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ON FRACTIONAL CALCULUS OPERATOR N^{ν_1,ν_2} AND P-TRANSFORMATION

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In this paper, the set $\{N^{\nu_1\nu_2}\}$ of fractional calculus is discussed. It is shown that the set is an Abelian product group for $f(z_1, z_2) = f \in \mathbb{F} = \{f; 0 \neq |f_{\nu_1,\nu_2}| < \infty, \nu_1, \nu_2 \in \mathbb{R}\}$ with continuous indexes ν_1, ν_2, [1]. A new P-transformation is being introduced for the functions of two variables.

1. Introduction.

Fractional and partial differintegration are defined in [3], { Part II chapter I, Definition and some properties of fractional calculus of the functions of many variables; [5]; pp. 160–175}.

Let $f = f(z_1, z_2)$ be a regular function in $D = D_1 \times D_2$,

$$N^{\nu_1,\nu_2} f(z_1, z_2) = f_{\nu_1,\nu_2} =_{c_2,c_1} f_{\nu_1(z_1)\nu_2(z_2)} = \tag{1.1}$$

$$= \frac{\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)}{(2\pi i)^2} \int_{C_2} \int_{C_1} \frac{f(\xi_1, \xi_2)}{(\xi_2 - z_2)^{\nu_2+1}(\xi_1 - z_1)^{\nu_1+1}} \, d\xi_1 d\xi_2$$

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$(\nu_1, \nu_2 \notin \mathbb{Z}^-)$ and

$$f_{-m_1,-m_2} = \lim_{\nu_k \to -m_k} f_{\nu_1,\nu_2} \quad (m_1, m_2 \in \mathbb{Z}^+), (k = 1, 2) \tag{1.2}$$

where

$$\xi_k \neq z_k,\ z_k \in \mathbb{C},\ \nu_k \in \mathbb{R}$$

$$-\pi \leq \arg(\xi_k - z_k) \leq \pi \text{ for } {}_-C_k$$

$$0 \leq \arg(\xi_k - z_k) < 2\pi \text{ for } {}_+C_k$$

${}_-C_k$ is a curve along the cut joining two points z_k and $-\infty + i\mathrm{Im}(z_k)$,
${}_+C_k$ is a curve along the cut joining two points z_k and $\infty + i\mathrm{Im}(z_k)$,
${}_-D_k$ is a domain surrounded by ${}_-C_k$,
${}_+D_k$ is a domain surrounded by ${}_+C_k$,
$D_k = \{{}_-D_k, {}_+D_k\}$ and $C_k = \{{}_-C_k, {}_+C_k\}$.

Then $f_{\nu_1,\nu_2}(\nu_1, \nu_2 > 0)$ is fractional and partial derivatives of order ν_1, ν_2 with respect to z_1, z_2 respectively of the function f and $f_{\nu_1,\nu_2}(\nu_1, \nu_2 < 0)$ is fractional and partial integral of order ν_1, ν_2 with respect to z_1, z_2 respectively of the function f, if $|f_{\nu_1,\nu_2}| < \infty$.

2. Fractional Calculus Operator N^{ν_1,ν_2}.

Theorems 1∼3 on the function of two variables given by Nishimoto [3] are as:

1. *Index Law*

Theorem 1. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, then we have*

$$(f_{\alpha_1(z_1),\alpha_2(z_2)}(z_1, z_2))_{\beta_1(z_1),\beta_2(z_2)} = f_{(\alpha_1+\beta_1)(z_1),(\alpha_2+\beta_2)(z_2)}(z_1, z_2) = \tag{2.1}$$

$$= f_{\alpha_1+\beta_1,\alpha_2+\beta_2}$$

Corollary 1. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, then we have*

$$f_{(\alpha_1+\alpha_2)(z_1),(\beta_1+\beta_2)(z_2)} = f_{(\beta_1+\beta_2)(z_2),(\alpha_1+\alpha_2)(z_1)} \tag{2.2}$$

2. *Linearity*

Theorem 2. *We have*

$$N^{\nu_1,\nu_2}(\alpha \cdot f(z_1, z_2)) = (\alpha \cdot f(z_1, z_2))_{\nu_1,\nu_2} = \tag{2.3}$$

$$= \alpha \cdot (f(z_1, z_2))_{\nu_1,\nu_2} = (f(z_1, z_2) \cdot \alpha)_{\nu_1,\nu_2}$$

where α is a constant.

Theorem 3. *Let U and V are the function of two variables and a, b are constants then*

$$N^{\nu_1,\nu_2}\{a \cdot U(z_1, z_2) + b \cdot V(z_1, z_2)\} = \tag{2.4}$$

$$= (a \cdot U(z_1, z_2) + b \cdot V(z_1, z_2))_{\nu_1\nu_2} = a \cdot U_{\nu_1,\nu_2} + b \cdot V_{\nu_1,\nu_2},$$

where $U = U(z_1, z_2)$ and $V = V(z_1, z_2)$.

Theorem 4. *Let binary operation $*$ be defined as*

$$N^{\alpha_1,\alpha_2} * N^{\beta_1,\beta_2} f(z_1, z_2) = N^{\alpha_1,\alpha_2} \cdot N^{\beta_1,\beta_2} f(z_1, z_2) = N^{\alpha_1,\alpha_2} \cdot (N^{\beta_1,\beta_2} f) \tag{2.5}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, and N^{ν_1,ν_2} is the operator of fractional calculus for the functions of two variables such as

$$f_{\nu_1,\nu_2}(z_1, z_2) = N^{\nu_1,\nu_2} f = \tag{2.6}$$

$$= \frac{\Gamma(\nu_1 + 1)\Gamma(\nu_2 + 1)}{(2\pi i)^2} \int_{C_2} \int_{C_1} \frac{f(\xi_1, \xi_2) \cdot d\xi_1 \cdot d\xi_2}{(\xi_1 - z_1)^{\nu_1+1}(\xi_2 - z_2)^{\nu_2+1}} ,$$

then the set

$$\{N^{\nu_1,\nu_2}\} = \{N^{\nu_1,\nu_2};\ \nu_k \in \mathbb{R}, k = 1, 2\} \tag{2.7}$$

is an Abelian product group and $f(z_1, z_2) = f \in \mathbb{F}$ is a function of two variables.

Proof.

(i) *Closure*: We have

$$\begin{aligned} N^{\alpha_1,\alpha_2} N^{\beta_1,\beta_2} f(z_1, z_2) &= N^{\alpha_1,\alpha_2}(N^{\beta_1,\beta_2} f) \\ &= N^{\alpha_1,\alpha_2} f_{\beta_1,\beta_2} \qquad \text{by (1.1))} \\ &= (f_{\beta_1,\beta_2})_{\alpha_1,\alpha_2} \\ &= f_{\alpha_1+\beta_1,\alpha_2+\beta_2} \qquad \text{by (2.1))} \\ &= N^{\alpha_1+\beta_1,\alpha_2+\beta_2} f(z_1, z_2). \end{aligned}$$

Therefore setting $\gamma_1 = \alpha_1 + \beta_1$, $\gamma_2 = \alpha_2 + \beta_2 \in \mathbb{R}$, we have

$$N^{\alpha_1,\alpha_2} N^{\beta_1,\beta_2} = N^{\alpha_1+\beta_1,\alpha_2+\beta_2} = N^{\gamma_1,\gamma_2} \in \{N^{\nu_1,\nu_2}\} \text{ if } f(z_1, z_2) \in \mathbb{F}$$

i.e. binary operation is closure for fractional operator of two variables.

(ii) *Associative Law*: We have

$$\begin{aligned} N^{\alpha_1,\alpha_2}(N^{\beta_1,\beta_2} N^{\gamma_1,\gamma_2}) f &= N^{\alpha_1,\alpha_2}(N^{\beta_1+\gamma_1,\beta_2+\gamma_2} f) \qquad \text{(by Index law)} \\ &= N^{\alpha_1,\alpha_2} f_{\beta_1+\gamma_1,\beta_2+\gamma_2} \qquad \text{(by (1.1))} \\ &= (f_{\beta_1+\gamma_1,\beta_2+\gamma_2})_{\alpha_1,\alpha_2} \\ &= f_{\alpha_1+\beta_1+\gamma_1,\alpha_2+\beta_2+\gamma_2} \\ &= N^{\alpha_1+\beta_1+\gamma_1,\alpha_2+\beta_2+\gamma_2} f \end{aligned} \tag{2.8}$$

On the other hand we have

$$\begin{aligned} (N^{\alpha_1,\alpha_2} N^{\beta_1,\beta_2}) N^{\gamma_1,\gamma_2} f &= N^{\alpha_1+\beta_1,\alpha_2+\beta_2} \cdot f_{\gamma_1,\gamma_2} \qquad \text{by (1.1) and (2.1))} \\ &= f_{\alpha_1+\beta_1+\gamma_1,\alpha_2+\beta_2+\gamma_2} \\ &= N^{\alpha_1+\beta_1+\gamma_1,\alpha_2+\beta_2+\gamma_2} f \end{aligned} \tag{2.9}$$

Therefore, from (2.8) and (2.9), we get

$$N^{\alpha_1,\alpha_2}(N^{\beta_1,\beta_2} N^{\gamma_1,\gamma_2}) = (N^{\alpha_1,\alpha_2} N^{\beta_1,\beta_2}) N^{\gamma_1,\gamma_2} = N^{\alpha_1+\beta_1+\gamma_1,\alpha_2+\beta_2+\gamma_2} \in \{N^{\nu_1,\nu_2}\}$$

for $f \in \mathbb{F}$ and $\alpha_k, \beta_k, \gamma_k \in \mathbb{R}$, $(k = 1, 2)$.

(iii) *Identity Element*: We have

$$N^{\alpha_1,\alpha_2} N^{0,0} f = N^{\alpha_1+0,\alpha_2+0} f = N^{\alpha_1,\alpha_2} f$$

and

$$N^{0,0}N^{\alpha_1,\alpha_2}f = N^{0+\alpha_1,0+\alpha_2}f = N^{\alpha_1,\alpha_2}f$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. Then we obtain

$$N^{\alpha_1,\beta_1}N^{0,0} = N^{0,0}N^{\alpha_1,\alpha_2} = N^{\alpha_1,\alpha_2}$$

This implies that

$$N^{0,0} = 1 \in \{N^{\nu_1,\nu_2}\}$$

or $N^{0,0} = 1$ is an identity element of the set $\{N^{\nu_1\nu_2}\}$ for $f(z_1, z_2) \in \mathbb{F}$.

(iv) *Inverse Element*: If $\alpha_1, \alpha_2, -\alpha_1, -\alpha_2 \in \mathbb{R}$, then we have from (2.1) of index law.

(2.10) $$N^{\alpha_1,\alpha_2}N^{-\alpha_1,\alpha_2}f = N^{\alpha_1-\alpha_1,\alpha_2-\alpha_2}f \qquad (f \in \mathbb{F})$$
$$= N^{0,0}f = f$$

and

(2.11) $$N^{-\alpha_1,-\alpha_2}N^{\alpha_1,\alpha_2}f = N^{-\alpha_1+\alpha_1,+\alpha_2-\alpha_2}f$$
$$= N^{0,0}f = f$$

From (2.10) and (2.11), we have

(2.12) $$N^{-\alpha_1,-\alpha_2}N^{\alpha_1,\alpha_2} = N^{\alpha_1,\alpha_2}\ ,\ N^{-\alpha_1-\alpha_2} = N^{0,0} = 1$$

Moreover, let $(N^{-\alpha_1,-\alpha_2})^{-1}$ be the inverse element to N^{α_1,α_2}, we have then

(2.13) $$(N^{-\alpha_1,-\alpha_2})^{-1}N^{\alpha_1,\alpha_2} = N^{\alpha_1,\alpha_2}(N^{-\alpha_1,-\alpha_2})^{-1} = 1.$$

Therefore, we have

(2.13)′ $$(N^{\alpha_1,\alpha_2})^{-1} = N^{-\alpha_1,-\alpha_2}$$

from (2.12 and (2.13).

(v) *Commutative Law*: Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, then by index Law

(2.14) $$N^{\alpha_1,\alpha_2}N^{\beta_1,\beta_2}f = N^{\alpha_1+\beta_1,\alpha_2+\beta_2}f$$

and

(2.15) $$N^{\beta_1,\beta_2}N^{\alpha_1,\alpha_2}f = N^{\beta_1+\alpha_1,\beta_2\alpha_2}f$$

for $f \in \mathbb{F}$, then we have

$$N^{\alpha_1,\alpha_2}N^{\beta_1,\beta_2} = N^{\beta_1,\beta_2}N^{\alpha_1,\alpha_2} \tag{2.16}$$

from (2.14) and (2.15).

Hence $\{N^{\nu_1,\nu_k}\}$ satisfies commutative property for $f(z_1, z_2) \in \mathbb{F}$.

(vi) *Continuity of Index* ν_k: As $\nu_k \in \mathbb{R}$ with $-\infty < \nu_k < \infty$, $(k = 1, 2)$, so N^{ν_1,ν_2} has continuous index ν_k respectively if $f(z_1, z_2) \in \mathbb{F}$. Therefore $\{N^{\nu_1,\nu_2}\}$ is a set of $N^{\nu_1\nu_2}$ with continuous index ν_k.

From (i)$\sim$(vi) all properties have been satisfied by N^{ν_1,ν_2} under binary operation $*$.

So, the set $\{N^{\nu_1,\nu_2}\} = N^{\nu_1,\nu_2}; \nu_1, \nu_2 \in \mathbb{R}\}$ is an Abelian product group for the functions of two variables.

Theorem 5. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$*, then we have (Associative law for addition)*

$$(N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2}) + N^{\gamma_1,\gamma_2} = N^{\alpha_1,\alpha_2} + (N^{\beta_1,\beta_2} + N^{\gamma_1\gamma_2}) \tag{2.17}$$

for

$$f(z_1, z_2) = f \in \mathbb{F}.$$

Proof. For $f \in \mathbb{F}$, we have

$$\begin{aligned}((N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2}) + N^{\gamma_1,\gamma_2})f &= (N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2})f + N^{\gamma_1\gamma_2}f \\ &= (N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2} + N^{\gamma_1\gamma_2})f\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}(N^{\alpha_1,\alpha_2} + (N^{\beta_1,\beta_2} + N^{\gamma_1,\gamma_2}))f &= (N^{\alpha_1,\alpha_2}f + (N^{\beta_1,\beta_2} + N^{\gamma_1\gamma_2})f \\ &= (N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2} + N^{\gamma_1\gamma_2})f\end{aligned} \tag{2.19}$$

Therefore from (2.18) and (2.19), we have (2.17).

Theorem 6. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$*, then we have (Commutative law for addition)*

$$N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2} = N^{\beta_1,\beta_2} + N^{\alpha_1,\alpha_2} \tag{2.20}$$

for

$$f(z_1, z_2) = f \in \mathbb{F}.$$

Proof. As we have

$$(N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2})f = N^{\alpha_1,\alpha_2} f + N^{\beta_1,\beta_2} f \tag{2.21}$$

and

$$(N^{\beta_1,\beta_2} + N^{\alpha_1,\alpha_2})f = N^{\beta_1,\beta_2} f + N^{\alpha_1,\alpha_2} f \tag{2.22}$$

Therefore from (2.21) and (2.22), we get (2.20).

Theorem 7. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, and $f(z_1, z_2) = f \in \mathbb{F}$, then we have (Distributive law)*

$$\begin{aligned}(N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2}) \cdot N^{\gamma_1,\gamma_2} &= N^{\gamma_1,\gamma_2} \cdot (N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2}) \\ &= N^{\alpha_1+\gamma_1,\alpha_2+\gamma_2} + N^{\beta_1+\gamma_1,\beta_1+\gamma_2}\end{aligned} \tag{2.23}$$

Proof. We have

$$\begin{aligned}(N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2}) \cdot N^{\gamma_1,\gamma_2} f &= (N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2}) \cdot f_{\gamma_1,\gamma_2} \quad \text{(by (1.1))} \\ &= N^{\alpha_1,\alpha_2} f_{\gamma_1,\gamma_2} + N^{\beta_1,\beta_2} f_{\gamma_1,\gamma_2} \\ &= (f_{\gamma_1,\gamma_2})_{\alpha_1,\alpha_2} + (f_{\gamma_1,\gamma_2})_{\beta_1,\beta_2} \\ &= f_{\alpha_1+\gamma_1,\alpha_2+\gamma_2} + f_{\beta_1+\gamma_1,\beta_2+\gamma_2}\end{aligned} \tag{2.24}$$

from (2.1) of index law.

Next we have

$$\begin{aligned}N^{\gamma_1,\gamma_2} \cdot (N^{\alpha_1,\alpha_2} + N^{\beta_1,\beta_2})f &= N^{\alpha_1+\gamma_1,\alpha_2+\gamma_2} f + N^{\beta_1+\gamma_1,\beta_2+\gamma_2} f \\ &= f_{\alpha_1+\gamma_1,\alpha_2+\gamma_2} + f_{\beta_1+\gamma_1,\beta_2+\gamma_2}\end{aligned} \tag{2.25}$$

Similarly

$$\begin{aligned}(N^{\alpha_1+\gamma_1,\alpha_2+\gamma_2} + N^{\beta_1+\gamma_1,\beta_2+\gamma_2})f &= N^{\gamma_1+\alpha_1,\gamma_2+\alpha_2} f + N^{\gamma_1+\beta_1,\gamma_2+\beta_2} f \\ &= f_{\gamma_1+\alpha_1,\gamma_2+\alpha_2} + f_{\gamma_1+\beta_1,\gamma_2+\beta_2}\end{aligned} \tag{2.26}$$

Therefore we have (2.23) from (2.24), (2.25) and (2.26).

Theorem 8. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$*, then we have*

$$(N^{\alpha_1,\alpha_2} N^{\beta_1,\beta_2})^{-1} = N^{-\alpha_1,-\alpha_2} N^{-\beta_1,-\beta_2} \tag{2.27}$$

for

$$f(z_1, z_2) \in \mathbb{F}$$

Proof. Since we have

$$\begin{aligned}(N^{\alpha_1,\alpha_2} N^{\beta_1,\beta_2})^{-1} f &= (N^{\alpha_1+\beta_1,\alpha_2+\beta_2})^{-1} f \quad \text{(by index law)}\\ &= N^{-(\alpha_1+\beta_1),-(\alpha_2+\beta_2)} f \quad \text{(by (2.13)')}\\ &= N^{-\alpha_1-\beta_1,-\alpha_2-\beta_2} f = N^{-\alpha_1,-\alpha_2,-\beta_1,-\beta_2} f\end{aligned}$$

Using index law, we get (2.27).

Theorem 9. *Let* $n \in \mathbb{Z}^+$, $\nu_1, \nu_2 \in \mathbb{R}$ *and* $f \in \mathbb{F}$*, we have then*

$$\textit{(i)}\ (N^{\nu_1,\nu_2})^n = N^{n\nu_1,n\nu_2} \tag{2.28}$$

$$\textit{(ii)}\ \left((N^{\nu_1,\nu_2})^{-1}\right)^n = N^{-n\nu_1,-n\nu_2} \tag{2.29}$$

$$\textit{(iii)}\ (N^{n\nu_1,n\nu_2})^{-1} = (N^{-\nu_1,-\nu_2})^n \tag{2.30}$$

Proof. of (i): We have

$$\begin{aligned}(N^{\nu_1,\nu_2})^n f &= (N^{\nu_1,\nu_2})^{n-1}(N^{\nu_1,\nu_2} f) = (N^{\nu_1,\nu_2})^{n-1} f_{\nu_1,\nu_2} \quad \text{by (1.1))}\\ &= (N^{\nu_1,\nu_2})^{n-2}(N^{\nu_1,\nu_2} f_{\nu_1,\nu_2}) = (N^{\nu_1,\nu_2})^{n-2} f_{2\nu_1,2\nu_2}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &= (N^{\nu_1,\nu_2})^{n-n} f_{n\nu_1,n\nu_2}\\ &= f_{n\nu_1,n\nu_2} = N^{n\nu_1,n\nu_2} f\end{aligned} \tag{2.31}$$

Proof. of (ii): We have

$$(N^{\nu_1,\nu_2})^{-1} f = N^{-\nu_1,-\nu_2} f \qquad \text{(by 2.13)}$$

and

$$\begin{aligned}\left((N^{\nu_1,\nu_2})^{-1}\right)^n f &= (N^{-\nu_1,-\nu_2})^n f\\ &= N^{-n\nu_1,-n\nu_2} f \quad \text{(by 2.28))}\end{aligned}$$

Proof. of (iii): We have

$$(N^{n\nu_1,n\nu_2})^{-1} f = N^{-n\nu_1,-n\nu_2} f \qquad \text{(by (2.13))} \tag{2.32}$$

and

$$N^{-n\nu_1,-n\nu_2} f = (N^{-\nu_1,-\nu_2})^n f \qquad \text{(by (2.28))} \tag{2.33}$$

Therefore, from (2.32) and (2.33), we get (2.31).

Theorem 10. *Let* $f(z_1, z_2) \in \mathbb{F}$ *and* $N^{\nu_1,\nu_2>0} f = f_{\nu_1,\nu_2}, (\nu_1, \nu_2 > 0)$ *and* $N^{\nu_1,\nu_2<0} f = f_{\nu_1,\nu_2}, (\nu_1, \nu_2 < 0)$ *then the sets* $\{N^{\nu_1,\nu_2>0}\} = \{N_{\nu_1,\nu_2}; \nu_1, \nu_2 \in \mathbb{R}^+\}$ *and* $\{N^{\nu_1,\nu_2<0}\} = \{N_{\nu_1,\nu_2}; \nu_1, \nu_2 \in \mathbb{R}^-\}$ *are Abelian product semi groups.*

Proof. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 > 0$ in (i), (ii), (v) and (vi) of the proof of theorem 4. Since $\{N^{\nu_1,\nu_2>0}\}$ has no unit element i.e. $\{N^{0,0} = 1\}$ and inverse element $\{N^{-\nu_1,-\nu_2}\}$. So $\{N^{\nu_1,\nu_2>0}\}$ is an Abelian semi group with continuous $\nu_1, \nu_2 > 0$ for $f(z_1, z_2) \in \mathbb{F}$.

Similarly let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 < 0$ in (i), (ii) (v) and (vi) of the proof of theorem 4. Same as in $\{N^{\nu_1,\nu_2>0}\}$, $\{N^{\nu_1,\nu_2<0}\}$ has no unit element i.e. $\{N^{0,0} = 1\}$ and inverse element $\{N^{\nu_1,\nu_2}\}$. By satisfying the other properties $\{N^{\nu_1,\nu_2<0}\}$ is an Abelian semi group with continuous $\nu_1, \nu_2 < 0$ for $f(z_1, z_2) \in \mathbb{F}$.

3. Integral Transformation and Its Inverse Transformation.

Theorem 1. *Let Nishimoto's complex transformations, [4], be*

$$R\{f(\zeta)\} = \frac{\Gamma(\mu+1)}{2\pi i} \int_c \frac{f(\zeta)}{(\zeta - z)^{\mu+1}} \, d\zeta = F(z) \tag{3.1}$$

for a given constant $\mu \in \mathbb{R}$, *then inverse to* $F(z)$ *is given by*

$$R^{-1}\{F(\zeta)\} = \frac{\Gamma(-\mu+1)}{2\pi i} \int_c \frac{F(z)}{(z - \zeta)^{-\mu+1}} \, dz \tag{3.2}$$

Theorem 2. *Let complex integral transfomations of two variables i.e.* P*-transformation be defined as*

$$\begin{aligned} P\{f(\xi_1, \xi_2)\} &= \\ &= \frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{(2\pi i)^2} \int_{c_2}\int_{c_1} \frac{f(\xi_1, \xi_2)}{(\xi_1 - z_1)^{\mu_1+1}(\xi_2 - z_2)^{\mu_2+1}} \, d\xi_1 d\xi_2 \\ &= F(z_1, z_2) \end{aligned} \tag{3.3}$$

for $\mu_1, \mu_2 \in \mathbb{R}$, *then the inverse to* $F(z_1, z_2)$ *is given by*

$$\begin{aligned} P^{-1}\{F(z_1, z_2)\} &= \\ &= \frac{\Gamma(-\mu_1+1)\Gamma(-\mu_2+1)}{(2\pi i)^2} \int_{c_2}\int_{c_1} \frac{F(z_1, z_2)}{(z_1 - \xi_1)^{-\mu_1+1}(z_2 - \xi_2)^{-\mu_2+1}} \, dz_1 dz_2 \\ &= f(\xi_1, \xi_2) \end{aligned} \tag{3.4}$$

where $f(\xi_1, \xi_2)$ *is a regular functions in* $D_1 \times D_2$.

Proof. Substituting (3.3) in (3.4), we get

$$P^{-1}\{F(z_1, z_2)\} = \tag{3.5}$$

$$= \frac{\Gamma(-\mu_1+1)\Gamma(\mu_1+1)}{(2\pi i)^2}\int_{c_1}\int_{c_1}\frac{f(\eta_1,\eta_2)}{(\xi_1-z_1)^{-\mu_1+1}(\eta_1-z_1)^{\mu_1+1}}\,d\eta_1 dz_1$$

$$\cdot\frac{\Gamma(-\mu_2+1)\Gamma(\mu_2+1)}{(2\pi i)^2}\int_{c_2}\int_{c_2}\frac{d\eta_2\cdot dz_2}{(\xi_2-z_2)^{-\mu_2+1}(\eta_2-z_2)^{-\mu_2+1}}$$

Now we have from [1] (pp. 10),

$$\frac{\Gamma(-\mu_1+1)\Gamma(-\mu_1+1)}{(2\pi i)^2}\int_{c_1}\int_{c_1}\frac{f(\eta_1,\eta_2)}{(\xi_1-z_1)^{-\mu_1+1}(\eta_1-z_1)^{\mu_1+1}}\,d\eta_1 dz_1 = \tag{3.6}$$

$$= \frac{1}{2\pi i}\int_{c_1}\frac{f(\eta_1,\eta_2)}{(\eta_1-\xi_1)}\,d\eta_1$$

$$= f(\xi_1, \eta_2) \tag{3.7}$$

Hence we have

$$P^{-1}\{F(z_1, z_2)\} =$$

$$\frac{\Gamma(-\nu_2+1)\Gamma(\nu_2+1)}{(2\pi i)^2}\int_{c_2}\int_{c_2}\frac{f(\xi_1,\eta_2)}{(\xi_2-z_2)^{-\nu_2+1}(\eta_2-z_2)^{\nu_2+1}}\,d\eta_2 dz_2$$

from (3.5) and (3.7).

Now again using the result of [1] (pp. 10) i.e. (3.7) for ξ_2, we get

$$P^{-1}\{F(z_1, z_2)\} = \frac{1}{2\pi i}\int_{c_2}\frac{f(\xi_1,\eta_2)}{(\eta_2-\xi_2)}\,d\eta_2 = f(\xi_1, \xi_2)$$

Therefore

$$P^{-1}\{F(z_1, z_2)\} = f(\xi_1, \xi_2) \tag{3.8}$$

when

$$P\{f(\xi_1, \xi_2)\} = F(z_1, z_2) \tag{3.9}$$

for

$$0 \neq F(z_1, z_2)| < \infty.$$

Theorem 3. *If* $P\{f(\xi_1, \xi_2)\} = F(z_1, z_2)$ *and* $F(z_1, z_2) \neq 0$*, then*

$$PP^{-1} = P^{-1}P = 1 \tag{3.10}$$

Proof. We have

$$P\{f(\xi_1,\xi_2)\} = P\left\{P^{-1}\{F(z_1,z_2)\}\right\} = PP^{-1}\{F(z_1,z_2)\} \tag{3.11}$$

from (3.8).

Therefore, we have

$$PP^{-1} = 1 \tag{3.12}$$

from (3.11) and (3.9).

Next, we have

$$P^{-1}\{F(z_1,z_2)\} = P^{-1}\{P\{f(\xi_1,\xi_2)\}\} = P^{-1}P\{f(\xi_1,\xi_2)\} \tag{3.13}$$

from (3.9), hence

$$P^{-1}P = 1 \tag{3.14}$$

from (3.8) and (3.13).

Therefore, finally we have

$$PP^{-1} = P^{-1}P = 1$$

from (3.14) and (3.12).

Theorem 4. *If* $\alpha \neq 0$, $z_1, z_2 \in \mathbb{C}$ *and* $\mu_1, \mu_2 \in \mathbb{R}$ *then*

$$\textit{(i)} \quad P\{e^{-\alpha(\xi_1+\xi_2)}\} = e^{-i\pi(\mu_1+\mu_2)}a^{\mu_1+\mu_2} \cdot e^{-\alpha(z_1+z_2)} \tag{3.15}$$

$$\textit{(ii)} \quad P^{-1}\{e^{-i\pi(\mu_1+\mu_2)}a^{\mu_1+\mu_2} \cdot e^{-\alpha(z_1+z_2)}\} = e^{-\alpha(\xi_1+\xi_2)} \tag{3.16}$$

Proof. of (i): Letting $f(\xi_1,\xi_2) = e^{-\alpha(\xi_1+\xi_2)}$ in (3.3), we have

$$P\{e^{-\alpha(\xi_1+\xi_2)}\} =$$

$$= \frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{(2\pi i)^2}\int_{c_2}\int_{c_1}\frac{e^{-\alpha\xi_1}\cdot e^{-\alpha\xi_2}}{(\xi_1-z_1)^{\mu_1+1}(\xi_2-z_2)^{\mu_2+1}}d\xi_1 d\xi_2$$

$$= R\{e^{-\alpha\xi_1}\}\cdot R\{e^{-\alpha\xi_2}\} \qquad \text{(using (3.1) [4])}$$

$$= e^{-i\pi\mu_1} a^{\mu_1} e^{-\alpha z_1} \cdot e^{-i\pi\mu_2} a^{\mu_2} e^{-\alpha z_2}$$

and so get (3.15).

Proof. *of* (ii): Letting $F(z_1, z_2) = e^{-i\pi(\mu_1+\mu_2)} a^{\mu_1+\mu_2} e^{-\alpha(z_1+z_2)}$ in (3.4), we have

$$P^{-1}\big\{F(z_1, z_2)\big\} = P^{-1}\big\{e^{-i\pi(\mu_1+\mu_2)} \cdot a^{\mu_1+\mu_2} \cdot e^{-\alpha(z_1+z_2)}\big\} \tag{3.17}$$

$$= e^{-i\pi(\mu_1+\mu_2)} \cdot a^{\mu_1+\mu_2} \cdot \frac{\Gamma(-\mu_1+1)\Gamma(-\mu_2+1)}{(2\pi i)^2} \cdot$$

$$\cdot \int_{c_1} \int_{c_2} \frac{e^{-\alpha z_1} \cdot e^{-\alpha z_2}}{(z_1-\xi_1)^{-\mu_1+1}(z_2-\xi_2)^{-\mu_2+1}}\, dz_1 dz_2$$

$$= e^{-i\pi(\mu_1+\mu_2)} \cdot a^{\mu_1+\mu_2} \cdot \frac{\Gamma(-\mu_1+1)}{(2\pi i)} \int_{c_1} \frac{e^{-\alpha z_1}}{(z_1-\xi_1)^{-\mu_1+1}}\, dz_1$$

$$\cdot \frac{\Gamma(-\mu_2+1)}{(2\pi i)} \int_{c_2} \frac{e^{-\alpha z_2}}{(z_2-\xi_2)^{-\mu_2+1}}\, dz_2$$

$$= e^{-i\pi(\mu_1+\mu_2)} a^{\mu_1+\mu_2} R^{-1}\{e^{-\alpha z_1}\} \cdot R^{-1}\{e^{-\alpha z_2}\} \qquad \text{(by (3.2) [4])}$$

$$= R^{-1}\{e^{-i\pi\mu_1} a^{\mu_1} e^{-\alpha z_1}\} \cdot R^{-1}\{e^{-i\pi\mu_2} a^{\mu_2} e^{-\alpha z_2}\}$$

$$= e^{-\alpha\xi_1} \cdot e^{-\alpha\xi_2} \quad = e^{-\alpha(\xi_1+\xi_2)}$$

Theorem 5. *If* $a \neq 0$, $z_1, z_2 \in \mathbb{C}$ *and* $\mu_1\mu_2 \in \mathbb{R}$, *then*

$$(i) \quad P\{e^{\alpha(\xi_1+\xi_2)}\} = a^{\mu_1+\mu_2} \cdot e^{\alpha(z_1+z_2)} \tag{3.18}$$

$$(ii) \quad P^{-1}\{a^{\mu_1+\mu_2} \cdot e^{\alpha(z_1+z_2)}\} = e^{\alpha(\xi_1+\xi_2)} \tag{3.19}$$

Proof. *of* (i): Letting $f(\xi_1, \xi_2) = e^{\alpha(\xi_1+\xi_2)}$ in (3.3), we get

$$P\big\{f(\xi_1, \xi_2)\big\} = P\big\{e^{\alpha(\xi_1+\xi_2)}\big\} = \frac{\Gamma(\mu_1+1)\Gamma(\mu_2+1)}{(2\pi i)^2}$$

$$\cdot \int_{c_2} \int_{c_1} \frac{e^{\alpha\xi_1} \cdot e^{\alpha\xi_2}}{(\xi_1-z_1)^{\mu_1+1}(\xi_2-z_2)^{\mu_2+1}} d\xi_1 d\xi_2 \qquad \text{using (3.1))}$$

$$= R\big\{e^{\alpha\xi_1}\big\} \cdot R\big\{e^{\alpha\xi_2}\big\}$$

Using the result in [4], we obtain (3.18).

Proof. *of* (ii): $F(z_1, z_2) = a^{\mu_1+\mu_2} e^{\alpha(z_1+z_2)}$ in (3.4), we have

$$P^{-1}\{F(z_1, z_2)\} = P^{-1}\{a^{\mu_1+\mu_2} \cdot e^{\alpha(z_1+z_2)}\}$$

By using (3.2) and the result in [4], we get (3.19).

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