LE MATEMATICHE
Vol. LXXV (2020) – Issue I, pp. 313–329
doi: 10.4418/2020.75.1.14

ANISOTROPIC ESTIMATES OF SUBELLIPTIC TYPE

ANNAMARIA MONTANARI - DANIELE MORBIDELLI

We discuss some estimates of subelliptic type related with vector fields satisfying the Hörmander condition. Our approach makes use of a class of approximate exponentials studied in our previous papers [7–9, 11]. Such kind of estimates arises naturally in the study of regularity theory of weak solutions of degenerate elliptic equations.

1. Introduction

In this note we review and slightly improve some estimates of subelliptic type for a family $X_1,\dots,X_m$ of smooth vector fields of Hörmander type in $\mathbb{R}^n$. We mainly use the analysis of a class of approximate exponential maps appearing in some previous papers of the authors and of Ermanno Lanconelli. See [7–9, 11]. We shall formulate our estimates making use of a family of fractional Sobolev norms modeled on the subRiemannian geometry defined by $X_1,\dots,X_m$. These norms have been analyzed in [11].

Let $X_1,\dots,X_m$ be a family of vector fields satisfying the Hörmander condition of step $\kappa \in \mathbb{N}$ in $\mathbb{R}^n$. A classical estimate states that given a bounded set

Received on October 19, 2019

AMS 2010 Subject Classification: 53C17, 35R03, 35A23, 46E35.
Keywords: Hörmander vector fields; subRiemannian distance; subelliptic estimates; approximate exponential map; ball-box Theorem.
The authors are members of the *Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni* (GNAMPA) of the *Istituto Nazionale di Alta Matematica* (INdAM).

$\Omega \subset \mathbb{R}^n$ and $p \in [1, +\infty[$, there is a positive constant C such that

$$[f]_{W^{1/k,p}} := \left(\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p(1/\kappa)}} dxdy \right)^{1/p} \leq C(\|f\|_{L^p} + \sum_j \|X_j f\|_{L^p}), \quad (1)$$

for all smooth f compactly supported in Ω. Here $[f]_{W^{1/\kappa,p}}$ denotes the L^p (semi)norm of the fractional derivative of order $1/\kappa$ of f. Classical references for this inequality are [3, 13]. Various versions of estimate (1) have been used extensively starting from the seminal paper of Hörmander [4] in the theory of hypoelliptic operators.

Estimate (1) inherently gives the same order $1/\kappa$ of Euclidean fractional differentiability in all directions, for a function $f \in L^p$ with derivatives $X_1 f, \dots$, $X_m f \in L^p$. Note that in typical situations of interest in subelliptic analysis, the dimension of the subspace generated by $X_1(x), \dots, X_m(x)$ is strictly less than the topological dimension n. It is known that (1) is sharp, as far as we refer to Euclidean fractional derivatives. However, it does not capture the fact that given f in the first order Folland–Stein space,[1] one can expect a better regularity of f along the directions of commutators of lower order compared with the regularity expected along the directions of commutators of higher order. This motivates the introduction of a different notion of fractional differentiability, based on the data of the metric measure space $(\mathbb{R}^n, \mathcal{L}^n, d)$, where $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$ and d the subRiemannian distance. Namely, following [11], given $\Omega \subset \mathbb{R}^n$, we define for any $s \in]0,1[$ and $p \in [1, +\infty[$, the seminorm

$$[f]_{W_d^{s,p}(\Omega)} := \left(\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{\mathcal{L}^n(B(x,d(x,y)))d(x,y)^{ps}} dxdy \right)^{1/p}, \quad (2)$$

where d denotes the subRiemannian distance associated with the vector fields X_1, $\dots, X_m$. Here and hereafter, $B(x,r)$ will denote the ball of center $x \in \mathbb{R}^n$ and radius r with respect to the subRiemannian distance d. Observe that, by known properties in subRiemannian geometry, we have the local estimates $\mathcal{L}^n(B(x,r)) \leq Cr^n$ and $d(x,y) \leq C|x-y|^{1/\kappa}$, where κ is the step of the vector fields. See (19) in Section 2. Then we have the trivial local embedding

$$\|f\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+p(s/\kappa)}} dxdy \right)^{1/p} \leq C\|f\|_{W_d^{s,p}(\Omega)},$$

where $\|f\|_{W_d^{s,p}(\Omega)} := \|f\|_{L^p(\Omega)} + [f]_{W_d^{s,p}(\Omega)}$ and the positive constant C depends on the bounded set $\Omega \subset \mathbb{R}^n$ and on $s \in]0,1[$.

[1] The functional space defined by the norm in the right-hand side of (1).

Note also that the embedding is somehow strict. In order to explain our comment, given a bounded set Ω, we define the fractional derivative of order $\varepsilon \in]0,1[$ of a function f along a vector field Z as

$$[f]_{W_Z^{\varepsilon,p}(\Omega)} := \left(\int_\Omega dx \int_{\{t\in[0,1]\,:\,e^{tZ}(x)\in\Omega\}} \frac{dt}{|t|^{1+p\varepsilon}} |f(e^{tZ}x) - f(x)|^p \right)^{1/p}, \qquad (3)$$

where as usual we denote by $e^{tZ}(x)$ the value at time t of the integral curve of Z starting from x when $t = 0$.

Let us introduce the notation $X_w := [X_{w_1}, \dots, [X_{w_{\ell-1}}, X_{w_\ell}] \dots]$ to denote nested commutators of length $|w| = \ell \le \kappa$. Then, we show that given vector fields of step $\kappa \in \mathbb{N}$, $p \in [1,+\infty[$ and a triple $\Omega_1 \Subset \Omega_2 \Subset \Omega_3$ of bounded sets, for all $s \in]0,1[$ we have the equivalence

$$C^{-1}\|f\|_{W_d^{s,p}(\Omega_1)} \le \|u\|_{L^p(\Omega_2)} + \sum_{|w|\le\kappa} [f]_{W_{X_w}^{s/|w|,p}(\Omega_2)} \le C\|f\|_{W_d^{s,p}(\Omega_3)}. \qquad (4)$$

This means that given a function $f \in L^p(\Omega)$ and $s < 1$, the seminorm $[f]_{W_d^{s,p}(\Omega)}$ is finite if and only if f has $s/|w|$ derivatives in L^p along commutators X_w of length $|w| = 1,2,3,\dots,\kappa$. Note that in [11] the first author proved the equivalence (4) for commutators X_w with length $|w| = 1$ only. Here we show that the argument in [11] provides also the inequality

$$[f]_{W_{X_w}^{s/|w|,p}(\Omega_2)} \le C\|f\|_{W_d^{s,p}(\Omega_3)}$$

for commutators X_w of arbitrary length $|w| \le \kappa$.

In Section 3 we shall prove the following anisotropic subelliptic estimate.

Theorem 1.1. *Given Hörmander vector fields $X_1,\dots,X_m$ in $\mathbb{R}^n$, for all $p \in [1,+\infty[$, for all $s \in]0,1[$ and for any pair of nested bounded sets $\Omega \Subset \Omega_0$ there is $C > 0$ such that for all C^1 function f we have the inequality*

$$[f]_{W_d^{s,p}(\Omega)} \le C\Big(\sum_{j=1}^m \|X_j f\|_{L^p(\Omega_0)} + \|f\|_{L^p(\Omega_0)} \Big). \qquad (5)$$

Roughly speaking, the inequality (5) means that the L^p norm of the derivatives of order $s < 1$ in the subRiemannian metric measure space $(\mathbb{R}^n, \mathcal{L}^n, d)$ can be estimated from above with the derivatives of order 1 in the Folland–Stein space. This estimate is trivial in the Euclidean case, see the discussion in Remark 3.1. Surprisingly, the proof of the subRiemannian statement (5) becomes less trivial and requires a certain amount of work.

Inequality (5) has been proved in [11, Theorem 5.1] for $1 < p < \frac{Q}{1-s}$ and for an appropriate Q, by using some nice properties of the fundamental solution of

Hörmander operators. Here we extend this inequality to every $p \geq 1$ by using a completely different technique, whose main tool is the approximate exponential map. The proof of Theorem 1.1 will be presented in Section 3 and it is inspired to the argument of the Lanconelli's unpublished proof of the nonsharp version of (1). See [9, Proposition 6.2]. Let us mention that for the case $s = 1$ our seminorm (2) is not useful, but there is a rich theory of Sobolev spaces of order $s = 1$ defined on metric measure spaces. See the Hajłasz spaces [5] and the Newtonian spaces [14], just to quote a few.

In the subsequent Section 4, following [11], we describe the proof of the equivalence (4). As a corollary, we obtain estimates in the directions of commutators.

Namely, we will get the following statement.

Theorem 1.2. *Let $X_1,\dots,X_m$ be Hörmander vector fields of step κ in $\mathbb{R}^n$ and take $p \in [1,+\infty[$. Let $s \in]0,1[$ and consider a commutator X_w of length $|w| \leq \kappa$. Then, given bounded open sets $\Omega \Subset \Omega_0$ there is $C > 0$ such that*

$$[f]_{W^{s/|w|,p}_{X_w}(\Omega)} := \left(\int_\Omega dx \int_{\{t\in[0,1]\,:\, e^{tX_w}(x)\in\Omega\}} \frac{dt}{|t|^{1+ps/|w|}} \left|f(e^{tX_w}x) - f(x)\right|^p dx \right)^{1/p}$$
$$\leq C\Big(\|f\|_{L^p(\Omega_0)} + \sum_{j=1}^m \|X_j f\|_{L^p(\Omega_0)} \Big).$$

The idea of using anisotropic estimates along different directions in a subelliptic context arises naturally in the study of pointwise estimates for weak solutions of degenerate elliptic equations with measurable coefficients and it was exploited long ago by Franchi and Lanconelli [2] in the setting of the diagonal vector fields $X_j = \lambda_j \frac{\partial}{\partial x_j}$, where $j = 1,\dots,n$ and $\lambda_1,\dots,\lambda_n$ are suitable functions. Here we formulate a family of anisotropic inequalities in the setting of Hörmander vector fields with their commutators. Again, our techniques do not provide a proof of the borderline case $s = 1$.

To motivate Theorem 1.2, let us give a formulation of its L^∞ version, starting from a well known fact. Let $X = \partial_x + 2y\partial_t$ anf $Y = \partial_y - 2x\partial_t$ be the vector fields of the Heisenberg group with coordinates (x,y,t). Then, by an easy computation, we have the exact formula

$$e^{-sY}e^{-sX}e^{sY}e^{sX}(x,y,t) = (x,y,t-4s^2) = e^{s^2[X,Y]}(x,y,t), \tag{6}$$

for all $(z,t) := (x,y,t) \in \mathbb{R}^3$ and $s > 0$. Then, given any regular function $f = f(z,t)$, we have an estimate of the $\frac{1}{2}$-Hölder seminorm of f on a bounded set Ω:

$$\sup_{(z,t)\in\Omega,\ |\tau|\leq r_0} \frac{\left|f(e^{\tau[X,Y]}(z,t)) - f(z,t)\right|}{|\tau|^{1/2}} \leq C \sup_{(z,t)\in\Omega_0} \big(|Xf(z,t)| + |Yf(z,t)|\big), \tag{7}$$

where $\Omega_0 \supset \overline{\Omega}$ and $r_0 > 0$ is small enough.

A natural generalization of (6) to more general vector fields $X_1, \dots, X_m$ would be a formula of the following form. Let X_w be any commutator of length $|w| \le \kappa$ constructed from a family $X_1, \dots, X_m$ of Hörmander vector fields of step κ in $\mathbb{R}^n$. Given a bounded set Ω and $\Omega_0 \supset \overline{\Omega}$ there is $r_0 > 0$ and a positive C such that for all nested commutator $X_w := [X_{w_1}, \dots [X_{w_{\ell-1}}, X_{w_\ell}] \dots]$ of length $|w| \in \{1, 2, \dots, \kappa\}$, we have

$$\sup_{x \in \Omega,\, |\tau| \le r_0} \frac{|f(e^{\tau X_w} x) - f(x)|}{|\tau|^{1/|w|}} \le C \sup_{y \in \Omega_0} \sum_{j=1}^m |X_j f(y)|. \tag{8}$$

If we try to prove (8) by generalizations of formula (6), we encounter some remainders which can not be controlled with elementary methods. However, estimate (8) does hold as a consequence of the ball-box Theorem presented in Section 2. See the explanation in Remark 2.4.

We conclude the Introduction by remarking that in this paper, for the sake of clarity, we consider smooth vector fields satisfying Hörmander condition. However, by the ball-box Theorem in [9], we expect that all the results of the present paper can be extended to nonsmooth vector fields of arbitrary step $\kappa \in \mathbb{N}$ and with coefficients in some regularity class related with the step κ appearing in the Hörmander condition.

2. Preliminaries

Consider smooth vector fields $X_1, \dots, X_m$ in $\mathbb{R}^n$. Given a word $w = w_1 \cdots w_\ell$ in the alphabeth $\{1, \dots, m\}$, let us introduce the commutator

$$X_w := [X_{w_1}, [X_{w_2}, \dots [X_{w_{\ell-1}}, X_{w_\ell}] \dots].$$

The number $|w| = |w_1 w_2 \cdots w_\ell| =: \ell$ is called the *length* of the commutator X_w. Let us define the *subRiemannian distance*

$$\begin{aligned} d(x,y) := \inf\bigl\{ r > 0 :\ &\text{there is } \gamma \in \mathrm{Lip}((0,1), \mathbb{R}^n) \text{ with } \gamma(0) = x, \gamma(1) = y \\ &\text{and } \dot\gamma(t) = \sum_{1 \le j \le m} u_j(t) r X_j(\gamma(t)) \text{ with } |u(t)|_{\mathrm{Euc}} \le 1 \text{ for a.e. } t \in [0,1] \bigr\}. \end{aligned}$$

Given a fixed $\kappa \ge 1$, denote by $Y_1, \dots, Y_q$ an enumeration of $\{X_w : 1 \le |w| \le \kappa\}$, the family of commutators of length at most κ. Let $\ell_j \le \kappa$ be the length of Y_j. Define the distance ρ

$$\begin{aligned} \rho(x,y) := \inf\bigl\{ r \ge 0 : &\text{ there is } \gamma \in \mathrm{Lip}((0,1), \mathbb{R}^n) \text{ such that } \gamma(0) = x \\ &\gamma(1) = y \text{ and } \dot\gamma(t) = \sum_{j=1}^q b_j(t) r^{\ell_j} Y_j(\gamma(t)) : |b(t)|_{\mathrm{Euc}} \le 1 \text{ for a.e. } t \in [0,1] \bigr\}. \end{aligned} \tag{9}$$

The Hörmander condition of step κ reads as $\operatorname{span}\{X_w(x) : |w| \le \kappa\} = \mathbb{R}^n$ for all $x \in \mathbb{R}^n$.

We denote by $B_\rho(x,r)$, $B(x,r)$ and $B_{\mathrm{Euc}}(x,r)$ the balls of center x and radius r with respect to ρ, d and the Euclidean distance respectively. Sometimes, to avoid any confusion, we denote by $|\cdot|_{\mathrm{Euc}}$ the Euclidean norm.

Since the vector fields $Y_1,\dots,Y_q$ span $\mathbb{R}^n$ at any point, it is easy to see that for all pair of points $x,y \in \mathbb{R}^n$, the set of competitors defining $\rho(x,y)$ is nonempty and then $\rho(x,y) < +\infty$ for any pair of points. Furthermore, an elementary argument shows that a local estimate of the form $\rho(x,y) \le C|x-y|^{1/\kappa}$ holds. Trivially, by definition, we have $\rho(x,y) \le d(x,y)$. The classical Chow's theorem implies also that for any pair of points x and $y \in \mathbb{R}^n$ the set of competitors defining $d(x,y)$ is nonempty. Then $d(x,y) < \infty$. Finally, both ρ and d satisfy the axioms of a distance.

Ball-box Theorem We recapitulate here the statement of the ball-box Theorem. Let us consider a family of vector fields $X_1,\dots,X_m$ satisfying the Hörmander condition of step κ in $\mathbb{R}^n$. Let us fix an enumeration $Y_1,\dots,Y_q$ of all the commutators X_w with length $|w| \le \kappa$. Let ℓ_i be the length of Y_i. If the Hörmander condition of step κ is fulfilled, then the vector fields $Y_1,\dots,Y_q$ span $\mathbb{R}^n$ at any point. Given a multi-index $I = (i_1,\dots,i_n) \in \{1,\dots,q\}^n$ and its corresponding $n-$tuple $Y_{i_1},\dots,Y_{i_n}$, let

$$\lambda_I(x) = \det(Y_{i_1}(x),\dots,Y_{i_n}(x)), \quad \text{and} \quad \ell(I) \; = \ell_{i_1} + \cdots + \ell_{i_n}. \tag{10}$$

The first "ball-box Theorem" was proved by Nagel, Stein and Wainger in [12]. Namely, in that paper, the authors introduced the exponential map related with an n-tuple $I = (i_1,\dots,i_n) \in \{1,\dots,q\}^n$ in the form

$$\Phi_{I,x}(u) := \exp\Big(\sum_{j=1}^{n} u_j Y_{i_j}\Big)(x),$$

where u belongs to a neighborhood of the origin in $\mathbb{R}^n$ and $\exp(Z)(x)$ or $e^Z x$ denotes the value at time $t = 1$ of the integral curve of the vector field Z starting from $x \in \mathbb{R}^n$ at $t = 0$. Since we are interested in local estimates, we may without loss of generality assume that $e^Z x$ is well defined in all situations of our interest.

To give the statement of the Nagel–Stein–Wainger ball-box Theorem, we introduce a definition.

Definition 2.1 (η-maximal triple)**.** Let $I = (i_1,\dots,i_n) \in \{1,\dots,q\}^n$, $x \in \mathbb{R}^n$, $\eta \in]0,1[$ and $r > 0$. We say that (I,x,r) is η-maximal if

$$|\lambda_I(x)| r^{\ell(I)} > \eta \max_{K \in \{1,\dots,q\}^n} |\lambda_K(x)| r^{\ell(K)}. \tag{11}$$

Define also for all $I \in \{1, \dots, q\}^n$

$$\|h\|_I = \max_{j=1,\dots,n} |h_j|^{1/\ell_{i_j}}, \qquad Q_I(r) = \{h \in \mathbb{R}^n : \|h\|_I < r\}. \tag{12}$$

Then, in [12] the authors proved that if (I,x,r) is η-maximal, x belongs to a compact set and r is sufficiently small, then we have the double inclusion

$$\Phi_{I,x}(Q_I(c_1 r)) \subset B_\rho(x,r) \subset \Phi_{I,x}(Q_I(c_2 r)), \tag{13}$$

where the constants c_1 and c_2 depend on η and on the compact set where x lies. This estimate together with some Jacobian estimates have the consequence that

$$\mathcal{L}^n(B_\rho(x,r)) \simeq \sum_{K \in \{1,\dots,q\}^n} |\lambda_K(x)| r^{\ell(K)}, \tag{14}$$

where the equivalence holds for compact sets and sufficiently small r. We have presented the statements of this part in an informal way. Below we shall give precise statements of similar results which are needed in this paper.

After [12], the analysis of the maps Φ_I was carried out by several authors in subsequent years. See [1, 15, 16]. Since $\ell(I) \geq n$ for all I, we always have by (14) the local estimate $\mathcal{L}^n(B_\rho(x,r)) \leq Cr^n$, where x belongs to a bounded set and $r > 0$ is sufficiently small.

Approximate exponentials of commutators and the corresponding ball-box Theorem. In [12], the authors gave also a sketch of the proof of the fact that the distance ρ is equivalent to d locally. This was done introducing a class of maps which we are now going to call "approximate exponentials".

Consider vector fields $X_{w_1}, \dots, X_{w_\ell}$, and their commutator X_w, which has length $\ell = |w|$. Let us define the approximate exponential $\exp_{\mathrm{ap}}(tX_w)$. For $\tau > 0$, we define, as in [12], [11] and [9],

$$\begin{aligned}
C_\tau(X_{w_1}) &= \exp(\tau X_{w_1}),\\
C_\tau(X_{w_1}, X_{w_2}) &= \exp(-\tau X_{w_2})\exp(-\tau X_{w_1})\exp(\tau X_{w_2})\exp(\tau X_{w_1}),\\
&\ \ \vdots\\
C_\tau(X_{w_1}, \dots, X_{w_\ell}) &= C_\tau(X_{w_2}, \dots, X_{w_\ell})^{-1}\exp(-\tau X_{w_1}) C_\tau(X_{w_2}, \dots, X_{w_\ell})\exp(\tau X_{w_1}).
\end{aligned}$$

Then let

$$\exp_{\mathrm{ap}}(tX_w) = \begin{cases} C_{t^{1/\ell}}(X_{w_1}, \dots, X_{w_\ell}), & \text{if } t > 0,\\ C_{|t|^{1/\ell}}(X_{w_1}, \dots, X_{w_\ell})^{-1}, & \text{if } t < 0. \end{cases} \tag{15}$$

By standard ODE theory, if x belongs to a bounded set Ω_0, there is $r_0 > 0$ so that if $|t| \leq r_0$ and $x \in \Omega_0$ the approximate exponential is well defined.

Let us assume that the system $X_1,\dots,X_m$ has step $\kappa\in\mathbb{N}$ and introduce the family $Y_1,\dots,Y_q$ of their nested commutators of length at most κ. Define, given $I=(i_1,\dots,i_n)\in\{1,\dots,q\}^n$, for $x\in K$ and $h\in\mathbb{R}^n$, h in a neghborhood of the origin

$$E_{I,x}(h)=\exp_{\mathrm{ap}}(h_1Y_{i_1})\cdots\exp_{\mathrm{ap}}(h_nY_{i_n})(x). \tag{16}$$

Theorem 2.2 (Ball-box). *Let $X_1,\dots,X_m$ be Hörmander vector fields of step κ in $\mathbb{R}^n$. Fix an open bounded set $\Omega_0\subset\mathbb{R}^n$. Then there is $r_0>0$ such that for all $\eta\in\,]0,1[$ there are constants $\varepsilon_\eta<1<C_\eta$ such that for any η-maximl triple (I,x,r) with $r\le r_0$ and $x\in\Omega_0$, we have*

(i) the map $h\mapsto E_{I,x}(h)$ is one-to-one on the box $Q_I(\varepsilon_\eta r)$ defined in (12)*;*

(ii) we have the inclusion

$$E_{I,x}(Q_I(\varepsilon_\eta r))\supseteq B_\rho(x,C_\eta^{-1}r); \tag{17}$$

(iii) The Jacobian of the map $E_{I,x}$ admits the following estimate

$$C_\eta^{-1}|\lambda_I(x)|r^{\ell(I)}\le\left|\det\frac{\partial E_{I,x}(h)}{\partial h}\right|\le C_\eta|\lambda_I(x)|r^{\ell(I)}\quad \textit{for all } h\in Q_I(\varepsilon_\eta r).$$

Theorem 2.2 has been proved and used in various regularity conditions in [9–11]. In this paper we will work in the smooth case and we shall choose always $\eta=\frac{1}{2}$, to make statements clean.

A first consequence of Theorem 2.2 is the following volume estimate. For all bounded set $\Omega\subset\mathbb{R}^n$ there is $r_0>0$ and $C>0$ such that

$$\begin{aligned}C^{-1}\mathcal{L}^n(B(x,r))\le\max_{K\in\{1,\dots,q\}^n}|\lambda_K(x)|r^{\ell(K)}\le C\mathcal{L}^n(B(x,r))\\ \text{for all } x\in\Omega \text{ and } r<r_0.\end{aligned} \tag{18}$$

A couple of further consequences are the estimates

$$\begin{aligned}&\mathcal{L}^n(B(x,r))\le Cr^n\quad\text{for all } x\in\Omega\text{ and } r<r_0;\\ &d(x,y)\le C|x-y|^{1/\kappa}\quad\text{for all } x,y\in\Omega.\end{aligned} \tag{19}$$

Remark 2.3. Let us observe the following consequence of Theorem 2.2 and of the construction of the maps $E_{I,x}$. Under the hypotheses of Theorem 2.2, for any open bounded set $\Omega\subset\mathbb{R}^n$ there are $r_0>0$ and $C_0>0$ so that any pair of points $x,y\in\Omega$ with $|x-y|<r_0$ can be connected with a piecewise integral curve of the vector fields $\pm X_1,\dots,\pm X_m$. The number of pieces is bounded by an universal algebraic constant M depending on m and κ, while each piece has length $\le C\rho(x,y)$. (Recall that by definition we always have $\rho(x,y)\le d(x,y)$).

In the following remark, we go back briefly to the discussion of the introduction, concerning estimate (8).

Remark 2.4. Let X_w be any nested commutator of length $|w| \le \kappa$ constructed from a family $X_1, \dots, X_m$ of Hörmander vector fields of step κ in $\mathbb{R}^n$. Given a bounded set Ω and $\Omega_0 \Supset \Omega$, there is $r_0 > 0$ and a positive C such that

$$\sup_{x\in\Omega,\ |\tau|\le r_0} \frac{|f(e^{\tau X_w}x) - f(x)|}{|\tau|^{1/|w|}} \le C \sup_{y\in\Omega_0} \sum_{j=1}^m |X_j f(y)|. \tag{20}$$

To check (20), it suffices to observe that, by definition of ρ, we have that $e^{\tau X_w}(x) \in B_\rho(x, 2|\tau|^{1/|w|})$ for all $x \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$. Note explicitly that we must use here the distance ρ defined using commutators. Then, by Remark 2.3, we conclude that if $x \in \Omega$ and $|\tau| < r_0$ for small r_0,

$$|f(e^{\tau X_w}x) - f(x)| \le C \sup_{\Omega_0} \sum_j |X_j f| \cdot |\tau|^{1/|w|}.$$

Informally speaking, if we choose r_0 small enough, then Ω_0 can be made a small open neighborhood of $\overline{\Omega}$.

3. Anisotropic subelliptic estimates

In this section we prove inequality (4) and consequently Theorem 1.1. Namely we show the local estimate

$$[f]_{W_d^{s,p}(\Omega)} \le C\Big(\sum_{j=1}^m \|X_j u\|_{L^p(\Omega_0)} + \|u\|_{L^p(\Omega_0)} \Big), \tag{21}$$

for all $f \in C^1$. Here Ω is bounded, $\Omega_0 \supset \overline{\Omega}$, $p \in [1, +\infty[$ and $s \in]0,1[$.

Remark 3.1. Let us look inequality (21) in the Euclidean case. Roughly speaking, it says that derivatives of order $s < 1$ in L^p can be estimated with derivatives of order 1 in L^p. The inequality is essentially trivial. Namely, in order to get the estimate

$$\int\limits_{\substack{\Omega\times\Omega \\ |x-y|<r_0}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \le C \int_{\Omega_0} |\nabla f(x)|^p dx,$$

where Ω_0 is an open neighborhood of Ω depending on r_0, it suffices to apply the fundamental theorem of calculus and then Minkowski inequality

$$\begin{aligned}
&\int_\Omega dx \int_0^{r_0} \frac{dh}{|h|^{n+ps}} |u(x) - u(x+h)|^p dx \\
&= \int_0^{r_0} \frac{dh}{|h|^{n+ps}} \int_\Omega dx \Big| \int_0^{|h|} \Big|\nabla f\Big(x + t\frac{h}{|h|}\Big)\Big| dt \Big|^p \\
&\le \int_0^{r_0} \frac{dh}{|h|^{n+ps}} \Big\{ \int_0^{|h|} dt \Big[\int_\Omega \Big|\nabla f\Big(x+t\frac{h}{|h|}\Big)\Big|^p dx \Big]^{1/p} \Big\}^p \\
&\le C \int_0^{r_0} \frac{dh}{|h|^{n+ps}} C|h|^p \int_{\Omega_0} |\nabla f(z)|^p dz \le C \int_{\Omega_0} |\nabla f|^p,
\end{aligned}$$

because the integral in dh converges for $s<1$. The argument of the inequalities above is based on integration of f along the curves $t\mapsto\gamma(t):=x+t\frac{h}{|h|}$, which is not available in the Hörmander setting. The machinery of the approximate exponentials $E_{I,x}$ makes possible to prove inequality (21) following the scheme of the chain of inequalities above.

Proof. Before starting the proof, we recall some useful consequences of the ball-box Theorem for vector fields stated in Section 2. We use below the statement with $\eta=\frac{1}{2}$ of Theorem 2.2 and we let $\widehat{C}:=C_{1/2}$ and $\widehat{\varepsilon}:=\varepsilon_{1/2}$.

Let Ω be a bounded open set and let $r_0>0$ be a small number so that Theorem 2.2 applies. Let us define, given $x\in\Omega$ and $I\in\{1,\dots,q\}^n$, the set

$$M_{I,x}:=\Big\{y\in\mathbb{R}^n : d(x,y)<r_0 \quad \text{and } (I,x,\widehat{C}d(x,y)) \text{ is } \frac{1}{2}\text{-maximal}\Big\}.$$

It is easy to see that the set $M_{I,x}$ is a metric annulus of the form

$$M_{I,x}=\Big\{y\in\mathbb{R}^n : r_{I,x}<d(x,y)<R_{I,x}\Big\}.$$

We have defined for all $I\in\{1,\dots,q\}^n$ and x an open set $M_{I,x}$ which can be empty. Furthermore, different choices of I can give overlapping annuli. Finally, $\cup_I M_{I,x}=B(x,r_0)$. The radius $R_{I,x}$ satisfies also the condition

$$|\lambda_I(x)|(\widehat{C}R_{I,x})^{\ell(I)}\geq\frac{1}{2}\max_{K\in\{1,\dots,q\}^n}|\lambda_K(x)|(\widehat{C}R_{I,x})^{\ell(K)}.$$

Therefore, Theorem 2.2 gives the estimates

$$\frac{1}{\widehat{C}}|\lambda_I(x)|\leq\Big|\det\frac{\partial E_{I,x}(h)}{\partial h}\Big|\leq\widehat{C}|\lambda_I(x)| \qquad \text{if } \|h\|_I\leq\widehat{\varepsilon}\widehat{C}R_{I,x} \tag{22}$$

$$B(x,R_{I,x})\subset E_{I,x}(Q_I(\widehat{C}\widehat{\varepsilon}R_{I,x})) \tag{23}$$

$$h\mapsto E_{I,x}(h) \text{ is one-to-one on } Q_I(\widehat{C}\widehat{\varepsilon}R_{I,x}). \tag{24}$$

Recall again that $Q_I(\widehat{C}\widehat{\varepsilon}R_{I,x}):=\{h\in\mathbb{R}^n:\|h\|_I\leq\widehat{\varepsilon}\widehat{C}R_{I,x}\}$. On the set $M_{I,x}$ we also have a lower estimate on the distance $d(x,y)$ in terms of the variable h. To get this bound, let us consider a point $y\in M_{I,x}$ and choose any number $\rho\in\,]d(x,y),R_{I,x}[$. For any such choice of ρ, we have obviously $y\in B(x,\rho)$. Furthermore, the triple $(I,x,\widetilde{C}\rho)$ is $1/2$-maximal. Then we can write $y=E_{I,x}(h)$ for a unique $h=E^{-1}_{I,x}(y)\in Q_I(\widehat{C}\widehat{\varepsilon}\rho)$. Therefore, given $y\in B(x,\rho)$, the unique h satisfying $E_{I,x}(h)=y$ belongs to $Q_I(\widehat{C}\widehat{\varepsilon}\rho)$, i.e. satisfies $\|h\|_I\leq\widehat{C}\widehat{\varepsilon}\rho$. Thus,

$$\|h\|_I=\|E^{-1}_{I,x}(y)\|\leq\widetilde{C}\widetilde{\varepsilon}\rho,$$
$$\text{for all } y\in M_{I,x} \text{ and } \rho\in\,]d(x,y),R_{I,x}[\,=\,]d(x,E_{I,x}(h)),R_{I,x}[.$$

Letting $\rho \searrow d(x, E_{I,x}(h))$, we get the useful lower bound

$$d(x, E_{I,x}(h)) \geq (\widehat{C}\varepsilon)^{-1}\|h\|_I \quad \text{for all } h \in E_{I,x}^{-1}(M_{I,x}). \tag{25}$$

Let now $p \in [1, +\infty[$ and $s \in]0,1[$. Then we start with the estimate

$$\begin{aligned}\int_\Omega dx \int_{B(x,r_0)\cap\Omega} &\frac{|f(x)-f(y)|^p}{d(x,y)^{ps}\mathcal{L}^n(B(x,d(x,y)))}dy \\ &\leq \sum_{I\in\{1,\dots,q\}^n} \int_\Omega dx \int_{M_{I,x}\cap\Omega} \frac{|f(x)-f(y)|^p}{d(x,y)^{ps}\mathcal{L}^n(B(x,d(x,y)))}dy.\end{aligned}$$

By the change of variable $y = E_{I,x}(h) \in M_{I,x} \cap \Omega$, the last integral is

$$\sum_I \int_\Omega dx \int_{E_{I,x}^{-1}(M_{I,x}\cap\Omega)} \frac{|f(x)-f(E_{I,x}(h))|^p}{d(x,E_{I,x}(h))^{ps}\mathcal{L}^n(B(x,d(x,E_{I,x}(h))))}\left|\det\frac{\partial E_{I,x}(h)}{\partial h}\right| dh$$

$$\leq C\sum_I \int_\Omega dx \int_{E_{I,x}^{-1}(M_{I,x}\cap\Omega)} \frac{|f(x)-f(E_{I,x}(h))|^p}{\|h\|_I^{\ell(I)+ps}} dh =: (*)$$

where we have used the equivalence $\mathcal{L}^n(B(x,r)) \simeq \max_K |\lambda_K(x)| r^{\ell(K)}$, the $\frac{1}{2}$-maximality of I and estimates (25) and (22). Note also that $E_{I,x}^{-1}(M_{I,x}\cap\Omega) \subset Q_I(\widehat{C}\widehat{\varepsilon}R_{I,x}) \subset Q_I(\widehat{C}\widehat{\varepsilon}r_0)$. In the previous chain of inequalities we denoted $E_{I,x}^{-1} = \left(E_{I,x}\big|_{Q_I(\widehat{C}\widehat{\varepsilon}R_{I,x})}\right)^{-1}$

Next recall that we can write in an obvious way

$$E_{I,x}(h) = \exp_{\mathrm{ap}}(h_1 Y_{i_1}) \cdots \exp_{\mathrm{ap}}(h_n Y_{i_n}) =: \gamma_{I,x,h}(T_I(h)), \tag{26}$$

where the curve $t \mapsto \gamma_{I,x,h}(t)$ is parametrized on the interval $[0, T_I(h)]$ and by construction of $\exp_{\mathrm{ap}}$ it is a concatenation of integral curves of the vector fields $X_1, \dots, X_m$. Furthermore $|T_I(h)| \leq C\|h\|_I$ for some absolute constant C. The map $x \mapsto \gamma_{I,x,t}$ is a change of variable by classical properties of flows of ODEs (see the discussion in [7]) and

$$C^{-1} \leq \left|\det\frac{\partial\gamma_{I,x,h}(t)}{\partial x}\right| \leq C,$$

uniformly in x on compact sets, $|h| < r_0$ and $t \in [0, T_I(h)]$.

Then, letting $|Xf| = |(X_1 f, \dots, X_m f)|$ we get

$$
\begin{aligned}
(*) &\leq \sum_I \int_\Omega dx \int_{Q_I(\widehat{C}\varepsilon R_{I,x})} \frac{dh}{\|h\|_I^{\ell(I)+ps}} \Big| \int_0^{T_I(h)} |Xf(\gamma_{I,x,h}(t))| dt \Big|^p \\
&\leq C \sum_I \int_{Q_I(\widehat{C}\varepsilon r_0)} \frac{dh}{\|h\|_I^{\ell(I)+ps}} \int_\Omega dx \Big| \int_0^{T_I(h)} |Xf(\gamma_{I,x,h}(t))| dt \Big|^p \\
&\leq C \sum_I \int_{Q_I(\widehat{C}\varepsilon r_0)} \frac{dh}{\|h\|_I^{\ell(I)+ps}} \Big\{ \int_0^{T_I(h)} dt \Big[\int_\Omega |Xf(\gamma_{I,x,h}(t))|^p dx \Big]^{1/p} \Big\}^p \\
&\leq C \sum_I \int_{Q_I(\widehat{C}\varepsilon r_0)} \frac{dh}{\|h\|_I^{\ell(I)+ps}} \|h\|_I^p \int_{\Omega_0} |Xf(z)|^p dz = C \int_{\Omega_0} |Xf|^p,
\end{aligned}
$$

as required. We have used the fact that for all $I \in \{1, 2, \dots, q\}^n$,

$$
\int_{\{\|h\|_I < 1\}} \frac{dh}{\|h\|_I^{\ell(I)+ps-p}} < \infty, \text{ for all } s < 1,
$$

which can be proved by a standard decomposition as a disjoint union of sets of the form $\{2^{-k} < \|h\|_I \leq 2^{-k+1}\}$ with $k \in \mathbb{N}$. □

4. Estimates along commutators

In this section we prove Theorem 1.2. Namely given Hörmander vector fields of step κ, $s < 1$ and $1 \leq p < \infty$, we show the estimate

$$
\int_0^{r_0} \frac{dt}{|t|^{1+ps/|w|}} \int_\Omega \big| f(e^{tX_w} x) - f(x) \big|^p dx \leq C \Big(\sum_{j=1}^m \|X_j f\|_{L^p(\Omega_0)} + \|f\|_{L^p(\Omega_0)} \Big)^p, \tag{27}
$$

for any nested commutator X_w with $|w| \leq \kappa$. Here, as usual Ω is a bounded open set, r_0 is a suitable small constant and $\Omega_0 \Supset \Omega$ is an enlarged set. As we already observed, we do not reach the optimal exponent $s = 1$. An estimate in the same spirit was proved by Franchi and Lanconelli in [2, Theorem] for diagonal vector fields.

The proof of the inequality (27) is an immediate consequence of Lemma 4.1 and of Theorem 1.1.

Lemma 4.1. *Let $p \geq 1$. Given Hörmander vector fields $X_1, \dots, X_m$ of step κ, for any nested commutator X_w with $|w| \leq \kappa$, for each $s \in]0,1[$ and for any $x_0 \in \mathbb{R}^n$ there is a neighborhood Ω of x_0 in $\mathbb{R}^n$ such that given $\widetilde{\Omega} \Supset \Omega$ there is $r_0 > 0$ and $C > 0$ such that*

$$
\int_0^{r_0} \frac{dt}{|t|^{1+ps/|w|}} \int_\Omega \big| f(e^{tX_w} x) - f(x) \big|^p dx \leq C \int_{\widetilde{\Omega}} dx \int_{\widetilde{\Omega}} dy \frac{|f(x) - f(y)|^p}{d(x,y)^{ps} \mathcal{L}^n(B(x, d(x,y)))} \tag{28}
$$

for any C^1 function f.

This statement was proved in [11] for commutators X_w of length $|w| = 1$. Here we provide a sketch of the proof and we show that the argument works for any commutator X_w of any length $1 \le |w| \le \kappa$.

The argument is based on the well known lifting procedure by Rothschild and Stein, which we now briefly describe. Let $X_1, \dots, X_m$ be vector fields of step κ at any point of $\mathbb{R}^n$. Let us fix a point $x_0 \in \mathbb{R}^n$. In [13] it was proved that there exists a neighborhood $U \times V$ of $(x_0, 0) \in \mathbb{R}^n \times \mathbb{R}^d =: \mathbb{R}^N$ such that on $U \times V \ni (x, \tau)$ we can define new vector fields

$$\widetilde{X}_j = X_j + \sum_{\beta=1}^{d} a_{j,\beta}(x,\tau) \frac{\partial}{\partial \tau_\beta}, \quad \text{where } (x,\tau) \in U \times V, \tag{29}$$

which are free up to order κ in $U \times V$. This means that the only linear relations among commutators of order $\le \kappa$ of the vector fields $\widetilde{X}_j$ have constant coefficients in $U \times V$ and are given by the antisymmetry and the Jacobi identity. Note also that N is the dimension of the nilpotent free Lie algebra of step κ with m generators. However, usually the Lie algebra generated by $\widetilde{X}_1, \dots, \widetilde{X}_m$ is not nilpotent.

Let us go back to the family $Y_1, \dots, Y_q$ introduced in the previous section as an enumeration of the family X_w as $1 \le |w| \le \kappa$. Note incidentally that it is $q > N$ for vector fields of step ≥ 2. For a given $Y_k = X_w$ with $w = w(k) = w_1 w_2 \dots w_\ell$ belonging to such family we define the lifted commutator $\widetilde{Y}_j = [\widetilde{X}_{w_1}, [\widetilde{X}_{w_2}, \dots, [\widetilde{X}_{w_{\ell-1}}, \widetilde{X}_{w_\ell}] \dots]]$.

Up to reordering the commutators $\widetilde{Y}_1, \dots, \widetilde{Y}_q$ we can assume that the subfamily of the first N commutators,

$$\widetilde{Y}_1 = \widetilde{X}_1, \widetilde{Y}_2 = \widetilde{X}_2, \dots, \widetilde{Y}_m = \widetilde{X}_m, \widetilde{Y}_{m+1}, \dots, \widetilde{Y}_N \tag{30}$$

are linearly independent. The lifted vector fields define a distance $\widetilde{d}$ whose properties are established in [12, Lemma 3.2] and [6, Lemma 4.4]. Then we get the following lemma.

Lemma 4.2 (See [11, Lemma 4.3]). *Let $x_0 \in \mathbb{R}^n$ and let U and V be the set arising in the lifting procedure. Given compact sets $E \subset U$ and $H \subset V$, there is $\delta_0 > 0$ and $C > 0$ such that for all $x, y \in E$ with $d(x,y) < \delta_0$, we have*

$$\int_{\{(\tau,\sigma) \in H \times H : \widetilde{d}((x,\tau),(y,\sigma)) \le \delta_0\}} \frac{d\tau d\sigma}{\widetilde{d}((x,\tau),(y,\sigma))^{Q+ps}} \le C \frac{1}{d(x,y)^{ps} \mathcal{L}^n(B(x, d(x,y)))} \tag{31}$$

where $Q = \ell(\widetilde{Y}_1) + \cdots + \ell(\widetilde{Y}_N)$ *is the homogeneous dimension of the free Carnot group of step* κ *with* m *generators.* [2]

We do not include the proof of Lemma 4.2, whose argument is based on [12, Lemma 3.2] or [6, Lemma 4.4]. See [11, Lemma 4.3] for the details.

Then we have the following Lemma.

Lemma 4.3. *Let* $\widetilde{Y}_1, \dots, \widetilde{Y}_q$ *be the commutators introduced in* (30) *and let* $U \times V \subset \mathbb{R}^n \times \mathbb{R}^d = \mathbb{R}^N$ *be the sets appearing in* (29). *Then, given* $s \in]0,1[$ *and* $p \in [1,+\infty[$, *fixed open sets* $G \Subset \widetilde{G} \Subset U \times V$, *and given a commutator* X_w *with* $|w| \le \kappa$, *there are* $\delta_0 > 0$ *and* $\alpha > 0$ *such that*

$$\int_G d\xi \int_{\{|t|<\delta_0^{|w|},\ e^{\alpha t X_w}(\xi)\in G\}} \frac{dt}{|t|^{1+ps/|w|}} |u(e^{t\alpha X_w}(\xi)) - u(\xi)|^p \le C \int_{\widetilde{G}\times\widetilde{G}} \frac{|u(\xi)-u(\eta)|^p d\xi d\eta}{\widetilde{d}(\xi,\eta)^{Q+ps}}.$$

In the statement we denoted with $\xi = (x,\tau)$ variables in the lifted space $\mathbb{R}^n \times \mathbb{R}^d$.

Sketch of the proof of Lemma 4.3. The argument is similar to the one appearing in [11, Lemma 4.4]. We sketch it, because there are some slight differences of notation and because here we consider commutators of any step. First of all, in view of the discussion in [13, p. 272], we can rearrange the choice of the basis $\widetilde{Y}_1, \dots, \widetilde{Y}_N$ in (30) in such a way that $\widetilde{X}_w = \widetilde{Y}_j$ for some $j \in \{1,\dots,N\}$ with $\ell_j = |w|$. Denote by $Q := \sum_{j=1}^N \ell_j$ the homogeneous dimension. Fix then an open set G^* such that $G \Subset G^* \Subset \widetilde{G}$ and define the exponential map $\widetilde{\Phi}_\xi(h) := \exp(h_1\widetilde{Y}_1 + \cdots + h_N\widetilde{Y}_N)(\xi)$. Taking δ_0 small enough we may assume that $\widetilde{\Phi}_\xi(h) \in G^*$ if $\|h\| = \max_{j\le N} |h_j|^{1/\ell_j} < \delta_0$. In particular, if $\alpha \in [0,1]$ we also have $e^{\alpha h_j \widetilde{Y}_j}\xi \in G^*$, if $\|h\| < \delta_0$. Let us start from the inequality (44) in [11], which reads as

$$\begin{aligned}\int_G d\xi \int_{\{|t|<\delta_0^{|w|},\ e^{\alpha t \widetilde{X}_w}(\xi)\in G\}} &\frac{dt}{|t|^{1+ps/|w|}} |u(e^{t\alpha \widetilde{X}_w}(\xi)) - u(\xi)|^p \\ &= \int_G d\xi \int_{\{|h_j|<\delta_0^{\ell_j},\ e^{\alpha h_j \widetilde{Y}_j}(\xi)\in G\}} \frac{dh_j}{|h_j|^{1+ps/|w|}} |u(e^{h_j\alpha \widetilde{Y}_j}(\xi)) - u(\xi)|^p \\ &\le C \int_G d\xi \int_{\|h\|\le\delta_0} \frac{dh}{\|h\|^{Q+ps}} |f(e^{h_j\alpha\widetilde{Y}_j}\xi) - f(\xi)|^p = (*).\end{aligned}$$

[2] In our notation, the free Carnot group of step κ with m generators has topological dimension N. The homogeneous dimension Q has the property that $\mathcal{L}^N(B_r) = Cr^Q$, where B_r is a Carnot–Carathéodory ball of radius $r > 0$ centered at any point.

We used the equivalence

$$\int_{|h_j|\le\delta_0^{\ell_j}} \frac{\psi(h_j)dh_j}{|h_j|^{1+ps/\ell_j}} \simeq \int_{\|h\|<\delta_0} \frac{\psi(h_j)dh}{\|h\|^{Q+ps}}, \tag{32}$$

valid for all $\psi = \psi(h_j)$ nonnegative and measurable, with $h = (h_1,\dots,h_N)$, $\|h\| = \max_{k=1,\dots,N}|h_k|^{1/\ell_k}$. See [11, (40) and (41)]. To conclude the proof, introducing the exponential map $\widetilde{\Phi}_\xi(h) = e^{h_1\widetilde{Y}_1+\cdots+h_N\widetilde{Y}_N}(\xi)$, by the triangle inequality, we have

$$\begin{aligned}(*) \le \int_G d\xi \int_{\|h\|\le\delta_0} \frac{dh}{\|h\|^{Q+ps}} |f(\widetilde{\Phi}_\xi(h)) - f(\xi)|^p \\ + \int_G d\xi \int_{\|h\|\le\delta_0} \frac{dh}{\|h\|^{Q+ps}} |f(\widetilde{\Phi}_\xi(h)) - f(e^{h_j\alpha\widetilde{Y}_j}\xi)|^p.\end{aligned}$$

To estimate the first term we just use the change of variable $h \mapsto \widetilde{\Phi}_\xi(h) =: \eta$, which is nonsingular because the vector fields $\widetilde{Y}_1,\dots,\widetilde{Y}_N$ are linearly independent. To estimate the second one, we must choose a sufficiently small $\alpha > 0$ so that, roughy speaking, $e^{h_j\alpha\widetilde{Y}_j}(\xi)$ stays rather close to ξ and the second term admits an analogous estimate

$$(*) \le \int_{\widetilde{G}\times\widetilde{G}} \frac{|u(\xi)-u(\eta)|^p}{\widetilde{d}(\xi,\eta)^{Q+ps}} d\xi d\eta. \tag{33}$$

See [11] for a detailed explaination. □

Proof of Lemma 4.1. We follow the argument of the proof of Proposition 4.2 in [11, p. 237-238]. Let $X_1,\dots,X_m$ be a family of Hörmander vector fields of step $\kappa \in \mathbb{N}$. Let us choose a commutator X_w with length $|w| \le \kappa$. Fix $x_0 \in \mathbb{R}^n$ and introduce the vector fields $\widetilde{X}_1,\dots,\widetilde{X}_m$ on the sets $U \times V \subset \mathbb{R}^N$ appearing in (29) and (30). Here N and Q are the topological and homogeneous dimension of the free Carnot group of step κ with m generators. By properties of free Lie algebras, see [13, p. 272], we may choose the vector fields $\widetilde{Y}_1,\dots,\widetilde{Y}_N$ in (30) in the family $\widetilde{Y}_1,\dots\widetilde{Y}_q$ assuming that $\widetilde{X}_w = \widetilde{Y}_j$ for some $j \le N$ with $\ell_j = |w|$.

We apply first Lemma 4.3 to a set of the form $G = O \times H \Subset \widetilde{O} \times \widetilde{H} \Subset U \times V$, where $\widetilde{H}$ is a small open neighborhood of the origin in $\mathbb{R}^{N-n}$. Recall also that the function f does not depend on the additional variables in V. Then we get the inequality

$$
\begin{aligned}
&\int_O dx \int_{|t|<\delta_0^{|w|},\, e^{\alpha t X_w}x\in O} \frac{dt}{|t|^{1+ps/|w|}}|f(e^{\alpha t X_w}(x))-u(x)|^p \\
&\quad\le C\int_{\widetilde{O}\times\widetilde{H}} dx d\tau \int_{\widetilde{O}\times\widetilde{H}} dy d\sigma \frac{|f(x)-f(y)|^p}{\widetilde{d}((x,\tau),(y,\sigma))^{Q+ps}} \\
&\quad= C\int_{\widetilde{O}\times\widetilde{O}} dx dy |f(x)-f(y)|^p \int_{\widetilde{H}\times\widetilde{H}} d\tau d\sigma \frac{1}{\widetilde{d}((x,\tau),(y,\sigma))^{Q+ps}} \\
&\quad\le C\int_{\widetilde{O}\times\widetilde{O}} dx dy \frac{|f(x)-f(y)|^p}{d(x,y)^{ps}\mathcal{L}^n(B(x,d(x,y)))},
\end{aligned}
$$

by Lemma 4.2.

Covering any given open bounded Ω set with a finite family of open sets O of the form appearing in the discussion above, we obtain the proof of the inequality (28) on Ω. □

REFERENCES

[1] Marco Bramanti, Luca Brandolini, and Marco Pedroni, *Basic properties of non-smooth Hörmander's vector fields and Poincaré's inequality*, Forum Math. **25** (2013), no. 4, 703–769.

[2] Bruno Franchi and Ermanno Lanconelli, *An embedding theorem for Sobolev spaces related to nonsmooth vector fields and Harnack inequality*, Comm. Partial Differential Equations **9** (1984), no. 13, 1237–1264.

[3] C. Fefferman and D. H. Phong, *Subelliptic eigenvalue problems*, Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 590–606. MR 730094 (86c:35112)

[4] Lars Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147–171.

[5] Piotr Hajłasz, *Sobolev spaces on an arbitrary metric space*, Potential Anal. **5** (1996), no. 4, 403–415.

[6] David Jerison, *The Poincaré inequality for vector fields satisfying Hörmander's condition*, Duke Math. J. **53** (1986), no. 2, 503–523.

[7] Ermanno Lanconelli and Daniele Morbidelli, *On the Poincaré inequality for vector fields*, Ark. Mat. **38** (2000), no. 2, 327–342.

[8] Annamaria Montanari and Daniele Morbidelli, *Balls defined by nonsmooth vector fields and the Poincaré inequality*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 2, 431–452.

[9] Annamaria Montanari and Daniele Morbidelli, *Nonsmooth Hörmander vector fields and their control balls.*, Trans. Am. Math. Soc. **364** (2012), no. 5, 2339–2375.

[10] Annamaria Montanari and Daniele Morbidelli, *Almost exponential maps and integrability results for a class of horizontally regular vector fields*, Potential Anal. **38** (2013), no. 2, 611–633.

[11] Daniele Morbidelli, *Fractional Sobolev norms and structure of Carnot-Carathéodory balls for Hörmander vector fields*, Studia Math. **139** (2000), no. 3, 213–244.

[12] Alexander Nagel, Elias M. Stein, and Stephen Wainger, *Balls and metrics defined by vector fields. I. Basic properties*, Acta Math. **155** (1985), no. 1-2, 103–147.

[13] Linda Preiss Rothschild and E. M. Stein, *Hypoelliptic differential operators and nilpotent groups*, Acta Math. **137** (1976), no. 3-4, 247–320.

[14] Nageswari Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana **16** (2000), no. 2, 243–279.

[15] Brian Street, *Multi-parameter Carnot-Carathéodory balls and the theorem of Frobenius*, Rev. Mat. Iberoam. **27** (2011), no. 2, 645–732.

[16] Terence Tao and James Wright, *L^p improving bounds for averages along curves*, J. Amer. Math. Soc. **16** (2003), 605–638.

ANNAMARIA MONTANARI
Dipartimento di Matematica
Università di Bologna

DANIELE MORBIDELLI
Dipartimento di Matematica
Università di Bologna