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BOUNDARY REGULARITY PROBLEMS FOR SOME ELLIPTIC-PARABOLIC EQUATIONS

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In this note we review some recent results in [64, 95, 96] concerning necessary and sufficient conditions for the regularity of boundary points relatively to the Dirichlet problem for linear degenerate-parabolic operators with well-behaved fundamental solutions. The main focus is on Wiener-type criteria for a class of operators whose degeneracy is controlled by Hörmander vector fields.

1. Introduction, and an overview of the related literature

The question whether the Perron-Wiener solution of the Dirichlet problem for the Laplace operator attains its boundary datum continuously at a particular boundary point is, obviously, very classical. The points where the boundary datum is attained are called regular in classical potential theory. The regular points were characterized by Wiener in [100, 101], who proved that a boundary point x_0 of any bounded open set $\Omega \subset \mathbb{R}^N$ is regular for the Laplace operator Δ if and only if, for fixed $\mu \in (0,1)$, we have

$$\sum_{k=1}^{\infty} \frac{\operatorname{cap}\left(\left(\overline{B}(x_0,\mu^k) \smallsetminus B(x_0,\mu^{k+1})\right) \smallsetminus \Omega\right)}{\mu^{k(N-2)}} = +\infty, \tag{1}$$

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where $\mathrm{cap}(E)$ indicates the Newtonian capacity of a set E and $B(x_0,r)$ denotes the ball centered at x_0 of radius $r>0$. The Wiener criterion (1) has been stated and proved in several ways in [17, 23, 33, 52, 53]. Furthermore, many generalizations to a wide range of elliptic operators have been obtained after the work by Wiener. We mention here the results in [47, 81, 84, 91, 92] where it was proved that the condition (1) is necessary and sufficient also for the regularity of boundary points related to uniformly elliptic operators with variable coefficients with various degree of regularity. In this direction, the most important contribution is due to Littman, Stampacchia and Weinberger who proved in [67] that, also for the case of divergence-form uniformly elliptic operators with bounded measurable coefficients, the boundary regularity is characterized by the Wiener condition (1). In other words, a large class of uniformly elliptic operators share the same notion of *regularity* with the Laplace operator. After [67] it is known that this is strongly related with the fact that the Green kernel of the operator under consideration behaves near the diagonal as $|x-y|^{2-N}$. For similar reasons it is also known that, for nondivergence-form operators with no control on the modulus of continuity of the matrix coefficients, the notion of regularity fails to be characterized by condition (1) (see [74], and also [58] for recent developments). The proper characterization of the regularity for uniformly elliptic operators in nondivergence-form with continuous coefficients has been obtained in [8] through the study of normalized adjoint solutions (we refer the reader to [19, 54] for related discussions for the case of bounded measurable coefficients). Wiener tests adapted to several classes of linear degenerate-elliptic operators are present in the literature. The first instance is probably the one in [30] where the degeneracy of the ellipticity is controlled by Muckenhoupt weights. Other type of operators and degeneracies were considered, e.g., in [9, 14, 46, 48, 78, 79, 94].

Let us turn our attention on the case of parabolic operators, for which the characterization of the regularity of boundary point appears to be more delicate. As a matter of fact, using the classical Petrowski's regularity criterion in [82] (see also [27, Theorem 8.1]) one can find an open set $\Omega\subset\mathbb{R}^{N+1}$ and a point $z_0\in\partial\Omega$ which is regular for $\alpha_1\Delta-\partial_t$ and not regular for $\alpha_2\Delta-\partial_t$ (if $\alpha_1>\alpha_2$). This represents a striking difference with the elliptic case: one way to see this is to recall that the two explicit Gaussian-type fundamental solutions for $\alpha_1\Delta-\partial_t$ and $\alpha_2\Delta-\partial_t$ have different singularities on the diagonal (just a one-side comparison holds if $\alpha_1\neq\alpha_2$).
The first $(\Delta-\partial_t)$-regularity criterion involving heat-capacity and the level-rings of the fundamental solution is, to the best of our knowledge, due to Pini in [83]. Pini's result deals with special open sets Ω with continuous boundary, and it provides a sufficient regularity criterion for the heat equation in spatial dimen-

sion $N = 1$. Concerning the true analogue of the Wiener test (1) for the heat operator, the necessary condition was proved by Lanconelli in [60] and the sufficient condition by Evans and Gariepy in [28]. We can summarize the results in [28, 60] as follows: let Ω be a bounded open subset of $\mathbb{R}^{N+1}$, let $z_0 \in \partial\Omega$ and $\lambda \in (0,1)$, then

$$z_0 = (x_0,t_0) \text{ is } (\Delta - \partial_t)\text{-regular} \quad \text{if and only if} \quad \sum_{k=1}^{+\infty} \frac{\mathrm{cap}_G\left(\Omega_k'(z_0)\right)}{\lambda^k} = +\infty, \quad (2)$$

where

$$\Omega_k'(z_0) = \left\{ z = (x,t) \in \mathbb{R}^{N+1} \setminus \Omega \, : \, \lambda^{-k} \le G^\Delta(z_0,z) \le \lambda^{-(k+1)} \right\}, \quad \text{for } k \in \mathbb{N},$$

with $G^\Delta(\cdot,\cdot)$ being the fundamental solution of $\Delta - \partial_t$, i.e.

$$G^\Delta(x,t,\xi,\tau) = \frac{(4\pi)^{-\frac{N}{2}}}{(t-\tau)^{\frac{N}{2}}} \exp\left(-\frac{|x-\xi|^2}{4(t-\tau)} \right), \qquad \text{if } t > \tau \text{ (and 0 otherwise)},$$

and where cap_G indicates the capacity related to the Gauss-Weierstrass kernel G^Δ (see Section 2 for the definition). Because of the presence of Petrowski's counterexamples the notion of regularity cannot be independent of the choice of the parabolic operator, and this fact makes more interesting the analysis of Wiener criteria for variable coefficients parabolic operators. The case of uniformly parabolic operators in divergence form with smooth coefficients was settled by Garofalo and Lanconelli in [39]. In their result the dependence of the Wiener condition on the operator H under consideration is evident in the capacitary terms: compared with the one in (2), in [39] they defined the sequence of compact sets $\Omega_k'(z_0)$ through the difference of superlevel sets of the fundamental solution Γ of the operator H and they also dealt with the capacity cap_Γ related to the kernel Γ. Subsequently, in [29], Fabes, Garofalo and Lanconelli obtained Wiener criteria for uniformly parabolic operators in divergence form with C^1-Dini continuous coefficients. We also mention the works [10, 37] for boundary regularity conditions related to weak solutions to parabolic equations. Before the appearance of the Evans-Gariepy Wiener criterion, a different characterization of the $(\Delta - \partial_t)$-regularity had been proved by Landis in [66]. Landis's criterion involves a suitable series of caloric potentials, and it can be read as follows: let Ω be a bounded open subset of $\mathbb{R}^{N+1}$ and let $z_0 \in \partial\Omega$, then

$$z_0 = (x_0,t_0) \text{ is } (\Delta - \partial_t)\text{-regular} \quad \text{if and only if} \quad \sum_{k=1}^{+\infty} G^\Delta * \mu_k(z_0) = +\infty, \quad (3)$$

where, for any fixed $k \in \mathbb{N}$, $G^\Delta * \mu_k$ denotes the convolution of the Gauss-Weierstrass kernel G^Δ with the equilibrium measure of the compact set $\{z =$

$(x,t) \in \mathbb{R}^{N+1} \smallsetminus \Omega : \rho_k \leq G^{\Delta}(z_0, z) \leq \rho_{k+1}\} \cup \{z_0\}$ being $\{\rho_k\}_{k\in\mathbb{N}}$ a certain sequence of positive real numbers such that $\frac{\rho_{k+1}}{\rho_k}$ grows fast at ∞. The approach and the techniques developed by Landis for Wiener-type series have been carried out for uniformly parabolic operators in nondivergence form with coefficients with various degree of regularity (Hölder continuity [66], Dini continuity [69, 80], coefficients in special form around the boundary point [73], bounded coefficients in domains with peculiar symmetries [70]). Related investigations are the ones for the large-time stabilization of solutions to parabolic equations with rough coefficients in infinite cylinders [3, 24].
Finally, we point out the investigations developed by Lanconelli in [61–63], where he considered the behavior of Wiener-type series like

$$\sum_{h,k=1}^{+\infty} \frac{\mathrm{cap}_G\left(\Omega_k^h(z_0,\lambda)\right)}{\lambda^{\frac{kN}{2}}} \lambda^{\alpha h} \tag{4}$$

(where $\Omega_k^h(z_0,\lambda)$ can be defined as in (12) below), in order to obtain necessary and sufficient conditions for the regularity of a boundary point which are valid for a whole class of uniformly parabolic operators. For example, see [61, Teorema A and Teorema B] and [63], one can deduce that if the series in (4) is divergent then $z_0 \in \partial\Omega$ is $\left(\frac{1}{4\beta}\Delta - \partial_t\right)$-regular whenever $0 < \beta < \alpha$ and, viceversa, if z_0 is $\left(\frac{1}{4\beta}\Delta - \partial_t\right)$-regular then the series (4) is divergent for every $\alpha \leq \beta$.
The main focus of this note is to present some results contained in [64, 95, 96] concerning Wiener-type tests for degenerate-parabolic operators. The list of papers in the literature dealing with this topic is, as far as we know, not very long. Wiener characterizations as in (2) were proved for the heat-counterpart of the subLaplacian in the Heisenberg group [41] and, more recently, in H-type groups [85]. Landis-type characterizations as in (3) were obtained in [57, 89] for Kolmogorov-type operators (and very recently in [96]). Wiener-type series as (4) were considered in [64, 95].

Before presenting in details the results [64, 95, 96], we want to discuss which is the type of degeneracy in the operators we would like to analyze and why we feel it is interesting to investigate such class of equations. Consider then a system of real C^∞-smooth vector fields $X_1, X_2, \ldots, X_p$ in $\mathbb{R}^N$ satisfying the so-called Hörmander's condition, i.e. such that

$$\mathrm{rank}\,\mathrm{Lie}\{X_1,\ldots,X_p\}(x) = N \text{ at every point } x.$$

We want to study a class of degenerate equations whose degeneracy is controlled by such vector fields. More precisely, we can think of $X_1(x),\ldots,\ldots,X_p(x)$ as the directions of ellipticity of the operator at any point x and the directions of missing ellipticity (which exist if $p < N$) can be recovered via Lie-brackets commutations. The prototype operators are $\sum_i X_i^2$ and $\sum_i X_i^2 - \partial_t$. We mention the monographs and surveys [2, 11, 14, 18, 38, 43, 75] where these type of equations and their underlying geometry are explained under different perspectives. These operators arise in fact in many different settings, both theoretical and applied. Among others, these Hörmander-type equations have a central role in geometric analysis problems appearing in CR-geometry and sub-Riemannian settings, see e.g. [7, 20, 21, 35, 36, 50, 68, 72]. Speaking about regularity issues, after some pioneering works [31, 49, 51, 59, 87] treating regularity in Hölder classes and estimates for the fundamental kernels, there has been a huge amount of regularity results for solutions to linear and nonlinear equations with this type of degeneracies (with no aim of completeness, we mention the following works about the interior regularity [1, 5, 15, 25, 26, 44, 45, 76, 88, 93, 98] and the boundary behavior of solutions [6, 32, 65, 71, 77]). Wiener-type tests for linear degenerate-elliptic equations (the stationary case) with underlying sub-Riemannian structures appear to be well-settled in the literature mainly thanks to upper and lower bounds for the Green kernels near the pole (we refer the reader to the papers [46, 48, 78, 94]). In the parabolic case, the difficulties rely on the fact that any point z_0 belongs to the boundary of every superlevel set $\{z : \Gamma(z_0, z) > r\}$ and a very precise information on the kernel Γ is needed. In this respect, we highlight some technical details in the proof in [39] of the Wiener criterion for the case of smooth uniformly parabolic operators in divergence form: in that situation the fundamental solution Γ is not explicit and they were able to make use of a refined Gaussian expansion of Γ in terms of the underlying geodesic Riemannian distance (see [40]). A sub-Riemannian analogue of this noteworthy expansion seems to be currently not available. In the above mentioned papers [41, 85] the authors treated special operators in the form $\sum_i X_i^2 - \partial_t$ for which the fundamental solution is explicit up to Fourier transform in one distinguished variable. The lack of explicit formulas or refined estimates for the fundamental solution is in fact one of the major obstacles to obtain Wiener-type tests for the class of equations under discussion. The results we are presenting deal with necessary and sufficient conditions of type (3) and (4) for the regularity of boundary points related to the the class of Hörmander-operators

$$\sum_{i=1}^{p} X_i^2 - \partial_t, \quad \text{or more generally} \tag{5}$$

$$\sum_{i,j=1}^{p} a_{i,j}(x,t) X_i X_j + \sum_{k=1}^{p} a_k(x,t) X_k - \partial_t \tag{6}$$

with $(a_{i,j})_{i,j=1,\dots,p}$ symmetric, uniformly positive definite, and with smoothness properties for the coefficients. The extension of the Wiener criterion (2) to equations as (5) represents currently an open, and seemingly difficult, problem.

The note is organized as follows. In Section 2 we recall few classical notions of potential theory. In Section 3 we discuss the approaches in [64, 96] for establishing Wiener-type tests respectively of type (4) and (3) starting from 'rough' short-time upper and lower Gaussian bounds for the fundamental solution of hypoelliptic operators. Such Gaussians are built with respect to a distance function satisfying doubling conditions and segment properties. Two-sided Gaussian bounds for parabolic operators have been studied in very general settings (we recall [4, 42, 90, 99]), and for the class in (5)-(6) we refer the reader to the results in [12, 13, 16, 51, 86] for Gaussian bounds with respect to the Carnot-Caratheodory distance related to the vector fields. Thus, the results in Section 3 apply to the operators (6) as long as the coefficients $a_{i,j}(\cdot)$ and $a_k(\cdot)$ are C^∞-smooth functions. In Section 4 we deal with the non-hypoelliptic case of coefficients $a_{i,j}(\cdot), a_k(\cdot)$ being Hölder continuous with respect to the Carnot-Caratheodory distance, and we discuss the regularity criteria found in [95] which refer to the behavior of Wiener-series of type (4).

2. Potential theory recalling

In what follows, we always deal with second order linear operators $\mathcal{H}$ endowing a strip $S \subseteq \mathbb{R}^{N+1}$ with a structure of β-harmonic space satisfying the Doob convergence property. It is out of the purposes of this note to describe the classical axiomatization which characterizes these spaces. We refer the interested reader to the book [22] (see also [14, 56, 65, 95]).
It is important for our exposition that, being in such a β-harmonic space, for any bounded open set Ω with $\overline{\Omega} \subseteq S$ and for every continuous function $\varphi : \partial\Omega \to \mathbb{R}$, the Dirichlet problem

$$\begin{cases} \mathcal{H}u = 0 \text{ in } \Omega, \\ u|_{\partial\Omega} = \varphi \end{cases}$$

has a generalized solution H_φ^Ω in the Perron-Wiener sense. Thus, we can recall the notion of $\mathcal{H}$-regularity of a boundary point.

Definition 2.1. Let Ω be a bounded open set with $\overline{\Omega} \subseteq S$. A point $z_0 \in \partial\Omega$ is called $\mathcal{H}$-regular if $\lim_{z\to z_0} H_\varphi^\Omega(z) = \varphi(z_0)$ for every $\varphi \in C(\partial\Omega)$.

As we already recalled the classical results, we know that Wiener-type criteria are tests to prove or disprove the $\mathcal{H}$-regularity of a boundary point by checking whether a suitable series is divergent or convergent. Such series has to involve capacitary terms 'measuring' the behavior of $\partial\Omega$ around the point. The

definition of the relevant capacity has to change according to the operator $\mathcal{H}$ we are dealing with. We recall here the definition of capacity with respect to a kernel K as given e.g. by Fuglede in [34].
Let $K : S \times S \to [0,+\infty]$ be a lower semicontinuous function, and assume that $K(\cdot,\zeta) \not\equiv 0$ for any fixed $\zeta \in S$. Given a compact set $F \subseteq S$, we denote by $\mathcal{M}^+(F)$ the set of nonnegative Radon measures supported on F. We can define

$$\mathcal{C}_K(F) = \sup\Big\{ \mu(F) \quad : \quad \mu \in \mathcal{M}^+(F), \text{ and} \qquad\qquad (7)$$
$$K*\mu(z) = \int K(z,\zeta)d\mu(\zeta) \le 1 \ \ \forall z \in S \Big\}.$$

For every F compact subset of S there exists $\mu \in \mathcal{M}^+(F)$ with $K*\mu \le 1$ in S such that $\mu(F) = \mathcal{C}_K(F)$. The measure μ_K is called K-equilibrium measure of F.

Various classical characterizations of the regularity of boundary points are given in terms of balayages. Let us recall the definition for the reader's convenience.
For a given a compact set $F \subset S$, we denote $W_F = \inf\{v \in \overline{\mathrm{H}}(S) : v \ge 0 \text{ in } S,\ v \ge 1 \text{ in } F\}$ where $\overline{\mathrm{H}}(S)$ is the set of $\mathcal{H}$-superharmonic functions in S. Then we can define the $\mathcal{H}$-*balayage* potential of F as

$$V_F(z) = \liminf_{\zeta \to z} W_F(\zeta), \qquad z \in S.$$

If $\mathcal{H}$ is the operator with smooth coefficients described in Section 3, we can also talk about the Riesz-measure associated to V_F. We denote it by μ_F, and it can be defined as the nonnegative Radon measure satisfying $-\mathcal{H}V_F = \mu_F$ in $\mathcal{D}'$. For any compact set $F \subset S$, we can then define the capacity $\mathrm{cap}_{\mathcal{H}}$ of F as

$$\mathrm{cap}_{\mathcal{H}}(F) = \mu_F(F).$$

We refer the reader to [64, Section 2] for comparisons between these different notions of capacities. Since our primary interest concerns the case of evolution operators, it is worth mentioning the case of the pure Gaussian kernel, i.e. when $K(\cdot,\cdot)$ is a Gauss-Weierstrass function or (better, if you want) the fundamental solution of $\alpha\Delta - \partial_t$: for such a special choice, it was proved in [61, Proposizione 2] that Gaussian kernels with different α are equivalent on the class of compact subsets of $\mathbb{R}^{N+1}$.

3. Regularity characterizations under Gaussian-type estimates

Let us consider the second order linear operators in the following form

$$\mathcal{H} = \sum_{i,j=1}^{N} q_{i,j}(z)\partial^2_{x_i,x_j} + \sum_{k=1}^{N} q_k(z)\partial_{x_k} - \partial_t, \tag{8}$$

in the strip $S = \{z = (x,t) \in \mathbb{R}^{N+1} : x \in \mathbb{R}^N,\ T_1 < t < T_2\}$, with $-\infty \leq T_1 < T_2 \leq \infty$. We assume the coefficients $q_{i,j} = q_{j,i}, q_k$ of class C^∞, and the quadratic form relative to $(q_{i,j}(z))_{i,j}$ nonnegative definite and not totally degenerate. We also assume the hypoellipticity of $\mathcal{H}$ and of its adjoint $\mathcal{H}^*$, and the existence of a global fundamental solution $\Gamma(z,\zeta)$ smooth out of the diagonal of $S \times S$ satisfying the following:

(i) $\Gamma(\cdot,\zeta) \in L^1_{\mathrm{loc}}(S)$ and $\mathcal{H}(\Gamma(\cdot,\zeta)) = -\delta_\zeta$, the Dirac measure at $\{\zeta\}$, for every $\zeta \in S$; $\Gamma(z,\cdot) \in L^1_{\mathrm{loc}}(S)$ and $\mathcal{H}^*(\Gamma(z,\cdot)) = -\delta_z$ for every $z \in S$;

(ii) for every compactly supported continuous function φ on $\mathbb{R}^N$ and for every $x_0 \in \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} \Gamma(x,t,\xi,\tau)\,\varphi(\xi)\,\mathrm{d}\xi \to \varphi(x_0)$$

as $x \to x_0$, $t \searrow \tau \in]T_1,T_2[$ and also as $x \to x_0$, $\tau \nearrow t \in]T_1,T_2[$;

(iii) there exist constants $0 < a_0 \leq b_0$ and $\Lambda \geq 1$ such that the following Gaussian estimates hold

$$\frac{1}{\Lambda}\mathrm{G}^{(d)}_{b_0}(z,\zeta) \leq \Gamma(z,\zeta) \leq \Lambda \mathrm{G}^{(d)}_{a_0}(z,\zeta), \quad \forall z,\zeta \in S, \tag{9}$$

for some distance d in $\mathbb{R}^N$ for which $(\mathbb{R}^N, d)$ is a complete metric space topologically equivalent to the Euclidean space, such that for every fixed $x \in \mathbb{R}^N$ we have $d(x,\xi) \to \infty$ if and only if $|\xi| \to \infty$, and satisfies the global doubling condition and the segment property.

For any $a > 0$, we have just denoted by $\mathrm{G}^{(d)}_a$ the function

$$\mathrm{G}^{(d)}_a(z,\zeta) = \mathrm{G}^{(d)}_a(x,t,\xi,\tau) = \frac{1}{|B(x,\sqrt{t-\tau})|}\exp\left(-a\frac{d(x,\xi)^2}{t-\tau}\right)$$

if $t > \tau$, and 0 where $\{t \leq \tau\}$. Here $|A|$ stands for the Lebesgue measure of A and $B(x,r)$ denotes the d-ball of center x and radius $r > 0$. We also make use of the notation $\hat{d}$ for the 'parabolic distance' defined by the formula $\hat{d}(z,\zeta) = \left(d^4(x,\xi) + (t-\tau)^2\right)^{\frac{1}{4}}$.

As a consequence of the results in [65, Theorem 3.9], the operator $\mathcal{H}$ endows the strip S with a structure of β-harmonic space satisfying the Doob convergence property. As recalled in Section 2, we can thus talk about the $\mathcal{H}$-regularity of boundary points in the sense of Definition 2.1.
Under the above assumptions we have recently proved in [96] a Wiener-type characterization of boundary regularity in the spirit of the classical result by Landis [66]. Our condition is expressed in terms of a series of caloric Riesz potentials or equivalently a series of balayages. If $z_0 \in \partial\Omega$ and $\lambda \in]0,1[$ are fixed, we define for $k \in \mathbb{N}$

$$\Omega_k^c(z_0) = \left\{ z \in S \setminus \Omega : \left(\frac{1}{\lambda}\right)^{k \log k} \leq \Gamma(z_0, z) \leq \left(\frac{1}{\lambda}\right)^{(k+1)\log(k+1)} \right\} \cup \{z_0\}. \quad (10)$$

We also denote by $\mu_{\Omega_k^c(z_0)}$ the Riesz-measure associated to the compact set $\Omega_k^c(z_0)$.

Theorem 3.1. [96, Theorem 1.3] *Let Ω be a bounded open set with $\overline{\Omega} \subseteq S$, and let $z_0 \in \partial\Omega$. Then*

$$z_0 \text{ is } \mathcal{H}\text{-regular for } \partial\Omega \qquad \Longleftrightarrow \qquad \sum_{k=1}^{\infty} \Gamma * \mu_{\Omega_k^c(z_0)}(z_0) = +\infty.$$

We would like to make a couple of comments about Theorem 3.1. First, we want to stress the importance of the choice of the exponent $\alpha(k) = k \log k$ in the definition (10) of $\Omega_k^c(z_0)$. The superlinear growth of $\alpha(k)$ is in fact crucial for the proof in [96]. On the other hand, the exact analogue of the Evans-Gariepy criterion in (2) would have required the sequence of level sets with $\alpha(k) = k$ (see [96, Corollary 4.1]). This is why Theorem 3.1 is a Landis-type criterion (recalling (3)). The proof of Theorem 3.1 in [96] is not performed using the strategy of Landis. It is used instead the strategy developed in [57] for a class of homogeneous ultraparabolic equations of Kolmogorov type. In [57] it appears the same choice $\alpha(k) = k \log k$ as in Theorem 3.1. We feel it is interesting to remark that we can get the same accuracy in Theorem 3.1 knowing only Gaussian bounds from above and from below (different from each other) for the fundamental solution (and not an explicit expression as in [57]). Moreover, let us comment further on some technical details of the proof given in [96]. A crucial feature in the proof strategy is to choose appropriately subregions of $\Omega_k^c(z_0)$ where one can estimate uniformly the ratio $\frac{\Gamma(z,\zeta)}{\Gamma(z_0,\zeta)}$ via Hölder-type estimates. A delicate point in pursuing this strategy is the identification of the balayages V_F with their Riesz representatives $\Gamma * \mu_F$. In [96, Theorem 2.1] we approached such a Riesz representation theorem by making use of mean value formulas, even if the kernel may change sign, and we showed that

$$V_F(z) = \Gamma * \mu_F(z) \quad \text{for } every\ z \in S. \quad (11)$$

As corollaries of Theorem 3.1, we can deduce sufficient conditions for the regularity which can be concretely applied. For example, for the model case of heat operators in Carnot groups, we have proved in [96, Corollary 4.4] that a sharp geometric criterion like the existence of a suitable *(log log)-paraboloid* with vertex in $z_0 \in \partial\Omega$ contained in the exterior of Ω ensures the regularity of z_0. Such condition, in the case of the classical heat equation, is known to be optimal (see the classical counterexamples in [82], as well as the discussions in [27, Section 7]). We refer the interested reader to [96, Section 4] for the precise definitions, further details, and the complete proofs.

Let us now turn our attention to Wiener-type criteria which are suitable *at the same time* for the whole class of operators (8) we are considering. Keeping in mind the discussions in the Introduction, we cannot hope for a condition which is necessary and sufficient for every operator in our class. The sufficient condition and the necessary condition for the regularity we are now going to present are in fact different from each other (in the same spirit as (4)). On the other hand, all the quantities and the objects appearing in our Wiener series depend on the underlying metric d appearing in the Gaussian bounds for Γ, and not on the fundamental solution of the specific operator. We have to introduce the following notation. If $z_0 \in \partial\Omega$ and $\lambda \in]0,1[$ are fixed, we define or $h,k \in \mathbb{N}$

$$\begin{aligned}\Omega_k^h(z_0,\lambda) \;\; = \;\; &\Big\{ \zeta = (\xi,\tau) \in S \smallsetminus \Omega \,:\, \lambda^{k+1} \le t_0 - \tau \le \lambda^k, \\ &\left(\frac{1}{\lambda}\right)^{h-1} \le \exp\left(\frac{d^2(x_0,\xi)}{t_0-\tau}\right) \le \left(\frac{1}{\lambda}\right)^h,\; \hat{d}(z_0,\zeta) \le \sqrt{\lambda} \Big\}.\end{aligned} \tag{12}$$

Moreove, for any $a > 0$, we denote by $\mathcal{C}_a$ the capacity with respect to the Gaussian kernel $\mathrm{G}_a^{(d)}$ according to the definition (7).

Theorem 3.2. [64, Theorem 1.1] *Let Ω be a bounded open set with $\overline{\Omega} \subseteq S$, and let $z_0 \in \partial\Omega$.*

(i) If there exists $0 < a \le a_0$ and $b > b_0$ such that

$$\sum_{h,k=1}^{+\infty} \frac{\mathcal{C}_a\left(\Omega_k^h(z_0,\lambda)\right)}{\left|B\left(x_0,\sqrt{\lambda^k}\right)\right|} \lambda^{bh} = +\infty \tag{13}$$

then the point z_0 is $\mathcal{H}$-regular.

(ii) If the point z_0 is $\mathcal{H}$-regular, then

$$\sum_{h,k=1}^{+\infty} \frac{\mathcal{C}_b\left(\Omega_k^h(z_0,\lambda)\right)}{\left|B\left(x_0,\sqrt{\lambda^k}\right)\right|} \lambda^{ah} = +\infty \tag{14}$$

for every $b \geq b_0$ *and* $0 < a \leq a_0$.

The proof of the sufficient condition (i) is obtained in [64] by estimating the quantity $|H^\Omega_\varphi(z) - \varphi(z_0)|$ with a suitable Wiener-type series resembling the one in (13). We can also derive an integral estimate involving the Lebesgue measures of the following sections

$$E_\lambda(\rho,\tau) = \Big\{ x \in \mathbb{R}^N : z = (x,\tau) \in S \setminus \Omega,\ \hat{d}(z_0,z) \leq \sqrt{\lambda},\ \exp\left(\frac{d^2(x_0,x)}{t_0-\tau}\right) \leq \rho \Big\}.$$

As a matter of fact, it is proved in [64, Theorem 1.3] that the point $z_0 \in \partial\Omega$ is $\mathcal{H}$-regular if

$$\int_0^\lambda \int_1^{+\infty} \frac{|E_\lambda(\rho,t_0-\eta)|}{\left|B(x_0,\sqrt{\eta})\right|} \frac{d\rho}{\rho^{1+b}} \frac{d\eta}{\eta} = +\infty$$

for some $b > b_0$.
A concrete sufficient condition for the regularity of $z_0 = (x_0,t_0) \in \partial\Omega$ is provided by the following cone-type condition at z_0 (which resembles the classical *tusk condition* in [27]; see also the recent developments and the references in [55]): there exist $M_0, r_0, \theta > 0$ such that

$$\frac{\left|\left\{x \in \overline{B(x_0,M_0r)} : (x,t_0-r^2) \notin \Omega\right\}\right|}{|B(x_0,M_0r)|} \geq \theta \quad \text{for every } 0 < r \leq r_0. \tag{15}$$

The fact that (15) is enough to ensure the $\mathcal{H}$-regularity had been proved in [65, Theorem 4.11]. Under such geometrical condition, one can show that also the series in (13) diverges for any b. Moreover, it is possible to prove a quantitative rate for the divergence of that series which yields a Hölder-modulus of continuity at z_0 for the H^Ω_φ. This can be summarized in the following result

Theorem 3.3. [64, Theorem 1.4] *Assume the exterior d-cone condition* (15) *holds at* z_0. *Let* $\varphi \in C(\partial\Omega,\mathbb{R})$ *be such that*

$$[\varphi]_{z_0,\delta} = \sup_{\rho>0} \sup_{\hat{d}(z,z_0)\leq\rho} \frac{|\varphi(z)-\varphi(z_0)|}{\rho^\delta} < \infty$$

for some $\delta > 0$. *Then, there exist* $0 < \alpha_0 \leq 1$ *and* $c > 0$ *only depending on* $\Lambda, a_0, b_0, d, \delta, \Omega$, *and the constants* M_0, r_0, θ *in the d-cone condition* (15) *such that*

$$|H^\Omega_\varphi(z) - \varphi(z_0)| \leq c[\varphi]_{z_0,\delta} \left(\hat{d}(z_0,z)\right)^{\alpha_0} \quad \textit{for all } z \in \Omega.$$

4. Regularity criteria for Hörmander operators

Let $X_1, X_2, \dots, X_p$ be a system of real smooth vector fields which are defined in some bounded open set $D_0 \subset \mathbb{R}^N$, and satisfy the Hörmander's condition in D_0, i.e. $\operatorname{rank}\operatorname{Lie}\{X_1, \dots, X_p\}(x) = N$ at every point $x \in D_0$. Denote by d the related Carnot-Carathéodory control distance. Let us also fix D a bounded open set compactly contained in D_0, and $-\infty < T_1 < T_2 < +\infty$. Let us now consider the family of partial differential operators in the form

$$\mathcal{H} = \sum_{i,j=1}^{p} a_{i,j}(z) X_i X_j + \sum_{j=1}^{p} b_j(z) X_j - \partial_t,$$

where $a_{i,j}, b_j$ are d-Hölder continuous functions in $D_0 \times]T_1, T_2[$, and the matrix $(a_{i,j})_{i,j}$ is symmetric and uniformly positive definite. We want to study the Cauchy-Dirichlet problem associated with $\mathcal{H}$ in bounded sets $\Omega \subset D \times]T_1, T_2[$. It is possible to extend suitably the operator $\mathcal{H}$ in the whole $\mathbb{R}^{N+1}$ with a procedure described in [16, Section 2-19] (see also [95, Section 2]). In particular it is possible to construct a system of smooth Hörmander vector fields $\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_m$ in $\mathbb{R}^N$ ($m = p + N$) coinciding in D with the system $\{X_1, \dots, X_p, 0, \dots, 0\}$, and outside D_0 with the Euclidean system $\{0, \dots, 0, \partial_{x_1}, \dots, \partial_{x_N}\}$. Moreover, the Carnot-Carathéodory control distance $\tilde{d}$ relative to the new system of vector fields satisfies a global doubling condition, the segment property, and also the other conditions for the metric space $(\mathbb{R}^N, d)$ we required in the previous subsection. The main difference with respect to the equation (8) relies on the regularity of the coefficients $a_{i,j}, b_j$ which prevents these Hörmander-type operators $\mathcal{H}$ from being hypoelliptic. Nonetheless, in [16, Theorem 10.7] it is proved that such extended operators have a global fundamental solution Γ (of 'intrinsic' regularity $\mathcal{C}_X^{2+\alpha}$ out of the diagonal of $\mathbb{R}^{N+1} \times \mathbb{R}^{N+1}$) satisfying two sided Gaussian-type estimates on strips as in (9). This is the main tool which allowed us to prove in [95] the following result

Theorem 4.1. [95, Theorem 1.1] *Let Ω be a bounded open set with closure $\overline{\Omega}$ contained in $D \times]T_1, T_2[$. Let $z_0 = (x_0, t_0) \in \partial\Omega$, and $\lambda \in]0, 1[$. Then there exist positive numbers $a_0 \le b_0 \le \beta_0$, depending just on the vector fields $X_1, \dots, X_p$, on the eigenvalue bounds for the matrix $(a_{i,j})_{i,j}$, and on the Hölder norms of $a_{i,j}, b_j$, such that*

$$(i) \quad \sum_{h,k=1}^{+\infty} \frac{\mathcal{C}_a\left(\Omega_k^h(z_0, \lambda)\right)}{\left|B\left(x_0, \sqrt{\lambda^k}\right)\right|} \lambda^{bh} = +\infty \quad \textit{for some } 0 < a \le a_0 \textit{ and } b > b_0$$

$$\Longrightarrow \quad z_0 \textit{ is } \mathcal{H}\textit{-regular.}$$

(ii) $\quad z_0$ *is $\mathcal{H}$-regular* $\quad\Longrightarrow$

$$\sum_{h,k=1}^{+\infty} \frac{\mathcal{C}_b\left(\Omega_k^h(z_0,\lambda)\right)}{\left|B\left(x_0,\sqrt{\lambda^k}\right)\right|}\lambda^{ah} = +\infty \quad \textit{for every } b \geq \beta_0 \textit{ and } 0 < a \leq a_0.$$

Here we used the same notations of Theorem 3.2. Moreover, again as in Theorem 3.2, we can remark that the sufficient condition and the necessary condition are different. On the other hand, all the quantities and the objects appearing in our Wiener series depend on the underlying metric, and not on the fundamental solution of the specific operator $\mathcal{H}$.
With respect to the hypoelliptic case treated in the previous section, the presence of non-smooth coefficients forced us to follow in [95] a different approach for the proof of part (ii). As a consequence, the necessary condition (14) turns out to be slightly weaker. The main technical reason is the lack of suitable Riesz-type representations for $\mathcal{H}$-superharmonic functions (such as the one in (11)).
Analogously to Theorem 3.3, one can deduce as a by-product of Theorem 4.1 that the cone-type condition (15) ensures the Hölder-regularity of the Perron-Wiener solution at the boundary point. As a consequence, exploiting also the results in [16], we obtained the following result

Corollary 4.2. [95, Corollary 1.2] (see also [97, Theorem 4.1]) *Denote*

$$E = \left\{ z_0 \in \partial\Omega : \sum_{h,k=1}^{+\infty} \frac{\mathcal{C}_a\left(\Omega_k^h(z_0,\lambda)\right)}{\left|B\left(x_0,\sqrt{\lambda^k}\right)\right|}\lambda^{bh} = +\infty \ \textit{for} \right.$$
$$\left. \textit{some } 0 < a \leq a_0 \textit{ and } b > b_0 \right\}.$$

Then, for every continuous datum φ on $\partial\Omega$ and for every d-Hölder continuous function f in a neighborhood of $\overline{\Omega}$, there exists

$$u \in C(\Omega \cup E) \cap C^2_X(\Omega) \ \textit{such that} \quad \begin{cases} \mathcal{H}u = f & \textit{in } \Omega, \\ u = \varphi & \textit{on } E. \end{cases}$$

In particular the above statement holds true even letting E be the set of points of $\partial\Omega$ where the exterior d-cone condition (15) *is satisfied.*

From Corollary 4.2, one can deduce the existence of a unique solution to the Cauchy-Dirichlet problem related to $\mathcal{H}$ in the following cylindrical domains.

Suppose $\Omega = A\times]t_1,t_2[$ is a bounded open set compactly contained in $D\times]T_1,T_2[$ satisfying

$$\forall x_0 \in \partial A \ \ \exists r_0, \theta_0 > 0 \text{ such that } \frac{|B(x_0,r)\smallsetminus A|}{|B(x_0,r)|} \geq \theta_0 \ \text{ for all } 0 < r \leq r_0. \quad (16)$$

We can denote by $\partial_p\Omega$ the parabolic boundary of Ω, i.e. $\partial_p\Omega = (\partial A \times [t_1,t_2]) \cup \left(\overline{A} \times \{t_1\}\right)$. It is proved in [65, Proposition 6.1] that the condition (16) implies the validity of the cone condition (15) at any parabolic boundary point. Then, for every continuous function φ on $\partial_p\Omega$ and for every d-Hölder continuous function f in a neighborhood of $\overline{\Omega}$, there exists a unique solution $u \in C(\Omega\cup\partial_p\Omega)\cap\mathcal{C}^2_X(\Omega)$ to the problem

$$\begin{cases} \mathcal{H}u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial_p\Omega. \end{cases}$$

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