LE MATEMATICHE
Vol. LXXVI (2021) – Issue I, pp. 79–96
doi: 10.4418/2021.76.1.5

SEQUENTIAL EFFICIENCY OPTIMALITY CONDITIONS FOR MULTIOBJECTIVE FRACTIONAL PROGRAMMING PROBLEMS VIA SEQUENTIAL SUBDIFFERENTIAL CALCULUS

M. B. MOUSTAID - M. LAGHDIR - I. DALI - A. RIKOUANE

The purpose of this paper is to establish sequential efficient optimality conditions, without any constraint qualification, characterizing an efficient solution for multiobjective fractional programming problem. The approach used in this investigation is based on sequential subdifferential calculus. By using the same approach, we establish the standard optimality conditions under a constraint qualification. Finally, we present an example illustrating the main result of this paper.

1. Introduction

In this paper, we consider the following multiobjective fractional programming problem

$$(\mathrm{P}) \quad \inf_{\substack{x\in C\\ h(x)\in -Y_+}} \left\{ \frac{f_1(x)}{g_1(x)}, \dots, \frac{f_p(x)}{g_p(x)} \right\}$$

where $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are two Banach spaces, C is a nonempty convex subset of X, Y_+ is a nonempty closed convex cone of Y, f_i, $g_i : X \longrightarrow \mathbb{R}$, $i = 1, \dots, p$ are proper convex functions and $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is a proper and Y_+-convex mapping. Moreover, we suppose that $f_i(x) \geq 0$ and $g_i(x) > 0$ for all $x \in C \cap h^{-1}(-Y_+)$.

Fractional programming problems arise from many applied areas such as portfolio selection and game theory. So, in this paper, we consider solutions defined as follows: let $\bar{x}$ be a

Received on December 1, 2019

AMS 2010 Subject Classification: 90C32, 90C46

Keywords: Multiobjective fractional programming; Efficient solution; Subdifferential; Sequential optimality conditions

feasible point of (P) i.e. $\bar{x} \in C \cap h^{-1}(-Y_+)$. The point $\bar{x}$ is called an efficient solution of (P) if there is no $x \in C \cap h^{-1}(-Y_+)$ such that

$$\frac{f_i(x)}{g_i(x)} \le \frac{f_i(\bar{x})}{g_i(\bar{x})}, \text{ for all } i \in \{1,\ldots,p\}$$

with at least one strict inequality.

In order to investigate optimality conditions for a vector optimization, we often use a parametric approach in order to formulate a corresponding equivalent scalar convex problem and one needs to impose some kinds of constraint qualifications but the constraint qualifications do not always hold for finite-dimensional convex programs and frequently fail for infinite-dimensional convex programs. These drawbacks lead many authors to derive optimality conditions for convex optimization problems without any constraint qualifications (see [2–4, 6, 8, 10–13]).

The purpose of this paper is to establish sequential optimality conditions in the absence of any constraint qualification for multiobjective fractional optimization problems characterizing completely an efficient solution by using a new approach based on sequential subdifferential calculus.

The paper is structured as follows. In Section 2, we recall some basic definitions, notations from convex analysis and auxiliary results describing important properties of conjugate functions and subdifferentials that will be used later in the paper. Section 3, is devoted to provide sequential subdifferential calculus rule for the sums of p $(p \ge 2)$ scalar functions and the composition of a scalar and vector mapping under convexity and lower semicontinuity hypotheses without assuming qualification conditions. In Section 4, we develop sequential efficiency optimality conditions for multiobjective fractional programming problem (P). In Section 5, we establish the standard optimality conditions under a constraint qualification and we present an example illustrating the main result of this paper.

2. Preliminaries

Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two Banach spaces and $(X^*, \|.\|_{X^*})$ and $(Y^*, \|.\|_{Y^*})$ be their topological dual spaces paired in duality by $\langle .,. \rangle$. Let $Y_+ \subset Y$ be a nontrivial convex cone. The positive polar cone Y_+^* of Y_+ is the set of $y^* \in Y^*$ such that $y^*(Y_+) \subset \mathbb{R}_+$. The space Y is ordered by the relation

$$y_1,\, y_2 \in Y, \quad y_1 \le_{Y_+} y_2 \Longleftrightarrow y_2 - y_1 \in Y_+$$

and we adjoin to Y an element $+\infty_Y$, which is the supremum with respect to $\le_{Y_+}$. It holds that $y \le_{Y_+} +\infty_Y$ for every $y \in Y$. The algebraic operations of Y are extended as follows

$$y + (+\infty_Y) = (+\infty_Y) + y = +\infty_Y, \quad \alpha.(+\infty_Y) = +\infty_Y, \quad \forall y \in Y,\, \forall \alpha > 0.$$

For a given mapping $f : X \longrightarrow Y \cup \{+\infty_Y\}$, the sets

$$\begin{aligned} \text{dom} f &:= \{x \in X : \ f(x) \in Y\}, \\ \text{epi} f &:= \{(x,y) \in X \times Y : \ f(x) \le_{Y_+} y\}, \end{aligned}$$

are called respectively the effective domain and the epigraph of f. We say that f is proper if its domain is a nonempty set. The mapping f is said to be Y_+-convex if, for every $\lambda \in [0,1]$, $x_1, x_2 \in X$, we have $f(\lambda x_1 + (1-\lambda)x_2) \leq_{Y_+} \lambda f(x_1) + (1-\lambda) f(x_2)$. Further, f is said to be Y_+-epi-closed, if its epigraph epi$f$ is closed (see [1]).
A function $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is said to be Y_+-nondecreasing, if for each $y_1, y_2 \in Y$ we have

$$y_1 \leq_{Y_+} y_2 \Longrightarrow g(y_1) \leq g(y_2).$$

The composite function $g \circ f \ : \ X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is defined by

$$(g \circ f)(x) := \begin{cases} g(f(x)) & \text{if } x \in \text{domf} \\ \\ +\infty & \text{otherwise.} \end{cases}$$

Let $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a given function. The subdifferential of f at a point $\bar{x} \in \text{dom} f$ defined as follows

$$\partial f(\bar{x}) := \{x^* \in X^* : \ \langle x^*, x - \bar{x} \rangle + f(\bar{x}) \leq f(x), \quad \forall x \in X\}.$$

The ε-subdifferential ($\varepsilon \geq 0$) of f at a point $\bar{x} \in \text{dom} f$ is given by

$$\partial_\varepsilon f(\bar{x}) := \{x^* \in X^* : \ \langle x^*, x - \bar{x} \rangle + f(\bar{x}) - \varepsilon \leq f(x), \quad \forall x \in X\}.$$

The conjugate function of f is defined by

$$\begin{array}{rccl} f^* : & X^* & \longrightarrow & \overline{\mathbb{R}} \\ & x^* & \longmapsto & f^*(x^*) := \sup\limits_{x \in X} \{\langle x^*, x \rangle - f(x)\}. \end{array}$$

The scalar indicator function of a nonempty subset $C \subset X$, denoted by δ_C, is defined as $\delta_C : X \longrightarrow \mathbb{R} \cup \{+\infty\}$

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C \\ \\ +\infty & \text{otherwise.} \end{cases}$$

The normal cone of C at $\bar{x}$ is defined by

$$N(\bar{x}, C) := \{x^* \in X^* : \quad \langle x^*, x - \bar{x} \rangle \leq 0, \quad \forall x \in C\}.$$

Lemma 2.1. *Let $y^* \in Y^*$ and $z^* \in Z^*$. We have*

i) If $\bar{y} \in Y_+$, then

$$y^* \in N(\bar{y}, Y_+) \Longleftrightarrow \begin{cases} y^* \in -Y_+^* \\ \langle y^*, \bar{y} \rangle = 0. \end{cases}$$

ii) If $\bar{y} \in -Y_+$, then

$$y^* \in N(\bar{y}, -Y_+) \Longleftrightarrow \begin{cases} y^* \in Y_+^* \\ \langle y^*, \bar{y} \rangle = 0. \end{cases}$$

iii) If $\bar{y} \in Y_+$ and $\bar{z} \in Z_+$ then

$$(y^*, z^*) \in N((\bar{y}, \bar{z}), Y_+ \times Z_+) \Longleftrightarrow \begin{cases} y^* \in -Y_+^*, \; z^* \in -Z_+^* \\ \langle y^*, \bar{y} \rangle + \langle z^*, \bar{z} \rangle = 0. \end{cases}$$

iv) If $\bar{y} \in -Y_+$ and $\bar{z} \in -Z_+$ then

$$(y^*, z^*) \in N((\bar{y}, \bar{z}), -(Y_+ \times Z_+)) \Longleftrightarrow \begin{cases} y^* \in Y_+^*, \; z^* \in Z_+^* \\ \langle y^*, \bar{y} \rangle + \langle z^*, \bar{z} \rangle = 0. \end{cases}$$

Proof. i) We have

$$y^* \in N(\bar{y}, Y_+) \Longleftrightarrow \langle y^*, y - \bar{y} \rangle \leq 0, \quad \forall y \in Y_+. \tag{1}$$

As Y_+ is a convex cone, we have for any $y \in Y_+$, $y + \bar{y} \in Y_+$ and hence it follows from (1) that $\langle y^*, y \rangle \leq 0$, for any $y \in Y_+$ i.e. $y^* \in -Y_+^*$. By taking in (1) $y := 0_Y$, we obtain $0 \leq \langle y^*, \bar{y} \rangle$, then we have $\langle y^*, \bar{y} \rangle = 0$. Conversely, let $y^* \in -Y_+^*$, we have

$$\begin{cases} \langle y^*, y \rangle \leq 0, \quad \forall y \in Y_+ \\ -\langle y^*, \bar{y} \rangle = 0. \end{cases}$$

By adding them up, we have $\langle y^*, y - \bar{y} \rangle \leq 0$, for any $y \in Y_+$ i.e. $y^* \in N(\bar{y}, Y_+)$.
ii) It is immediate from i) by taking the convex cone $-Y_+$ instead of Y_+.
iii) and iv) follow by using the same arguments used in the proof of i).

□

Let us recall a version of the Brondsted-Rockafellar theorem which was established in [13].

Theorem 2.2. *Let X be a Banach space and $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Then for any real $\varepsilon > 0$ and $\bar{x}^* \in \partial_\varepsilon f(\bar{x})$, there exist $x \in \mathrm{dom} f$ and $x^* \in \partial f(x)$ such that*

i) $\|x - \bar{x}\| \leq \sqrt{\varepsilon}$,

ii) $\|x^* - \bar{x}^*\| \leq \sqrt{\varepsilon}$,

iii) $|f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle| \leq 2\varepsilon$.

In what follows, we will need two important contributions by Hiriart-Urruty et al.[7]. The first is given by the following proposition

Proposition 2.3. *([7]) Let f be a proper, convex and lower semicontinuous function, assume that $\bar{x} \in \mathrm{dom} f$ and $\varepsilon > 0$, then we have*

$$\delta^*_{\partial_\varepsilon f(\bar{x})}(d) = \inf_{t > 0} \left\{ \frac{f(\bar{x} + td) - f(\bar{x}) + \varepsilon}{t} \right\}.$$

The second expresses without qualification condition, the subdifferential $\partial(f_1+f_2)(\bar{x})$ of two convex proper and lower semicontinuous functions where $\bar{x}\in \mathrm{dom} f_1\cap \mathrm{dom} f_2$, in terms of the approximate subdifferentials of f_1 and f_2, given by

$$\partial(f_1+f_2)(\bar{x}) = \bigcap_{\varepsilon>0} \mathrm{cl}_{w^*}\left(\partial_\varepsilon f_1(\bar{x})+\partial_\varepsilon f_2(\bar{x})\right)$$

where the notation cl_{w^*} stands for the weak star closure.

In order to establish our main result, we will need to extend the above formula to the case of p functions $(p\geq 2)$.

Theorem 2.4. *Let X be a locally convex vector space and $f_1,\ldots,f_p : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be p proper, convex and lower semicontinuous functions. Let $\bar{x}\in \bigcap_{i=1}^{p} \mathrm{dom} f_i$, then we have*

$$\partial(\sum_{i=1}^{p} f_i)(\bar{x}) = \bigcap_{\varepsilon>0} \mathrm{cl}_{w^*}\left(\partial_\varepsilon f_1(\bar{x})+\ldots+\partial_\varepsilon f_p(\bar{x})\right).$$

Proof. We use the same arguments used in the proof of [Theorem 3.1, [7]] for the case of two convex functions. Suppose that $x^*\in\partial\left(\sum_{i=1}^{p} f_i\right)(\bar{x})$ and $x^*\notin \bigcap_{\varepsilon>0} \mathrm{cl}_{w^*}\left(\partial_\varepsilon f_1(\bar{x})+\ldots+\partial_\varepsilon f_p(\bar{x})\right)$ then there exists $\varepsilon>0$ such that $x^*\notin \mathrm{cl}_{w^*}\left(\partial_\varepsilon f_1(\bar{x})+\ldots+\partial_\varepsilon f_p(\bar{x})\right)$. By virtue of Hahn-Banach theorem's, there exists $d\in X$ such that

$$<x^*,d> \quad > \quad \delta^*_{\mathrm{cl}_{w^*}(\partial_\varepsilon f_1(\bar{x})+\ldots+\partial_\varepsilon f_p(\bar{x}))}(d) = \sum_{i=1}^{p}\delta^*_{\partial_\varepsilon f_i(\bar{x})}(d).$$

It follows from Proposition 2.3 that there exist strictly positive numbers $t_1,\ldots,t_p$ such that

$$<x^*,d> \quad > \quad \sum_{i=1}^{p}\frac{f_i(\bar{x}+t_i d)-f_i(\bar{x})+\varepsilon}{t_i}.$$

By taking $\eta = \underset{1\leq i\leq p}{min}\, t_i$ and $\tau = \underset{1\leq i\leq p}{max}\, t_i$, we have

$$\frac{f_i(\bar{x}+t_i d)-f(\bar{x})}{t_i} \geq \frac{f_i(\bar{x}+\eta d)-f(\bar{x})}{\eta} \text{ and } \frac{\varepsilon}{\eta}\geq\frac{\varepsilon}{t_i}, \quad \forall i\in\{1,\ldots,p\}.$$

Hence

$$<x^*,d> \quad > \quad \sum_{i=1}^{p}\frac{f_i(\bar{x}+\eta d)-f_i(\bar{x})}{\eta}+\frac{p\varepsilon}{\tau},$$

which yields

$$<x^*,\eta d> \quad > \quad \sum_{i=1}^{p} f_i(\bar{x}+\eta d)-\sum_{i=1}^{p} f_i(\bar{x}),$$

this contradicts the fact that $x^* \in \partial(\sum_{i=1}^{p} f_i)(\bar{x})$.
The reverse inclusion is easy to prove, since it suffices to observe that

$$\partial_\varepsilon f_1(\bar{x}) + \ldots + \partial_\varepsilon f_p(\bar{x}) \subset \partial f_1(\bar{x}) + \ldots + \partial f_p(\bar{x}) \subset \partial(\sum_{i=1}^{p} f_i)(\bar{x})$$

for any $\varepsilon > 0$ and $\partial(\sum_{i=1}^{p} f_i)(\bar{x})$ is weak star closure. The proof is then complete.

Remark 2.5. When X is a reflexive Banach space and as $\partial_\varepsilon f(\bar{x})$ is convex, the above theorem holds if we take the closure of the convex set

$$\partial_\varepsilon f_1(\bar{x}) + \ldots + \partial_\varepsilon f_p(\bar{x})$$

with respect to the norm closure $\mathrm{cl}_{\|\cdot\|_{X^*}}$ instead of the weak star closure cl_{w^*}.

□

3. Sequential subdifferential calculus

In this section, without considering any qualification condition, we establish sequential formula for the subdifferential of the convex function $(\sum_{i=1}^{p} f_i + g \circ h)$ in terms of the subdifferentials of the data functions at nearby points, where $f_i : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ $(i = 1, \ldots, p)$ are proper convex functions, $g : Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex and Y_+-nondecreasing function and $h : X \longrightarrow Y \cup \{+\infty_Y\}$ is a proper and Y_+-convex mapping. On $X \times Y$ we use the norm $\| (x,y) \|_{X \times Y} = \sqrt{\| x \|_X^2 + \| y \|_Y^2}$, for $(x,y) \in X \times Y$. Similarly, we define the norm on $X^* \times Y^*$. Let us consider the following auxiliary functions defined by

$$\begin{array}{rrcl} F_i : & X \times Y & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & (x,y) & \longrightarrow & F_i(x,y) := f_i(x), \end{array} \qquad (i = 1, \ldots, p)$$

$$\begin{array}{rrcl} G : & X \times Y & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & (x,y) & \longrightarrow & G(x,y) := g(y), \end{array}$$

$$\begin{array}{rrcl} H : & X \times Y & \longrightarrow & \mathbb{R} \cup \{+\infty\} \\ & (x,y) & \longrightarrow & H(x,y) := \delta_{\mathrm{epi}h}(x,y). \end{array}$$

Lemma 3.1. *([9]) For any $(\bar{x},\bar{y}) \in (\mathrm{dom} f_i \times \mathrm{dom} g) \cap \mathrm{epi} h$, $(i = 1, \ldots, p)$. We have*

i) $\partial F_i(\bar{x},\bar{y}) = \partial f_i(\bar{x}) \times \{0\}, \quad (i = 1, \ldots, p).$

ii) $\partial G(\bar{x},\bar{y}) = \{0\} \times \partial g(\bar{y}).$

iii)

$$(x^*,y^*) \in \partial H(\bar{x},\bar{y}) \Longleftrightarrow \begin{cases} (\bar{x},\bar{y}) \in \mathrm{epi}h \\ x^* \in \partial(-y^* \circ h)(\bar{x}) \\ y^* \in N(\bar{y}-h(\bar{x}),Y_+). \end{cases}$$

Theorem 3.2. *Let X and Y be two reflexive Banach spaces. Let $f_1,\dots,f_p : X \longrightarrow \mathbb{R}\cup\{+\infty\}$ be p proper, convex and lower semicontinuous functions, $g : Y \longrightarrow \mathbb{R}\cup\{+\infty\}$ be proper, convex, lower semicontinuous and Y_+-nondecreasing function and $h : X \longrightarrow Y\cup\{+\infty_Y\}$ be proper, Y_+-convex and Y_+-epi-closed mapping. Let $\bar{x} \in (\bigcap_{i=1}^{p} \mathrm{dom} f_i) \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g)$. Then, $x^* \in \partial\left(\sum_{i=1}^{p} f_i + g\circ h\right)(\bar{x})$ if and only if there exist $x_{i,n} \in \mathrm{dom} f_i$, $x^*_{i,n} \in X^*$ $(i=1,\dots,p)$, $y_n \in \mathrm{dom} g$, $(u_n,v_n) \in \mathrm{epi}h$, $u^*_n \in X^*$, $y^*_n \in Y^*$ and $v^*_n \in -Y^*_+$, satisfying*

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x}\ (i=1,\dots,p),\ u_n \xrightarrow{\|\cdot\|_X} \bar{x},\ y_n \xrightarrow{\|\cdot\|_Y} h(\bar{x}),\ v_n \xrightarrow{\|\cdot\|_Y} h(\bar{x}) \\ x^*_{i,n} \in \partial f_i(x_{i,n})\ (i=1,\dots,p),\ y^*_n \in \partial g(y_n)\ u^*_n \in \partial(-v^*_n \circ h)(u_n), \\ \langle v^*_n, h(u_n)-v_n\rangle = 0 \end{cases}$$

and

$$\begin{cases} (\sum_{i=1}^{p} x^*_{i,n}) + u^*_n \xrightarrow{\|\cdot\|_{X^*}} x^*,\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0 \\ f_i(x_{i,n}) - \langle x^*_{i,n}, x_{i,n}-\bar{x}\rangle \longrightarrow f_i(\bar{x}) \quad (i \in \{1,\dots,p\}) \\ g(y_n) - \langle y^*_n, y_n - h(\bar{x})\rangle \longrightarrow g(h(\bar{x})) \\ \langle u^*_n, u_n-\bar{x}\rangle + \langle v^*_n, v_n - h(\bar{x})\rangle \longrightarrow 0. \end{cases}$$

Proof. $(\Longrightarrow)$ For any $x \in X$, one has

$$\left(\sum_{i=1}^{p} f_i + g\circ h\right)(x) = \inf_{y\in Y}\left\{\sum_{i=1}^{p} F_i(x,y) + G(x,y) + H(x,y)\right\}.$$

Then, it is not difficult to see for $\bar{x} \in (\bigcap_{i=1}^{p} \mathrm{dom} f_i) \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g)$, that

$$x^* \in \partial\left(\sum_{i=1}^{p} f_i + g\circ h\right)(\bar{x}) \Longleftrightarrow (x^*,0) \in \partial(\sum_{i=1}^{p} F_i + G + H)(\bar{x},h(\bar{x})). \tag{2}$$

The functions F_i $(i=1,\ldots,p)$ and G are proper, convex and lower semicontinuous and as epi$h$ is nonempty, convex and closed, it follows that H is proper, convex and lower semicontinuous. The condition $\bar{x} \in (\bigcap_{i=1}^{p} \mathrm{dom} f_i) \cap \mathrm{dom} h \cap h^{-1}(\mathrm{dom} g)$ can be written equivalently as $(\bar{x}, h(\bar{x})) \in (\bigcap_{i=1}^{p} \mathrm{dom} F_i) \cap \mathrm{dom} G \cap \mathrm{dom} H$. Thus, the functions F_i $(i=1,\ldots,p)$, G and H, satisfy together all the assumptions of Theorem 2.4 and hence it follows from (2) that

$$(x^*,0) \in \bigcap_{n\in\mathbb{N}^*} \mathrm{cl}_{\|.\|_{X^*\times Y^*}} \left\{ \partial_{\frac{1}{n}} F_1(\bar{x},h(\bar{x})) + \ldots + \partial_{\frac{1}{n}} G(\bar{x},h(\bar{x})) + \partial_{\frac{1}{n}} H(\bar{x},h(\bar{x})) \right\}$$

and therefore, there exist $(\bar{x}^*_{i,n}, \bar{y}^*_{i,n}) \in \partial_{\frac{1}{n}} F_i(\bar{x},h(\bar{x}))$ $(i=1,\ldots,p)$, $(\bar{x}^*_n, \bar{y}^*_n) \in \partial_{\frac{1}{n}} G(\bar{x},h(\bar{x}))$ and $(\bar{u}^*_n, \bar{v}^*_n) \in \partial_{\frac{1}{n}} H(\bar{x},h(\bar{x}))$, satisfying

$$\sum_{i=1}^{p} (\bar{x}^*_{i,n}, \bar{y}^*_{i,n}) + (\bar{x}^*_n, \bar{y}^*_n) + (\bar{u}^*_n, \bar{v}^*_n) \xrightarrow{\|.\|_{X^*\times Y^*}} (x^*,0). \tag{3}$$

According to Theorem 2.2, there exist $(x_{i,n}, y_{i,n}) \in \mathrm{dom} F_i$ $(i=1,\ldots,p)$, $(x_n,y_n) \in \mathrm{dom} G$, $(u_n,v_n) \in \mathrm{dom} H$, $(x^*_{i,n}, y^*_{i,n})$, (x^*_n, y^*_n), $(u^*_n, v^*_n) \in X^* \times Y^*$ such that

$$(x^*_{i,n}, y^*_{i,n}) \in \partial F_i(x_{i,n}, y_{i,n}),\ (x^*_n, y^*_n) \in \partial G(x_n,y_n),\ (u^*_n, v^*_n) \in \partial H(u_n,v_n) \tag{4}$$

$$\| (x_{i,n}, y_{i,n}) - (\bar{x}, h(\bar{x})) \|_{X\times Y} \leq \frac{1}{\sqrt{n}} \tag{5}$$

$$\| (x_n, y_n) - (\bar{x}, h(\bar{x})) \|_{X\times Y} \leq \frac{1}{\sqrt{n}} \tag{6}$$

$$\| (u_n, v_n) - (\bar{x}, h(\bar{x})) \|_{X\times Y} \leq \frac{1}{\sqrt{n}} \tag{7}$$

$$\| (\bar{x}^*_{i,n}, \bar{y}^*_{i,n}) - (x^*_{i,n}, y^*_{i,n}) \|_{X^*\times Y^*} \leq \frac{1}{\sqrt{n}} \tag{8}$$

$$\| (\bar{x}^*_n, \bar{y}^*_n) - (x^*_n, y^*_n) \|_{X^*\times Y^*} \leq \frac{1}{\sqrt{n}} \tag{9}$$

$$\| (\bar{u}^*_n, \bar{v}^*_n) - (u^*_n, v^*_n) \|_{X^*\times Y^*} \leq \frac{1}{\sqrt{n}} \tag{10}$$

$$| F_i(x_{i,n}, y_{i,n}) - \langle (x^*_{i,n}, y^*_{i,n}), (x_{i,n}, y_{i,n}) - (\bar{x}, h(\bar{x})) \rangle - F_i(\bar{x}, h(\bar{x})) | \leq \frac{2}{n} \tag{11}$$

$$| G(x_n, y_n) - \langle (x^*_n, y^*_n), (x_n, y_n) - (\bar{x}, h(\bar{x}) \rangle - G(\bar{x}, h(\bar{x})) | \leq \frac{2}{n} \tag{12}$$

$$| H(u_n, v_n) - \langle (u^*_n, v^*_n), (u_n, v_n) - (\bar{x}, h(\bar{x})) \rangle - H(\bar{x}, h(\bar{x})) | \leq \frac{2}{n}. \tag{13}$$

By applying Lemma 3.1, the expression (4) can be expressed by means of data functions f_i, g and h as follow

$$\begin{cases} x^*_{i,n} \in \partial f_i(x_{i,n}), \quad y^*_{i,n} = 0, \quad (i=1,\ldots,p) \\ y^*_n \in \partial g(y_n), \quad x^*_n = 0 \\ u^*_n \in \partial(-v^*_n \circ h)(u_n),\ v^*_n \in N(v_n - h(u_n)), Y_+). \end{cases}$$

By letting $n \longrightarrow +\infty$, we get from (5), (6), (7), (11), (12), (13) that

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \overline{x}\ (i=1,\dots,p),\ u_n \xrightarrow{\|\cdot\|_X} \overline{x},\ y_n \xrightarrow{\|\cdot\|_Y} h(\overline{x}),\ v_n \xrightarrow{\|\cdot\|_Y} h(\overline{x}) \\ f_i(x_{i,n}) - \langle x_{i,n}^*, x_{i,n} - \overline{x}\rangle \longrightarrow f_i(\overline{x}) \quad (i=1,\dots,p) \\ g(y_n) - \langle y_n^*, y_n - h(\overline{x})\rangle \longrightarrow g(h(\overline{x})) \\ \langle u_n^*, u_n - \overline{x}\rangle + \langle v_n^*, v_n - h(\overline{x})\rangle \longrightarrow 0. \end{cases}$$

Moreover, since

$$\begin{aligned} &\| \sum_{i=1}^p x_{i,n}^* + u_n^* - x^* \|_{X^*} \\ &\quad = \| \sum_{i=1}^p x_{i,n}^* - \sum_{i=1}^p \overline{x}_{i,n}^* + u_n^* - \overline{u}_n^* - \overline{x}_n^* + \overline{x}_n^* + \sum_{i=1}^p \overline{x}_{i,n}^* + \overline{u}_n^* - x^* \|_{X^*} \\ &\quad \le \sum_{i=1}^p \|x_{i,n}^* - \overline{x}_{i,n}^*\|_{X^*} + \|u_n^* - \overline{u}_n^*\|_{X^*} + \| \sum_{i=1}^p \overline{x}_{i,n}^* + \overline{x}_n^* + \overline{u}_n^* - x^* \|_{X^*} + \|\overline{x}_n^*\|_{X^*}, \end{aligned}$$

and

$$\begin{aligned} \|y_n^* + v_n^*\|_{Y^*} &= \|y_n^* - \overline{y}_n^* + v_n^* - \overline{v}_n^* - \sum_{i=1}^p \overline{y}_{i,n}^* + \sum_{i=1}^p \overline{y}_{i,n}^* + \overline{y}_n^* + \overline{v}_n^*\|_{Y^*} \\ &\le \|y_n^* - \overline{y}_n^*\|_{Y^*} + \|v_n^* - \overline{v}_n^*\|_{Y^*} + \sum_{i=1}^p \|\overline{y}_{i,n}^*\|_{Y^*} + \| \sum_{i=1}^p \overline{y}_{i,n}^* + \overline{y}_n^* + \overline{v}_n^*\|_{Y^*}, \end{aligned}$$

it follows from (8), (9) and (10), by letting $n \longrightarrow +\infty$, that

$$(\sum_{i=1}^p x_{i,n}^*) + u_n^* \xrightarrow{\|\cdot\|_{X^*}} x^*,\ y_n^* + v_n^* \xrightarrow{\|\cdot\|_{Y^*}} 0.$$

By applying Lemma 2.1 to $v_n^* \in N(v_n - h(u_n), Y_+)$, we get

$$v_n^* \in -Y_+^*, \qquad \langle v_n^*, h(u_n) - v_n\rangle = 0,$$

and hence we obtain the desired result.

($\Longleftarrow$) Assume that the preceding conditions holds. Then, we have

$$\begin{aligned} \langle x_{i,n}^*, x - x_{i,n}\rangle + f_i(x_{i,n}) &\le f_i(x), \quad \forall x \in X,\ (i=1,\dots,p) \\ \langle y_n^*, y - y_n\rangle + g(y_n) &\le g(y), \quad \forall y \in Y \\ \langle u_n^*, u - u_n\rangle - (v_n^* \circ h)(u_n) &\le -(v_n^* \circ h)(u), \quad \forall u \in X \\ \langle v_n^*, h(u_n) - v_n\rangle &= 0. \end{aligned}$$

By summing the terms of the above inequalities, we obtain

$$\begin{aligned} \sum_{i=1}^p \langle x_{i,n}^*, x - x_{i,n}\rangle &+ \langle y_n^*, y - y_n\rangle + \langle u_n^*, u - u_n\rangle + \langle v_n^*, h(u_n) - v_n\rangle \\ &+ \sum_{i=1}^p f_i(x_{i,n}) + g(y_n) - (v_n^* \circ h)(u_n) \\ &\le \sum_{i=1}^p f_i(x) + g(y) - (v_n^* \circ h)(u), \quad \forall x, u \in X,\ \forall y \in Y. \end{aligned}$$

The above inequality may be rewritten as

$$\begin{aligned}
\sum_{i=1}^{p}[f_i(x_{i,n})-\langle x_{i,n}^*,x_{i,n}-\overline{x}\rangle] &+ g(y_n)-\langle y_n^*,y_n-h(\overline{x})\rangle \\
&- [\langle u_n^*,u_n-\overline{x}\rangle+\langle v_n^*,v_n-h(\overline{x})\rangle] \\
&+ \sum_{i=1}^{p}\langle x_{i,n}^*,x-\overline{x}\rangle+\langle y_n^*,y-h(\overline{x})\rangle+\langle u_n^*,u-\overline{x}\rangle \\
&+ \langle v_n^*,h(u_n)-h(\overline{x})\rangle+(v_n^*\circ h)(u)-(v_n^*\circ h)(u_n) \\
&\leq \sum_{i=1}^{p}f_i(x)+g(y), \quad \forall x,u\in X,\ \forall y\in Y.
\end{aligned} \tag{14}$$

By taking in (14) $u=x$ and $y=h(x)$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{p}[f_i(x_{i,n})-\langle x_{i,n}^*,x_{i,n}-\overline{x}\rangle]+g(y_n)-\langle y_n^*,y_n-h(\overline{x})\rangle-\langle u_n^*,u_n-\overline{x}\rangle \\
&+\langle v_n^*,v_n-h(\overline{x})\rangle+\langle\sum_{i=1}^{p}x_{i,n}^*+u_n^*,x-\overline{x}\rangle+\langle y_n^*+v_n^*,h(x)-h(\overline{x})\rangle \\
&\leq\sum_{i=1}^{p}f_i(x)+g(h(x)), \quad \forall x\in X.
\end{aligned}$$

Thus, by taking the limit in both terms ($n\longrightarrow+\infty$) of the above inequality, we deduce that

$$\langle x^*,x-\overline{x}\rangle+\sum_{i=1}^{p}f_i(\overline{x})+g(h(\overline{x}))\leq\sum_{i=1}^{p}f_i(x)+g(h(x)), \quad \forall x\in X,$$

i.e.

$$x^*\in\partial\left(\sum_{i=1}^{p}f_i+g\circ h\right)(\overline{x}).$$

The proof is complete.

□

4. Sequential efficient optimality conditions

In this section, by applying the previous results we present, without any constraint qualification, sequential efficient necessary and sufficient optimality conditions characterizing completely an efficient solution for multiobjective fractional programming problem (P). The following notation will be considered in what follows

$$\nu_i:=\tfrac{f_i(\overline{x})}{g_i(\overline{x})}.$$

We associate to problem (P) the scalar convex minimization problem $(\overline{x}\in X)$

$$(S_{\overline{x}}) \qquad \inf_{\substack{x\in C\cap S(\overline{x})\\ h(x)\in-Y_+}}\sum_{i=1}^{p}(f_i(x)-\nu_i g_i(x))$$

where

$$S(\overline{x}) := \{x \in X : f_i(x) - \nu_i g_i(x) \leq 0,\ \forall i \in \{1,,\ldots,p\}\}.$$

We will need the following lemma

Lemma 4.1. *A point $\overline{x} \in C \cap h^{-1}(-Y_+)$ is an efficient solution of (P) if and only if $\overline{x}$ is a solution of $(S_{\overline{x}})$.*

Proof. $(\Longleftarrow)$ Assume that $\overline{x}$ is not an efficient solution of (P), then there exists $x \in C \cap h^{-1}(-Y_+)$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\overline{x})}{g_i(\overline{x})},\ \forall i \in \{1,\ldots,p\}, \tag{15}$$

$$\frac{f_j(x)}{g_j(x)} < \frac{f_j(\overline{x})}{g_j(\overline{x})} \text{ for some } j \in \{1,\ldots,p\}. \tag{16}$$

Since $g_i(x) > 0$, it follows from (15) and (16) that

$$\begin{cases} f_i(x) - \nu_i g_i(x) \leq 0 = f_i(\overline{x}) - \nu_i g_i(\overline{x}), & \forall i \in \{1,,\ldots,p\} \\ \\ f_j(x) - \nu_j g_j(x) < 0 = f_j(\overline{x}) - \nu_j g_j(\overline{x}), & \text{for some } j \in \{1,\ldots,p\}, \end{cases}$$

which means that $x \in S(\overline{x})$ and adding them up, we get

$$\sum_{i=1}^{p}[f_i(x) - \nu_i g_i(x)] < \sum_{i=1}^{p}[f_i(\overline{x}) - \nu_i g_i(\overline{x})]$$

this leads to a contradiction.

$(\Longrightarrow)$ Conversely, assume that $\overline{x}$ is not a solution of $(S_{\overline{x}})$. Then, there exists $x \in C \cap S(\overline{x}) \cap h^{-1}(-Y_+)$ such that

$$\sum_{i=1}^{p}[f_i(x) - \nu_i g_i(x)] < 0 = \sum_{i=1}^{p}[f_i(\overline{x}) - \nu_i g_i(\overline{x})]. \tag{17}$$

Using the fact that $x \in S(\overline{x})$ we have $f_i(x) - \nu_i g_i(x) \leq 0$, for any $i \in \{1,\ldots,p\}$ and according to (17), it follows that there exists some $j \in \{1,\ldots,p\}$ such that $f_j(x) - \nu_j g_j(x) < 0$. As $g_i(x) > 0$, we obtain

$$\begin{cases} \frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\overline{x})}{g_i(\overline{x})}, & \forall i \in \{1,\ldots,p\} \\ \\ \frac{f_j(x)}{g_j(x)} < \frac{f_j(\overline{x})}{g_j(\overline{x})} & \text{for some } j \in \{1,\ldots,p\} \\ \\ x \in C \cap h^{-1}(-Y_+). \end{cases}$$

This contradicts the fact that $\overline{x}$ is an efficient solution of (P).

$\square$

Using Theorem 3.2 and Lemma 4.1, we obtain the following result

Theorem 4.2. *Let X and Y be two reflexive Banach spaces. Let $h : X \longrightarrow Y \cup \{+\infty_Y\}$ be a proper, Y_+-convex and Y_+-epi-closed mapping. Let f_i and $-g_i : X \longrightarrow \mathbb{R}$ be $2p$ convex and lower semicontinuous functions such that $f_i(x) \geq 0$ and $g_i(x) > 0$ for any $x \in C \cap h^{-1}(-Y_+)$ $(i = 1,\ldots,p)$. Then, $\bar{x} \in C \cap h^{-1}(-Y_+)$ is an efficient solution of (P) if and only if there exist $x_{i,n} \in \mathrm{dom} f_i = X$, $w_{i,n} \in \mathrm{dom}[\nu_i(-g_i)] = X$, $c_n \in \mathrm{dom}\delta_C = C$, $x^*_{i,n} \in X^*$, $w^*_{i,n} \in X^*$, $(i = 1,\ldots,p)$, $c^*_n \in X^*$, $y_n \in -Y_+$, $(\alpha_{1,n},\ldots,\alpha_{p,n}) \in -\mathbb{R}^p_+$, $(u_n, v_n) \in \mathrm{epi}h$, $(\beta_{1,n},\ldots,\beta_{p,n}) \in \mathbb{R}^p$, $u^*_n \in X^*$, $y^*_n \in Y^*_+$, $(\gamma_{1,n},\ldots,\gamma_{p,n}) \in \mathbb{R}^p_+$, $v^*_n \in -Y^*_+$, and $(\lambda_{1,n},\ldots,\lambda_{p,n}) \in -\mathbb{R}^p_+$, satisfying*

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x},\ w_{i,n} \xrightarrow{\|\cdot\|_X} \bar{x},\ c_n \xrightarrow{\|\cdot\|_X} \bar{x},\ u_n \xrightarrow{\|\cdot\|_X} \bar{x},\ (i = 1,\ldots,p) \\ y_n \xrightarrow{\|\cdot\|_Y} h(\bar{x}),\ v_n \xrightarrow{\|\cdot\|_Y} h(\bar{x}),\ \alpha_{i,n} \longrightarrow 0,\ \beta_{i,n} \longrightarrow 0\ (i = 1,\ldots,p) \\ x^*_{i,n} \in \partial f_i(x_{i,n}),\ w^*_{i,n} \in \partial(\nu_i(-g_i))(w_{i,n}),\ \ c^*_n \in N(c_n, C),\ (i = 1,\ldots,p) \\ \langle y^*_n, y_n\rangle + \sum_{i=1}^{p} \gamma_{i,n}\alpha_{i,n} = 0,\ u^*_n \in \partial(-v^*_n \circ h - \sum_{i=1}^{p} \lambda_{i,n} f_i + \sum_{i=1}^{p} \lambda_{i,n}\nu_i g_i)(u_n) \\ (v^*_n \circ h)(u_n) + \sum_{i=1}^{p} \lambda_{i,n}(f_i(u_n) - \nu_i g_i(u_n) - \beta_{i,n}) = 0 \end{cases}$$

and

$$\begin{cases} \sum_{i=1}^{p} x^*_{i,n} + \sum_{i=1}^{p} w^*_{i,n} + c^*_n + u^*_n \xrightarrow{\|\cdot\|_{X^*}} 0,\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0,\ \lambda_{i,n} + \gamma_{i,n} \longrightarrow 0\ (i = 1,\ldots,p) \\ f_i(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \bar{x}\rangle \longrightarrow f_i(\bar{x})\ (i = 1,\ldots,p) \\ \nu_i(-g_i)(w_{i,n}) - \langle w^*_{i,n}, w_{i,n} - \bar{x}\rangle \longrightarrow \nu_i(-g_i)(\bar{x})\ (i = 1,\ldots,p) \\ \langle c^*_n, c_n - \bar{x}\rangle \longrightarrow 0 \\ \langle y^*_n, y_n - h(\bar{x})\rangle + \sum_{i=1}^{p} \gamma_{i,n}\alpha_{i,n} \longrightarrow 0, \\ \langle u^*_n, u_n - \bar{x}\rangle + \langle v^*_n, v_n - h(\bar{x})\rangle + \sum_{i=1}^{p} \lambda_{i,n}\beta_{i,n} \longrightarrow 0. \end{cases}$$

Proof. By virtue of Lemma 4.1, $\bar{x}$ is an efficient solution of (P) if and only if $\bar{x}$ is a solution of $(S_{\bar{x}})$. The product cone $Y_+ \times \mathbb{R}^p_+$ induce a partial preorder on the product space $Y \times \mathbb{R}^p$ defined by: $(y_1, \alpha_1,\ldots,\alpha_p)$, $(y_2, \beta_1,\ldots,\beta_p) \in Y \times \mathbb{R}^p$

$$(y_1, \alpha_1,\ldots,\alpha_p) \leq_{(Y_+ \times \mathbb{R}^p_+)} (y_2, \beta_1,\ldots,\beta_p) \Longleftrightarrow \begin{cases} y_1 \leq_{Y_+} y_2 \\ \alpha_i \leq \beta_i,\ \forall i = 1,\ldots,p. \end{cases}$$

We adjoint to $Y \times \mathbb{R}^p$ an element $+\infty_{Y\times\mathbb{R}^p}$ which is the supremum with respect to $\leq_{Y\times\mathbb{R}^p}$. By introducing the following auxiliary mapping

$$\begin{array}{rcll} H: & X & \longrightarrow & (Y \times \mathbb{R}^p) \cup \{+\infty_{Y\times\mathbb{R}^p}\} \\ & x & \longrightarrow & H(x) := (h(x), f_1(x) - \nu_1 g_1(x), \ldots, f_p(x) - \nu_p g_p(x)), \end{array}$$

the problem $(P_{\bar{x}})$ may be written equivalently as

$$\inf_{\substack{x\in C \\ H(x)\in-(Y_+\times\mathbb{R}^p_+)}} \sum_{i=1}^{p}(f_i(x) - \nu_i g_i(x)).$$

By using the scalar indicator functions δ_C and $\delta_{-(Y_+\times\mathbb{R}^p_+)}$, we transform the problem $(P_{\bar{x}})$ into an unconstrained minimization problem

$$\inf_{x\in X}(\sum_{i=1}^{p} f_i + \sum_{i=1}^{p} \nu_i(-g_i) + \delta_C + \delta_{-(Y_+\times\mathbb{R}^p_+)} \circ H)(x)$$

and hence $\bar{x}$ is an efficient solution of (P) if and only if

$$0 \in \partial(\sum_{i=1}^{p} f_i + \sum_{i=1}^{p} \nu_i(-g_i) + \delta_C + \delta_{-(Y_+\times\mathbb{R}^p_+)} \circ H)(\bar{x}). \tag{18}$$

Let us consider the following scalar functions $l_i : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, $(i = 1, \ldots, 2p+1)$, defined by

$$l_i(x) := \begin{cases} f_i(x) & \text{if } i \in \{1, \ldots, p\} \\ \nu_{i-p}(-g_{i-p}(x)) & \text{if } i \in \{p+1, \ldots, 2p\} \\ \delta_C(x) & \text{if } i = 2p+1. \end{cases}$$

By means of these notations we can write

$$(18) \Longleftrightarrow 0 \in \partial\left(\sum_{i=1}^{2p+1} l_i + \delta_{-(Y_+\times\mathbb{R}^p_+)} \circ H\right)(\bar{x}).$$

We endow the product space $X \times Y \times \mathbb{R}^p$ with the norm

$$\| (x, y, \alpha_1, \ldots, \alpha_p) \| := \sqrt{\| x \|_X^2 + \| y \|_Y^2 + (\sum_{i=1}^{p} \alpha_i^2)^{\frac{1}{2}}},$$

for $(x, y, \alpha_1, \ldots, \alpha_p) \in X \times Y \times \mathbb{R}^p$. Let us note that the scalar functions l_i $(i = 1, \ldots, 2p+1)$ are proper, convex and lower semicontinuous since C is a nonempty convex closed subset of X, f_i and $-g_i$ are proper, convex and lower semicontinuous. Furthermore, by using the fact that epi$h$ is closed, it is easy to check that epi$H$ is a closed subset of $X \times Y \times \mathbb{R}^p$. Let us recall that the indicator function $\delta_{-(Y_+\times\mathbb{R}^p_+)}$ is $(Y_+ \times \mathbb{R}^p_+)$-nondecreasing (see [5]) and convex and as H is $(Y_+ \times \mathbb{R}^p_+)$-convex, it follows that all the assumptions of Theorem 3.2 are satisfied, hence

$x_{i,n} \in \mathrm{dom} l_i$, $x^*_{i,n} \in X^*$, $(i = 1,\dots,2p+1)$, $(y_n, \alpha_{1,n},\dots,\alpha_{p,n}) \in \mathrm{dom}\delta_{-(Y_+\times\mathbb{R}^p_+)} = -(Y_+\times\mathbb{R}^p_+)$, $(u_n, v_n, \beta_{1,n},\dots,\beta_{p,n}) \in \mathrm{epi} H$, $u^*_n \in X^*$, $(y^*_n, \gamma_{1,n},\dots,\gamma_{p,n}) \in Y^*\times\mathbb{R}^p$ and $(v^*_n, \lambda_{1,n},\dots,\lambda_{p,n}) \in -(Y^*_+\times\mathbb{R}^p_+)$, satisfying

$$\begin{cases} x_{i,n} \xrightarrow{\|\cdot\|_X} \overline{x},\ u_n \xrightarrow{\|\cdot\|_X} \overline{x},\ y_n \xrightarrow{\|\cdot\|_Y} h(\overline{x}),\ v_n \xrightarrow{\|\cdot\|_Y} h(\overline{x}),\ (i = 1,\dots,2p+1) \\ \alpha_{i,n} \longrightarrow 0,\ \beta_{i,n} \longrightarrow 0\ (i = 1,\dots,p) \\ x^*_{i,n} \in \partial l_i(x_{i,n}),\ (i = 1,\dots,2p+1), & (19) \\ (y^*_n, \gamma_{1,n},\dots,\gamma_{p,n}) \in N\big((y_n, \alpha_{1,n},\dots,\alpha_{p,n}), -(Y_+\times\mathbb{R}^p_+)\big) & (20) \\ u^*_n \in \partial\left(-(v^*_n, \lambda_{1,n},\dots,\lambda_{p,n})\circ H\right)(u_n) & (21) \\ \langle (v^*_n, \lambda_{1,n},\dots,\lambda_{p,n}), H(u_n) - (v_n, \beta_{1,n},\dots,\beta_{p,n})\rangle = 0 \end{cases}$$

and

$$\begin{cases} \sum_{i=1}^{2p+1} x^*_{i,n} + u^*_n \xrightarrow{\|\cdot\|_{X^*}} 0,\ y^*_n + v^*_n \xrightarrow{\|\cdot\|_{Y^*}} 0,\ \lambda_{i,n} + \gamma_{i,n} \longrightarrow 0\ (i = 1,\dots,p) \\ l_i(x_{i,n}) - \langle x^*_{i,n}, x_{i,n} - \overline{x}\rangle \longrightarrow l_i(\overline{x}),\ (i = 1,\dots,2p+1) & (22) \\ \langle y^*_n, y_n - h(\overline{x})\rangle + \sum_{i=1}^{p} \gamma_{i,n}\alpha_{i,n} \longrightarrow 0, \\ \langle u^*_n, u_n - \overline{x}\rangle + \langle v^*_n, v_n - h(\overline{x})\rangle + \sum_{i=1}^{p} \lambda_{i,n}\beta_{i,n} \longrightarrow 0. \end{cases}$$

For each $i \in \{1,\dots,2p+1\}$ the conditions $x_{i,n} \in \mathrm{dom} l_i$, (19) and (22) can be rewritten by means of data functions f_i, g_i $(i = 1,\dots,p)$ and δ_C as follow

$$x_{i,n} \in \mathrm{dom} l_i \Longleftrightarrow \begin{cases} x_{i,n} \in \mathrm{dom} f_i = X & \text{if } i \in \{1,\dots,p\} \\ w_{i,n} := x_{i+p,n} \in \mathrm{dom}(\nu_i(-g_i)) = X & \text{if } i \in \{1,\dots,p\} \\ c_n := x_{2p+1,n} \in \mathrm{dom}\delta_C = C & \text{if } i = 2p+1. \end{cases}$$

$$(19) \Longleftrightarrow \begin{cases} x^*_{i,n} \in \partial f_i(x_{i,n}) & \text{if } i \in \{1,\dots,p\} \\ w^*_{i,n} := x^*_{i+p,n} \in \partial(\nu_i(-g_i))(w_{i,n}) & \text{if } i \in \{1,\dots,p\} \\ c^*_n := x^*_{2p+1,n} \in \partial\delta_C(c_n) = N(c_n, C) & \text{if } i = 2p+1 \end{cases}$$

and

$$(22) \Longleftrightarrow \begin{cases} f_i(x_{i,n}) - \langle x_{i,n}^*, x_{i,n} - \bar{x}\rangle \xrightarrow[n\longmapsto+\infty]{} f_i(\bar{x}) & \text{if } i \in \{1,\ldots,p\} \\ v_i(-g_i)(w_{i,n}) - \langle w_{i,n}^*, w_{i,n} - \bar{x}\rangle \xrightarrow[n\longmapsto+\infty]{} v_i(-g_i)(\bar{x}) & \text{if } i \in \{1,\ldots,p\} \\ \langle c_n^*, c_n - \bar{x}\rangle \xrightarrow[n\longmapsto+\infty]{} 0 & \text{if } i = 2p+1. \end{cases}$$

The condition (21) is equivalent to

$$u_n^* \in \partial\left(-v_n^* \circ h - \sum_{i=1}^{p} \lambda_{i,n} f_i + \sum_{i=1}^{p} \lambda_{i,n} v_i g_i\right)(u_n).$$

From Lemma 2.1, the condition (20) may be rewritten as

$$\begin{cases} y_n^* \in Y_+^*,\ (\gamma_{1,n},\ldots,\gamma_{p,n}) \in \mathbb{R}_+^p \\ \langle y_n^*, y_n\rangle + \sum_{i=1}^{p} \gamma_{i,n}\alpha_{i,n} = 0 \end{cases}$$

which completes the proof.

□

5. Optimality conditions of problem (P) under a constraint qualification

In order to establish the standard necessary and sufficient optimality conditions for a feasible point $\bar{x}$ to be an efficient solution for problem (P) under a constraint qualification, we shall need a formula in [5] by Combari et al., concerning the computation of the subdifferential of the composite of a nondecreasing convex function with a convex mapping taking values in a partially ordered topological vector space. For this, let us consider the following constraint qualification called usually Moreau-Rockafellar qualification condition

$$(C.Q.M.R) \begin{cases} X \text{ and } Y \text{are locally convex spaces} \\ f : X \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ is convex and proper} \\ g : Y \longrightarrow \mathbb{R} \cup \{+\infty\} \text{ is convex, proper and } Y_+ - \text{nondecreasing} \\ h : X \longrightarrow Y \cup \{+\infty_Y\} \text{ is } Y_+ - \text{convex and proper} \\ \exists a \in \operatorname{dom} f \cap \operatorname{dom} h \text{ such that } g \text{ is finite and continuous at } h(a). \end{cases}$$

Theorem 5.1. *[5] If the condition (C.Q.M.R) holds, then we have*

$$\partial(f + g \circ h)(\bar{x}) = \bigcup_{y^* \in \partial g(h(\bar{x}))} \partial(f + y^* \circ h)(\bar{x})$$

for any $\bar{x} \in X$.

Remark 5.2. Notice that in [1] one can find more general qualification conditions for this result.

Now, we use Lemma 4.1 to derive necessary and sufficient optimality conditions for a feasible point $\bar{x}$ to be an efficient optimal solution for (P)

Theorem 5.3. *Let $h : X \longrightarrow Y \cup \{+\infty_Y\}$ be a proper and Y_+-convex mapping. Let f_i and $-g_i$ $: X \longrightarrow \mathbb{R}$ be $2p$ convex functions such that $f_i(x) \geq 0$ and $g_i(x) > 0$ for any $x \in C \cap h^{-1}(-Y_+)$ $(i = 1, \ldots, p)$. Let us consider the following constraint qualification*

$$(C.Q.M_0.R_0) \begin{cases} X \text{ and } Y \text{ are locally convex spaces} \\ \exists a \in C \cap \operatorname{dom} h \text{ such that} \\ h(a) \in -\operatorname{int} Y_+,\ f_i(a) - \nu_i g_i(a) < 0, \quad \forall i \in \{1, \ldots, p\}. \end{cases}$$

Suppose that $\operatorname{int} Y_+ \neq \emptyset$ ($\operatorname{int} Y_+$ stands for the topological interior of Y_+) is nonempty and the constraint qualification $(C.Q.M_0.R_0)$ is satisfied. Then $\bar{x} \in C \cap h^{-1}(-Y_+)$ is an efficient optimal solution to (P) if and only if there exist $y^* \in Y_+^*$ and $\lambda_i \geq 0$ $(i \in \{1, \ldots, p\})$ such that $\langle y^*, h(\bar{x}) \rangle = 0$ and

$$0 \in \partial\Big(\sum_{i=1}^{p} (1+\lambda_i)(f_i - \nu_i g_i) + \delta_C + y^* \circ h\Big)(\bar{x}).$$

Proof. Following the proof of Theorem 4.2, we have $\bar{x}$ is an efficient optimal solution of (P) if and only if

$$0 \in \partial\Big(\sum_{i=1}^{p} (f_i - \nu_i g_i) + \delta_C + \delta_{-(Y_+ \times \mathbb{R}_+^p)} \circ H\Big)(\bar{x}).$$

It was mentioned in this proof that the function $\delta_{-(Y_+ \times \mathbb{R}_+^p)} \circ H$ is convex. The constraint qualification $(C.Q.M_0.R_0)$ show that $H(a) = (h(a),\ f_1(a) - \nu_1 g_1(a), \ldots,\ f_p(a) - \nu_p g_p(a)) \in -\operatorname{int} Y_+ \times (]0, +\infty[)^p = -\operatorname{int}(Y_+ \times \mathbb{R}_+^p)$, which yields that the indicator function $\delta_{-(Y_+ \times \mathbb{R}_+^p)}$ is continuous at $H(a)$ and hence according to Theorem 5.1 there exist $(y^*, \alpha_1, \ldots, \alpha_p) \in \partial \delta_{-(Y_+ \times \mathbb{R}_+^p)}(H(\bar{x})) = N(H(\bar{x}), -(Y_+ \times \mathbb{R}_+^p))$ such that

$$0 \in \partial\Big(\sum_{i=1}^{p} (1+\lambda_i)(f_i - \nu_i g_i) + \delta_C + y^* \circ h\Big)(\bar{x}).$$

By virtue of Lemma 2.1 iv), the condition $(y^*, \alpha_1, \ldots, \alpha_p) \in N(H(\bar{x}), -(Y_+ \times \mathbb{R}_+^p))$ is equivalent to

$$\begin{cases} y^* \in Y_+^*,\ \lambda_i \geq 0, \quad \forall i \in \{1, \ldots, p\} \\ \langle y^*, h(\bar{x}) \rangle + \sum_{i=1}^{p} \lambda_i (f_i(\bar{x}) - \nu_i g_i(\bar{x})) = 0 \end{cases} \tag{23}$$

and then the expression (23) is reduced to $\langle y^*, h(\bar{x}) \rangle = 0$, since $f_i(\bar{x}) - \nu_i g_i(\bar{x}) = 0$ for any $i \in \{1, \ldots, p\}$. The proof is complet.

□

In the sequel we present an example of multiobjective fractional programming problem, where the standard optimality condition can not be derived due to the lack of constraint qualification and the sequential optimality conditions hold.

Example. Consider the following multiobjective fractional problem (Q).

$$(\mathrm{Q})\begin{cases}\inf\left(\dfrac{x}{2},\dfrac{x}{2(y+1)}\right)\\ \sqrt{x^2+y^2}-y\leq 0\\ (x,y)\in C:=\left\{(x,y)\in\mathbb{R}^2, x=0, y\geq 0\right\}.\end{cases}$$

Let $f_1(x,y)=f_2(x,y)=\frac{x}{2}, g_1(x,y)=1, g_2(x,y)=y+1$, and $h(x,y)=\sqrt{x^2+y^2}-y$. Let $(\bar{x},\bar{y})=(0,1)$ be a feasible point. Since $\nu_1=\frac{f_1(\bar{x},\bar{y})}{g_1(\bar{x},\bar{y})}=0,\ \nu_2=\frac{f_2(\bar{x},\bar{y})}{g_2(\bar{x},\bar{y})}=0$, then

$$\begin{aligned}S(\bar{x},\bar{y}) &= \left\{(x,y)\in\mathbb{R}^2 : (f_1-\nu_1 g_1)(x,y)\leq 0,\ (f_2-\nu_2 g_2)(x,y)\leq 0\right\}\\ &= \left\{(x,y)\in\mathbb{R}^2 : x\leq 0\right\}.\end{aligned}$$

Observe that $C\subset S(\bar{x},\bar{y})$ and hence the corresponding equivalent scalar minimization problem to (Q) is given by

$$(\mathrm{S}_{(\bar{x},\bar{y})})\begin{cases}\inf x\\ h(x,y)\leq 0\\ (x,y)\in C.\end{cases}$$

It is easy to check that the feasible point $(\bar{x},\bar{y})$ is an optimal solution of $(\mathrm{S}_{(\bar{x},\bar{y})})$. Observe that $h(x,y)=0$, for any $(x,y)\in C$, which yields that the constraint qualification $(C.Q.M_0.R_0)$ does not hold. For each $n\in\mathbb{N}$, if we take (for $i=1,2$), $x_{i,n}=w_{i,n}=c_n=u_n=(0,1)$, $y_n=v_n=h(0,1)=0$, $x^*_{i,n}\in\partial f_i(x_{i,n})=\{(\frac{1}{2},0)\}$, $w^*_{i,n}\in\partial(\nu_i(-g_i))(w_{i,n})=\{(0,0)\}$, $c^*_n=(-1,0)\in N(c_n,C)=\mathbb{R}\times\{0\}$, $\alpha_{i,n}=\beta_{i,n}=0$, $u^*_n=(0,0)$, $y^*_n=v^*_n=0$, $\gamma_{i,n}=\lambda_{i,n}=0$. Hence the sequential optimality conditions of Theorem 4.2, hold.

Acknowledgements

We are thankful to an anonymous reviewer for helping us to improve the quality of the paper.

REFERENCES

[1] Boţ RI. Conjugate duality in convex optimization. Lecture Notes in Economics and Mathematical Systems. 637. Springer-Verlag, Berlin, 2010.

[2] Boţ RI, Csetnek ER, Wanka G. Sequential optimality conditions in convex programming via perturbation approach. Journal of Convex Analysis. 2008;15(1):149–164.

[3] Boţ RI, Csetnek ER, Wanka G. Sequential optimality conditions for composed convex optimization problems. Journal of Mathematical Analysis and Applications. 2008;342(2):1015–1025.

[4] Boţ RI, Grad A, Wanka G. Sequential characterization of solutions in convex composite programming and applications to vector optimization. Journal of Industrial and Management Optimization. 2008;4(4):767–782.

[5] Combari C, Laghdir M, Thibault L. Sous-différentiels de fonctions convexes composées. Annales des Sciences Mathématiques du Québec. 1994;18(2):119–148.

[6] Fajri YA, Laghdir M. Hassouni A. Formulas for sequential Pareto subdifferentials of the sums of vector mappings and applications to optimality conditions. Applied Mathematics E-Notes. 2018;18:318–333.

[7] Hiriart-Urruty JB, Moussaoui M, Seeger A and Volle M. Subdifferential calculus without qaulification conditions, using approximate subdiferentials. Nonlinear Analysis, Theory, Methods and Applications. 1995;24(12):1727–175.

[8] Kim MH, Kim GS, Lee GM. On ε-optimality conditions for multiobjective fractional optimization problems. Fixed Point Theory and Applications.2011;2011(6):1–13.

[9] Laghdir M, Benabbou R. Sensitivity analysis in parametrized convex optimization. Applied Mathematical Sciences. 2007;1(49):2409–2419.

[10] Laghdir M, Rikouane A, Fajri YA, Tazi E. Sequential Pareto subdifferential sum rule and sequential efficiency. Applied Mathematics E-Notes. 2016;16:133–143.

[11] Laghdir M, Rikouane A, Fajri YA, Tazi E. Sequential Pareto subdifferential composition rule and sequential efficiency. Journal of Nonlinear and Convex Analysis. 2017;18(12):2177–2187.

[12] Penot JP. Subdifferential calculus without qualification assumptions. Journal of Convex Analysis. 1996;3(2):207–219.

[13] Thibault L. Sequential convex subdifferential calculus and sequential Lagrange multipliers. SIAM Journal on Control and Optimization. 1997;35(4):1434–1444.

M. B. MOUSTAID
Department of Mathematics, Faculty of Sciences
Chouaïb Doukkali University, BP. 20, El Jadida, Morocco
e-mail: bilalmoh39@gmail.com

M. LAGHDIR
Department of Mathematics, Faculty of Sciences
Chouaïb Doukkali University, BP. 20, El Jadida, Morocco
e-mail: laghdirm@yahoo.fr

I. DALI
Department of Mathematics, Faculty of Sciences
Chouaïb Doukkali University, BP. 20, El Jadida, Morocco
e-mail: dali.issam@gmail.com

A. RIKOUANE
Department of Mathematics, Faculty of Sciences
Ibn Zohr University, Addakhla, B.P. 8106, Agadir, Morocco
e-mail: rikouaneah@gmail.com