general than Theorem K and Theorem CL. Indeed, asking that the mean value formula holds for every non-negative, harmonic functions, is weaker than asking that the mean value formula holds true for every summable, harmonic functions, being

$$\begin{aligned}
&\{u \in \mathcal{H}(D) : u \geq 0, \fint_D u(x)\,dx = u(x_0)\} \\
&= \{u \in \mathcal{H}(D) \cap L^1(D) : u \geq 0, \fint_D u(x)\,dx = u(x_0)\} \\
&\subsetneq \{u \in \mathcal{H}(D) \cap L^1(D) : \fint_D u(x)\,dx = u(x_0)\}.
\end{aligned}$$

A result analogous to Corollary 3.3, stated for a bounded set D, is Corollary 3.5 below.

Corollary 3.5. *Let D be an open, bounded subset of $\mathbb{R}^n$ and $x_0 \in D$. Assume that*

$$\int_D \Gamma(y-x)\,dy = c\,\Gamma(x_0-x) \qquad \forall x \in \mathbb{R}^n \setminus D \tag{23}$$

for some $c>0$. Then $c=|D|$ and D is a Euclidean ball centered at x_0.

Proof. By (23)

$$c = \lim_{|x|\to+\infty} \int_D \frac{\Gamma(y-x)}{\Gamma(x_0-x)}\,dy = \int_D 1\,dy = |D|,$$

therefore

$$⨍_D \Gamma(y-x)\,dy = \Gamma(x_0-x) \qquad \forall x \in \mathbb{R}^n \setminus D.$$

The conclusion follows by Corollary 3.3. □

This last result is a version for open sets of the following result proved by Aharonov, Schiffer and Zalcman in [1] for closed sets:

Theorem 3.6 (Aharonov-Schiffer-Zalcman [1])**.** *Let P be a compact set in $\mathbb{R}^n$, $n \geq 3$, such that $P = \overline{\operatorname{int} P}$. Assume that*

$$\int_P \Gamma(y-x)\,dy = c\,\Gamma(x_0-x) \qquad \forall x \in \mathbb{R}^n \setminus P \tag{24}$$

for some $x_0 \in P$ and $c \in \mathbb{R}$. Then $c=|P|$ and P is a closed Euclidean ball centered at x_0.

The proof given in [1] uses in a crucial way the Kuran's result and it cannot be immediately deduced by our previous results, since, up to now, we dealt with open sets. Nevertheless our technique works equally well also for closed sets, by allowing to prove Theorem 3.6 by-passing the Kuran's result. For the sake of completeness, we will provide in Section 4 the complete proof of Theorem 3.6 by using our method.

Theorem 3.1, together with Theorem 2.2, gives the following *d-spherical* symmetry result.

Theorem 3.7. *Let d be any smooth homogeneous norm in $\mathbb{R}^n$.*
Let $D \subset \mathbb{R}^n$ be an open bounded set and $x_0 \in D$.

Using the notation and the definitions in Proposition 2.2, assume that for $\alpha > 0$ *and* $c > 0$

$$\Gamma(x_0 - x) = c \int_D \Gamma(y-x) \frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}}\, dy, \qquad \forall x \in \mathbb{R}^n \setminus D.$$

Then $c = \left(\int_D \frac{m_d(y-x_0)}{(d(y-x_0))^{n-\alpha}}\, dy \right)^{-1}$ *and* $D = B_r^d(x_0)$ *with* $r = (\frac{\alpha}{c})^{\frac{1}{\alpha}}$.

Note that if $\alpha = n$ and d is the Euclidean norm, then Theorem 3.7 gives back Corollary 3.5.

We conclude this section, by exhibiting examples to show that in Theorem 3.1 neither (ii) nor $\partial D \subseteq \operatorname{supp} \nu$ can be removed, see Examples 3.8 and 3.9. We also show that in the definition of rigidity triple, see (i) of Theorem 3.1, the property (8) cannot be removed, otherwise the rigidity result fails, see Example 3.10.

Example 3.8. Assumption (ii) in Theorem 3.1 cannot be removed.

For instance, if Ω is the ball $B_r(x_0)$ and D is the ball in $B_{r'}(x_0)$, with $r \neq r'$, and $\mu = \frac{1}{|B_r(x_0)|} m \llcorner B_r(x_0)$ and $\nu = \frac{1}{|B_{r'}(x_0)|} m \llcorner B_{r'}(x_0)$ then

$$\mu \llcorner (\Omega \cap D) \neq \nu \llcorner (\Omega \cap D).$$

It is easy to prove that all the other assumptions of Theorem 3.1 are satisfied.

Example 3.9. The assumption $\partial D \subseteq \operatorname{supp} \nu$ in Theorem 3.1 cannot be removed.

Indeed, consider $\Omega = B_r(x_0)$, $D = B_R(x_0)$, with $0 < r < R$. Define $\mu = \nu = \frac{1}{|B_r(x_0)|} m \llcorner B_r(x_0)$. Of course $\operatorname{supp}(\nu) = \overline{B_r(x_0)}$, therefore $\partial D \not\subseteq \operatorname{supp}(\nu)$. It is easy to prove that all the other assumptions of the theorems are satisfied. In particular, for every $x \in \mathbb{R}^n \setminus D \subset \mathbb{R}^n \setminus \Omega$

$$\Gamma(x_0 - x) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \Gamma(y-x)\, dy = \int_{B_R(x_0)} \Gamma(y-x)\, d\nu(y),$$

which implies that ν satisfies (21) with $\Omega = B_R(x_0)$.

Example 3.10. The request that μ satisfies (8) cannot be removed in Theorem 3.1.

Indeed, consider the balls $\Omega = B_R(x_0)$, $D = B_r(x_0)$, with $0 < r < R$. Define $\mu = \nu = \frac{1}{|B_r(x_0)|} m \llcorner B_r(x_0)$. Obviously μ satisfies (7) on the set $B_R(x_0)$ and ν satisfies (21) on the set $B_r(x_0)$, $\partial D \subseteq \operatorname{supp} \nu$. For every $x \in B_R(x_0) \setminus B_r(x_0)$, by (3)

$$\Gamma(x_0 - x) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \Gamma(y-x)\, dy = \int_{B_R(x_0)} \Gamma(y-x)\, d\mu(y),$$

which implies that (8) does not hold, so $(B_R(x_0), \mu, x_0)$ is not a rigidity-triple. All the other assumptions in Theorem 3.1 hold.

4. Proofs of Theorems 3.1 and 3.6

We begin by proving Theorem 3.1.

Proof of Theorem 3.1. We split the proof in four steps. Denote the Γ-*potentials* of μ and ν as follows:

$$\Gamma_\mu(x) := \int_\Omega \Gamma(y-x)\, d\mu(y) \qquad x \in \mathbb{R}^n,$$

$$\Gamma_\nu(x) := \int_D \Gamma(y-x)\, d\nu(y) \qquad x \in \mathbb{R}^n.$$

STEP 1. Let us prove that $\Gamma_\mu \le \Gamma_\nu$ in $\mathbb{R}^n \setminus \{x_0\}$.

Since (Ω, μ, x_0) is a rigidity triple, then (7) and (8) hold. Using also the assumption (21) we get

$$\Gamma_\mu(x) \le \Gamma(x_0 - x) < \infty \quad \forall x \in \mathbb{R}^n \setminus \{x_0\}, \quad \Gamma_\nu(x) = \Gamma(x_0 - x) \quad \forall x \in \mathbb{R}^n \setminus D. \tag{25}$$

Then, since $x_0 \in D$,

$$\Gamma_\mu(x) \le \Gamma_\nu(x) \qquad \forall x \in \mathbb{R}^n \setminus D.$$

It remains to prove that $\Gamma_\mu \le \Gamma_\nu$ in $D \setminus \{x_0\}$. We first remark that, by the first chain of inequalities in (25),

$$\Gamma_\mu - \Gamma_\nu \text{ is well defined and } < \infty \text{ in } D \setminus \{x_0\}.$$

Moreover, by using (ii), one easily recognizes that

$$\Gamma_\mu(x) - \Gamma_\nu(x) = \int_{\Omega \setminus D} \Gamma(y-x)\, d\mu(y) - \int_{D \setminus \Omega} \Gamma(y-x)\, d\nu(y) \qquad \forall x \in D \setminus \{x_0\}. \tag{26}$$

Hereafter we agree to let an integral be equal to zero, if the integration domain is empty.

The functions

$$h(x) := \int_{\Omega \setminus D} \Gamma(y-x)\, d\mu(y), \qquad x \in D$$

and

$$v(x) := \int_{D \setminus \Omega} \Gamma(y-x)\, d\nu(y), \qquad x \in D$$

are, respectively, harmonic and superharmonic in D, see e.g. Appendix in [7]. As a consequence,

$$\tilde{u} := h - v \text{ is subharmonic in } D;$$

moreover, keeping in mind (26),

$$\tilde{u} = \Gamma_\mu - \Gamma_\nu \quad \text{in } D \setminus \{x_0\}.$$

On the other hand, by the first item in (25), the lower semicontinuity of Γ_ν, and (21)

$$\begin{aligned}\limsup_{D \ni y \to x} \tilde{u}(y) &= \limsup_{D \ni y \to x} (\Gamma_\mu - \Gamma_\nu)(y) \le \limsup_{D \ni y \to x} (\Gamma(x_0 - y) - \Gamma_\nu(y)) \\ &\le \Gamma(x_0 - x) - \Gamma_\nu(x) = 0 \qquad \forall x \in \partial D.\end{aligned}$$

Moreover,

$$\limsup_{D \ni y \to \infty} \tilde{u}(y) \le \limsup_{D \ni y \to \infty} (\Gamma(x_0 - y) - \Gamma_\nu(y)) \le \limsup_{y \to \infty} \Gamma(x_0 - y) = 0.$$

By the maximum principle for subharmonic functions (see [4, Theorem 8.2.19 (ii)]) we get $\tilde{u} \le 0$ in D; hence $\Gamma_\mu \le \Gamma_\nu$ in $D \setminus \{x_0\}$.

STEP 2. Let us prove that $\partial D \subseteq \overline{\Omega}$.

By contradiction, assume there exists a point $x \in \partial D$ such that $x \notin \overline{\Omega}$. Then $x \in \operatorname{supp} \nu$ (by assumption $\partial D \subseteq \operatorname{supp} \nu$) and $\mathbb{R}^n \setminus \overline{\Omega}$ is an open set containing x. As a consequence

$$\nu(\mathbb{R}^n \setminus \overline{\Omega}) > 0. \tag{27}$$

Since μ has its support contained in $\overline{\Omega}$, Γ_μ is harmonic in $\mathbb{R}^n \setminus \overline{\Omega}$, see e.g. Appendix in [7], so that

$$\Gamma_\mu - \Gamma_\nu \text{ is subharmonic in } \mathbb{R}^n \setminus \overline{\Omega}.$$

On the other hand, by what we proved in Step 1, $\Gamma_\mu - \Gamma_\nu \le 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Moreover, since (7) and (21) imply

$$\Gamma_\mu = \Gamma_\nu \qquad \text{in } \Omega^c \cap D^c,$$

then

$$(\Gamma_\mu - \Gamma_\nu)(x) = 0.$$

Since Ω is a solid set, then $\mathbb{R}^n \setminus \overline{\Omega}$ is a connected set, so the strong maximum principle for subharmonic functions (see in [4, Theorem 8.2.19 (i)]) imply

$$\Gamma_\mu - \Gamma_\nu = 0 \qquad \text{in } \mathbb{R}^n \setminus \overline{\Omega},$$

so that

$$\Delta(\Gamma_\mu - \Gamma_\nu) = 0 \qquad \text{in } \mathbb{R}^n \setminus \overline{\Omega}.$$

On the other hand, in $\mathbb{R}^n \setminus \overline{\Omega}$, $\Delta(\Gamma_\mu - \Gamma_\nu) = \nu$. Therefore, $\nu(\mathbb{R}^n \setminus \overline{\Omega}) = 0$, in contradiction with (27).

STEP 3. Let us prove that $D \subseteq \Omega$.

We have

$$\mathbb{R}^n \setminus \overline{\Omega} = (D \cup D^c) \setminus \overline{\Omega} = (D \setminus \overline{\Omega}) \cup (D^c \cap (\overline{\Omega})^c) = (D \setminus \overline{\Omega}) \cup (D \cup \overline{\Omega})^c.$$

Notice that $D \setminus \overline{\Omega}$ and $(D \cup \overline{\Omega})^c$ are disjoint sets. By assumption (i), $(D \cup \overline{\Omega})^c$ is not empty and by Step 2, we have

$$(D \cup \overline{\Omega})^c = (\partial D \setminus \overline{\Omega}) \cup (\overline{D}^c \cap \overline{\Omega}^c) = (\overline{D} \cup \overline{\Omega})^c,$$

thus, $(D \cup \overline{\Omega})^c$ is an open set. Since $\mathbb{R}^n \setminus \overline{\Omega}$ is connected, because Ω is a solid set, the open set $D \setminus \overline{\Omega}$ must be empty. We have so proved that $D \subseteq \overline{\Omega}$. Since Ω is solid, then $\operatorname{int} \overline{\Omega} = \Omega$. We conclude that $D \subseteq \Omega$.

STEP 4. Let us prove that $\Omega \subseteq D$. We argue by contradiction; i.,e, we assume that there exists $x \in \Omega \setminus D$. In particular, $x \neq x_0$. By Step 3, $D \subseteq \Omega$. Therefore, since (Ω, μ, x_0) is a rigidity triple and using (ii) and (21), we have

$$\begin{aligned}\Gamma(x_0 - x) > \Gamma_\mu(x) &= \int_D \Gamma(y-x)\, d\mu(y) + \int_{\Omega \setminus D} \Gamma(y-x)\, d\mu(y)\\ &\geq \int_D \Gamma(y-x)\, d\mu(y) = \int_D \Gamma(y-x)\, d\nu(y) = \Gamma(x_0 - x).\end{aligned}$$

This is an absurd.

We have so proved that $D = \Omega$ and, consequently, that $\mu = \nu$. □

We now give our proof of the Aharonov-Schiffer-Zalcman's result, Theorem 3.6.

Proof of Theorem 3.6. Let r be the positive real number such that $|B(x_0, r)| = |P|$ and define the measures

$$\mu := \frac{1}{|B(x_0, r)|} m \llcorner B(x_0, r), \quad \text{and} \quad \nu := \frac{1}{|P|} m \llcorner P.$$

Denote the Γ-*potentials* of μ and ν as follows:

$$\Gamma_\mu(x) := \fint_{B(x_0,r)} \Gamma(y-x)\, dy \qquad x \in \mathbb{R}^n,$$

$$\Gamma_\nu(x) := \fint_P \Gamma(y-x)\, dy \qquad x \in \mathbb{R}^n.$$

The functions $\Gamma_\mu, \Gamma_\nu : \mathbb{R}^n \to [0, \infty[$ are continuous, being the convolution of a locally integrable function with a bounded measurable function of compact support. By (24) it follows that x_0 must lie in the interior of P, for otherwise the right-hand side of (24) would be unbounded as x goes to x_0, while the left hand side remains bounded. Moreover, if (x_n) is a sequence in $\mathbb{R}^n$, such that $|x_n| \to \infty$, by (24) and the dominated convergence theorem we get

$$c = \lim_{n\to\infty} \int_P \frac{\Gamma(y-x_n)}{\Gamma(x_0-x_n)}\,dy = |P|.$$

We now split the rest of the proof into steps.

STEP 1. Let us prove that $\Gamma_\mu \le \Gamma_\nu$ in $\mathbb{R}^n$.

Since $(B(x_0,r),\mu,x_0)$ is a rigidity triple, then (7) and (8) hold. Using also the assumption (24) we get

$$\Gamma_\mu(x) \le \Gamma(x_0-x) \quad \forall x\in\mathbb{R}^n, \quad \Gamma_\nu(x) = \Gamma(x_0-x) \quad \forall x\in\mathbb{R}^n\setminus P. \tag{28}$$

Then

$$\Gamma_\mu(x) \le \Gamma_\nu(x) \qquad \forall x\in\mathbb{R}^n\setminus P.$$

It remains to prove that $\Gamma_\mu \le \Gamma_\nu$ in P. One easily recognizes that

$$\Gamma_\mu(x)-\Gamma_\nu(x) = \frac{1}{|P|}\left(\int_{B(x_0,r)\setminus P}\Gamma(y-x)\,dy - \int_{P\setminus B(x_0,r)}\Gamma(y-x)\,dy\right) \quad \forall x\in\mathbb{R}^n, \tag{29}$$

where we agree to let an integral be equal to zero, if the integration domain is empty.

The functions

$$h(x) := \frac{1}{|P|}\int_{B(x_0,r)\setminus P}\Gamma(y-x)\,dy, \qquad x\in\mathbb{R}^n$$

and

$$v(x) := \frac{1}{|P|}\int_{P\setminus B(x_0,r)}\Gamma(y-x)\,dy, \qquad x\in\mathbb{R}^n$$

are, respectively, harmonic and superharmonic in $\operatorname{int}P$, thus

$$\tilde{u} := h-v \ \text{ is subharmonic in } \operatorname{int}P.$$

Moreover, by (29),

$$\tilde{u} = \Gamma_\mu - \Gamma_\nu \quad \text{in } \mathbb{R}^n.$$

Consider $x\in\partial P$ and let (y_h) be a sequence in P^c, convergent to x. By the first item in (28), and (24), we have

$$\begin{aligned}\lim_{y\to x}\tilde{u}(y) &= \lim_{h\to\infty}\tilde{u}(y_h) = \lim_{h\to\infty}(\Gamma_\mu-\Gamma_\nu)(y_h)\\ &\le \lim_{h\to\infty}(\Gamma(x_0-y_h)-\Gamma_\nu(y_h)) = \lim_{h\to\infty}(\Gamma(x_0-y_h)-\Gamma(x_0-y_h)) = 0.\end{aligned}$$

By the maximum principle for subharmonic functions, we get $\tilde{u}\le 0$ in $\operatorname{int}P$; hence $\Gamma_\mu \le \Gamma_\nu$ in $\operatorname{int}P$. By continuity, we conclude that $\Gamma_\mu \le \Gamma_\nu$ in P.

STEP 2.

Let us prove that $\partial P \subseteq \overline{B(x_0,r)}$.

By contradiction, assume there exists a point $x \in \partial P$ such that $x \notin \overline{B(x_0,r)}$. Γ_μ is harmonic in $\mathbb{R}^n \setminus \overline{B(x_0,r)}$, so that

$$\Gamma_\mu - \Gamma_\nu \text{ is subharmonic in } \mathbb{R}^n \setminus \overline{B(x_0,r)}.$$

On the other hand, by what we proved in Step 1, $\Gamma_\mu - \Gamma_\nu \leq 0$ in $\mathbb{R}^n \setminus \overline{B(x_0,r)}$. Moreover, since (7) and (24) imply

$$\Gamma_\mu = \Gamma_\nu = \Gamma \qquad \text{in } B(x_0,r)^c \cap P^c,$$

then, by continuity, $(\Gamma_\mu - \Gamma_\nu)(x) = 0$.

Since $\mathbb{R}^n \setminus \overline{B(x_0,r)}$ is a connected set, by the strong maximum principle for subharmonic functions we get

$$\Gamma_\mu - \Gamma_\nu = 0 \qquad \text{in } \mathbb{R}^n \setminus \overline{B(x_0,r)},$$

so that

$$\Delta(\Gamma_\mu - \Gamma_\nu) = 0 \qquad \text{in } \mathbb{R}^n \setminus \overline{B(x_0,r)}.$$

On the other hand, in $\mathbb{R}^n \setminus \overline{B(x_0,r)}$, $\Delta(\Gamma_\mu - \Gamma_\nu) = -\mu + \nu = \nu$. Therefore,

$$\nu(\mathbb{R}^n \setminus \overline{B(x_0,r)}) = 0.$$

This is an absurd. Indeed, $P = \overline{\operatorname{int} P}$ implies $\partial P \subseteq \operatorname{supp} \nu$. Since $\mathbb{R}^n \setminus \overline{B(x_0,r)}$ is an open set containing $x \in \partial P$, then $\nu(\mathbb{R}^n \setminus \overline{B(x_0,r)}) > 0$.

STEP 3. Let us prove that $P \subseteq \overline{B(x_0,r)}$.

We have

$$\begin{aligned}
\mathbb{R}^n \setminus \overline{B(x_0,r)} &= (P \cup P^c) \setminus \overline{B(x_0,r)} = (P \setminus \overline{B(x_0,r)}) \cup (P^c \cap (\overline{B(x_0,r)})^c) \\
&= (P \setminus \overline{B(x_0,r)}) \cup (P \cup \overline{B(x_0,r)})^c \\
&= (\operatorname{int} P \setminus \overline{B(x_0,r)}) \cup (P \cup \overline{B(x_0,r)})^c.
\end{aligned}$$

where the last equality follows by the inclusion $\partial P \subseteq \overline{B(x_0,r)}$ proved in Step 2.

Notice that $\operatorname{int} P \setminus \overline{B(x_0,r)}$ and $(P \cup \overline{B(x_0,r)})^c$ are open, disjoint sets. By the boundedness of P the set $(P \cup \overline{B(x_0,r)})^c$ is not empty. Since $\mathbb{R}^n \setminus \overline{B(x_0,r)}$ is a connected set, the set $\operatorname{int} P \setminus \overline{B(x_0,r)}$ must be empty; i.e. $\operatorname{int} P \subseteq \overline{B(x_0,r)}$. Since $P = \overline{\operatorname{int} P}$ we conclude that $P \subseteq \overline{B(x_0,r)}$.

STEP 4. Let us prove that $\overline{B(x_0,r)} \subseteq P$. It suffices to prove that $B(x_0,r) \subseteq P$. We argue by contradiction. Assume that there exists $x \in B(x_0,r) \setminus P$. By Step 3, $P \subseteq \overline{B(x_0,r)}$. Therefore, using that $(B(x_0,r), \frac{1}{B(x_0,r)} m \llcorner B(x_0,r), x_0)$ is a rigidity triple, (24) and recalling that $|B(x_0,r)| = |P|$, we have

$$\begin{aligned}\Gamma(x_0 - x) > \Gamma_\mu(x) &= \fint_{B(x_0,r)} \Gamma(y-x)\, dy \\ &= \fint_P \Gamma(y-x)\, dy + \frac{1}{|P|} \int_{B(x_0,r)\setminus P} \Gamma(y-x)\, d\mu(y) \\ &\geq \fint_P \Gamma(y-x)\, dy = \Gamma(x_0 - x).\end{aligned}$$

This is an absurd.

Collecting Steps 3 and 4, we get the conclusion. $\square$

REFERENCES

[1] D. AHARONOV, M.M. SCHIFFER, L. ZALCMAN: Potato kugel, *Israel J. Math.* **40** (1981), 331-339.

[2] H. AIKAWA: Integrability of superharmonic functions and subharmonic functions, *Proc. Amer. Math. Soc.* 120 (1994) 109-117.

[3] H. AIKAWA: Densities with the mean value property for harmonic functions in a Lipschitz domain, *Proc. Amer. Math. Soc.* 125 (1997) 229-234.

[4] A. BONFIGLIOLI, E. LANCONELLI, F. UGUZZONI: *Stratified Lie Groups and Potential Theory for their sub-Laplacians*. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[5] G. CUPINI, N. FUSCO, E. LANCONELLI, X. ZHONG: *A sharp stability result for the Gauss mean value formula,* J. Analyse Mathématiques, accepted.

[6] G. CUPINI, E. LANCONELLI: *On an inverse problem in potential theory,* Atti Acc. Sci. Mat. Ren. Lincei **27** (2016), 431-442.

[7] G. CUPINI, E. LANCONELLI: *Densities with the Mean Value Property for Sub-Laplacians: An Inverse Problem.* Harmonic Analysis, Partial Differential Equations and Applications. S. Chanillo, B. Franchi, G. Lu, C. Perez, E.T. Sawyer (eds.). Applied and Numerical Harmonic Analysis, Birkhäuser, 2017, 109-124.

[8] B. EPSTEIN: On the mean-value property of harmonic functions, *Proc. Amer. Math. Soc.***13** (1962) 830.

[9] B. EPSTEIN, M.M. SCHIFFER: On the mean-value property of harmonic functions, *J. Analyse Math.***14** (1965) 109-111.

[10] W. HANSEN, I. NETUKA: Volume densities with the mean value property for harmonic functions, *Proc. Amer. Math. Soc.* 123 (1995) 135-140.

[11] W.K. HAYMAN, P.B. KENNEDY: *Subharmonic functions Vol. I.* London Mathematical Society Monographs, No. 9. Academic Press, London-New York, 1976.

[12] Ü. KURAN: *On the mean-value property of harmonic functions,* Bull. London Math. Soc. **4** (1972), 311-312.

[13] E. LANCONELLI: "Potato kugel" for sub-Laplacians, *Israel J. Math.* 194 (2013) 277-283.

GIOVANNI CUPINI, ERMANNO LANCONELLI
Dipartimento di Matematica
Università di Bologna
giovanni.cupini@unibo.it, ermanno.lanconelli@unibo.it