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SOME MEASURABILITY AND CONTINUITY PROPERTIES OF ARBITRARY REAL FUNCTIONS

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Given an arbitrary real function f, the set $\mathcal{D}_f$ of all points where f admits approximate limit is the maximal (with respect to the relation of inclusion except for a nullset) measurable subset of the real line having the properties that the restriction of f to $\mathcal{D}_f$ is measurable, and f is approximately continuous at almost every point of $\mathcal{D}_f$. These results extend the well-known fact that a function is measurable if and only if it is approximately continuous almost everywhere. In addition, there exists a maximal G_δ-set $\mathcal{C}_f$ (which can be actually constructed from f) such that it is possible to find a function $g = f$ almost everywhere, whose set of points of continuity is exactly $\mathcal{C}_f$.

1. Introduction and Notation.

This paper is devoted to the investigation of some properties of real functions with respect to Lebesgue measure. We shall denote Lebesgue measure and Lebesgue outer measure by μ and μ^*, respectively. Recall that, for any subset A of the real line $\mathbb{R}$, the outer measure of A is given by

$$\mu^*(A) = \inf\{\mu(O) : O \supset A, \quad O \text{ open }\}.$$

For $\varepsilon > 0$ and $l \in \mathbb{R}$ we introduce the sets

$$I_\varepsilon(l) = \{x \in \mathbb{R} : |x - l| < \varepsilon\} \qquad \text{and} \qquad \Delta_\varepsilon(l) = \{x \in \mathbb{R} : |x - l| \geq \varepsilon\}.$$

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Also, for any A, $B \subset \mathbb{R}$ we agree to denote the complement of A by A^C, the characteristic function of A by χ_A, and the difference of A and B by $A \setminus B$.

Given an arbitrary set $A \subset \mathbb{R}$, the *Lebesgue upper density* $\Psi(A)$ of A is defined as follows:

$$\Psi(A) = \left\{ x \in \mathbb{R} : \lim_{\delta \to 0} \frac{\mu^*\big(I_\delta(x) \cap A^C\big)}{\delta} = 0 \right\}.$$

In literature it is usually preferred the *Lebesgue lower density* $\Phi(A)$ of A, defined by

$$\Phi(A) = \left\{ x \in \mathbb{R} : \lim_{\delta \to 0} \frac{\mu^*\big(I_\delta(x) \cap A\big)}{2\delta} = 1 \right\}.$$

Both $\Phi(A)$ and $\Psi(A)$ are measurable. Moreover, the inclusions $\Phi(A) \supset A \supset \Psi(A)$ hold except for a nullset. In fact, $\Phi(A)$ and $\Psi(A)$ are, respectively, the smallest (except for a nullset) measurable set containing A, and the largest (except for a nullset) measurable set contained in A (see, e.g., Theorem 2.9.11 in [1]). In particular, A is measurable if and only if $\Phi(A) = A = \Psi(A)$ neglecting nullsets. This fact is known as the Lebesgue density theorem.

Throughout the paper we consider functions $f : \mathbb{R} \to \mathbb{R}$ everywhere defined. On occurence, we shall highlight the possibility of extending the results for functions defined on certain subsets of $\mathbb{R}$. If f is summable in a neighborhood of $x \in \mathbb{R}$ (which implies that f is measurable in a neighborhood of x) and there exists $l \in \mathbb{R}$ such that

$$\lim_{\delta \to 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} |f(t) - l| \, dt = 0,$$

then x is said to be a *Lebesgue point* of f. In that case, we denote $l = Lf(x)$. If f is locally summable on $\mathbb{R}$, then the function Lf equals f almost everywhere (so, in particular, it is defined almost everywhere). For a detailed presentation of the subject, the reader is referred to any classical textbook of measure theory. See, for instance, [2,5], or [1,3,4,6] for more selected topics.

Definition 1.1. Given $f : \mathbb{R} \to \mathbb{R}, x, l \in \mathbb{R}$, and $\varepsilon > 0$ we introduce the quantity

$$\mathcal{M}_\varepsilon[f, l, x] = \limsup_{\delta \to 0} \frac{\mu^*\big(I_\delta(x) \cap f^{-1}(\Delta_\varepsilon(l))\big)}{\delta}.$$

We say that f has *M-limit* (or *approximate limit*) l at x, and write $l = M\text{-}\lim_{y \to x} f(y)$, if

$$\mathcal{M}_\varepsilon[f, l, x] = 0 \qquad \forall \varepsilon > 0.$$

If $f(x) = M\text{-}\lim_{y\to x} f(y)$, then we say that f is *M-continuous* (or *approximately continuous*) at x. We denote by $\mathcal{D}_f$ the subset of $\mathbb{R}$ consisting of all points where f admits M-limit. Also, we introduce the function

$$Mf(x) = \begin{cases} M\text{-}\lim\limits_{y\to x} f(y) & \text{if } x \in \mathcal{D}_f \\ f(x) & \text{otherwise}. \end{cases}$$

One could think of a different definition of "approximate" limit. Namely, f has *P-limit l* at x if

$$\lim_{\delta\to 0} \frac{\mu^*\big(I_\delta(x) \cap f^{-1}(I_\varepsilon(l))\big)}{2\delta} = 1 \qquad \forall \varepsilon > 0.$$

Analogously, f is *P-continuous* at x if $f(x) = P\text{-}\lim_{y\to x} f(y)$. However, this definition turns out to be of little interest. Indeed, Sierpinski proved that every function f (measurable or not) is P-continuous almost everywhere (see [3], Theorem 2.6.2).

Definition 1.2. We say that f has *C-limit l* at x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mu\big(I_\delta(x) \cap f^{-1}(\Delta_\varepsilon(l))\big) = 0.$$

We denote by $\mathcal{C}_f$ the subset of $\mathbb{R}$ consisting of all points where f admits C-limit.

It is apparent that $\mathcal{D}_f \supset \mathcal{C}_f$. It is also clear that if $g = f$ almost everywhere, then $\mathcal{D}_f = \mathcal{D}_g$, and $\mathcal{C}_f = \mathcal{C}_g$. Notice that if x is a continuity point of g, for some $g = f$ almost everywhere, then $x \in \mathcal{C}_f$ and conversely. Thus, for each function g lying in $[f]$, the equivalence class of all functions equal almost everywhere to f, the set of of all continuity points of g is a subset of $\mathcal{C}_f$. We shall see that in fact $[f]$ contains an element g such that $\mathcal{C}_f$ is exactly the set of continuity points of g.

2. Lebesgue Measurability of Real Functions.

The aim of this section is to find a relation between the measurability properties of a function $f : \mathbb{R} \to \mathbb{R}$ and the set $\mathcal{D}_f$ of its M-limit points. We begin with a well-known result, whose proof is almost immediate.

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$. Then if x is a Lebesgue point of f it follows that $x \in \mathcal{D}_f$, and $Mf(x) = Lf(x)$. Moreover if f is measurable and bounded in a neighborhood of x the reverse implication holds too.*

The following result is classical (see [1], Theorem 2.9.13). For the reader's convenience we provide a simple proof of one implication (which differs from the classical one and relies on Proposition 2.1).

Theorem 2.2. *A function $f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if f is M-continuous almost everywhere.*

Proof. We only show that, if f is measurable, then $\mu(D_f^C) = 0$ and $Mf(x) = f(x)$ almost everywhere. For every $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \le n \\ n & \text{otherwise .} \end{cases}$$

Since $f_n \in L^1_{loc}(\mathbb{R})$ then almost every point of $\mathbb{R}$ is a Lebesgue point of f_n, thus, by Proposition 2.1, $\mu(\mathcal{D}_{f_n}^C) = 0$ and $f_n = Mf_n$ almost everywhere. Denote

$$A = \bigcap_n \{x \in \mathcal{D}_{f_n} : f_n(x) = Mf_n(x)\}.$$

Notice that $\mu(A^C) = 0$. Let now $x \in A$ be fixed, and choose $n > |f(x)|$. Then $f(x) = f_n(x) = Mf_n(x)$. Select $\varepsilon < n - |f(x)|$. The equality

$$f_n^{-1}(\Delta_\varepsilon(f(x))) = f^{-1}(\Delta_\varepsilon(f(x)))$$

holds, which yields

$$\mathcal{M}_\varepsilon[f, f(x), x] = \mathcal{M}_\varepsilon[f_n, f(x), x] = 0,$$

i.e., $x \in \mathcal{D}_f$ and $Mf(x) = f(x)$. □

If f is measurable but not bounded in a neighborhood of x it can happen that $Mf(x)$ exists, but x is not a Lebesgue point. Indeed, there exist measurable functions (hence having M-limit almost everywhere) with no Lebesgue points, as the following (classical) example shows.

Example 2.3. Let $\{J_n\}_{n\in\mathbb{N}}$ be the rational endpoint intervals contained in $\mathbb{R}$. Then it is possible to find a sequence $\{T_n\}_{n\in\mathbb{N}}$ of pairwise disjoint set of positive measure such that $T_n \subset J_n$ for every $n \in \mathbb{N}$. This can be done by recalling that for every interval $J \subset \mathbb{R}$ there exists a compact set of positive measure $T \subset J \setminus \mathbb{Q}$. Notice that any interval $I \subset \mathbb{R}$ contains infinitely many sets T_n. The function

$$f(x) = \begin{cases} \dfrac{n}{\mu(T_n)} & \text{if } x \in T_n \\ 0 & \text{otherwise} \end{cases}$$

is clearly measurable, and, in force of the preceding result, almost every $x \in \mathbb{R}$ belongs to $\mathcal{D}_f$. On the other hand, fixed any interval $I \subset \mathbb{R}$, and any $n \in \mathbb{N}$, there exists $n_0 \geq n$ such that $T_{n_0} \subset I$. Thus

$$\int_I |f(t)|dt \geq \int_{T_{n_0}} |f(t)|dt = n_0 > n.$$

Letting $n \to \infty$ we realize that f is not summable on any interval I, and therefore no point of $\mathbb{R}$ is a Lebesgue point of f.

We now state the main result of this section.

Theorem 2.4. *For any function $f : \mathbb{R} \to \mathbb{R}$ the set $\mathcal{D}_f$ is measurable and the restriction of f to $\mathcal{D}_f$, denoted by $f_{|\mathcal{D}_f}$, is a measurable function. Moreover, $f = Mf$ almost everywhere.*

To prove the above result, we shall make use of the following two technical lemmas.

Lemma 2.5. *Let f and φ be two real functions on $\mathbb{R}$ and let h denote the composite function $\varphi \circ f$. If φ is continuous and strictly monotonic, then*

$$\mathcal{D}_f = \{x \in \mathcal{D}_h : \gamma < Mh(x) < \Gamma\},$$

where $\gamma = \inf \varphi(\mathbb{R})$, $\Gamma = \sup \varphi(\mathbb{R})$. Moreover, the following implication holds true:

$$M\text{-}\lim_{y \to x} f(y) = \lambda \quad \Longrightarrow \quad M\text{-}\lim_{y \to x} h(y) = \varphi(\lambda).$$

Proof. We first show the latter assertion. Assume that $M\text{-}\lim_{y \to x} f(y) = \lambda$. Then, owing to the continuity of φ at the point λ, for each $\sigma > 0$ there exists $\varepsilon > 0$ such that $I_\varepsilon(\lambda) \subset \varphi^{-1}(I_\sigma(\varphi(\lambda)))$, hence $f^{-1}(\Delta_\varepsilon(\lambda)) \supset h^{-1}(\Delta_\sigma(\varphi(\lambda)))$ and consequently

$$\frac{\mu^*\big(I_\delta(x) \cap f^{-1}(\Delta_\varepsilon(\lambda))\big)}{\delta} \geq \frac{\mu^*\big(I_\delta(x) \cap h^{-1}(\Delta_\sigma(\varphi(\lambda)))\big)}{\delta} \quad \forall \delta > 0.$$

Letting $\delta \to 0$, we get $M\text{-}\lim_{y \to x} h(y) = \varphi(\lambda)$.

To complete the proof it is now sufficient to show that also the implication

$$M\text{-}\lim_{y \to x} h(y) = w \in \varphi(\mathbb{R}) \quad \Longrightarrow \quad M\text{-}\lim_{y \to x} f(y) = \varphi^{-1}(w)$$

is true. Indeed, if $M\text{-}\lim_{y \to x} h(y) = w \in \varphi(\mathbb{R})$ and we assume, for instance, that φ is strictly increasing, then for each $\varepsilon > 0$, denoting

$$\lambda_1 = \varphi^{-1}(w) - \varepsilon, \qquad \lambda_2 = \varphi^{-1}(w) + \varepsilon, \qquad \sigma = \min\{w - \varphi(\lambda_1), \varphi(\lambda_2) - w\},$$

we have

$$\begin{aligned}\{t\in\mathbb{R} : \lambda_1 < f(t) < \lambda_2\} &= \{t\in\mathbb{R} : \varphi(\lambda_1) < h(t) < \varphi(\lambda_2)\}\\ &\supset \{t\in\mathbb{R} : w-\sigma < h(t) < w+\sigma\}\end{aligned}$$

and consequently

$$\frac{\mu^*\big(I_\delta(x)\cap f^{-1}(\Delta_\varepsilon(\varphi^{-1}(w)))\big)}{\delta} \le \frac{\mu^*\big(I_\delta(x)\cap h^{-1}(\Delta_\sigma(w))\big)}{\delta} \quad \forall\ \delta > 0.$$

So, letting $\delta \to 0$, we get $M\text{-}\lim_{y\to x} f(y) = \varphi^{-1}(w)$. □

Lemma 2.6. *Let a function $h : \mathbb{R} \to \mathbb{R}$ and a number $\beta \in \mathbb{R}$ be given. If $N = \{x \in \mathbb{R} : h(x) = \beta\}$ is a nullset, then also $L = \{x \in \mathbb{R} : Mh(x) = \beta\}$ is a nullset.*

Proof. Clearly, it is enough to prove that for every $\varepsilon > 0$ and every bounded open interval $I \subset \mathbb{R}$ the set $L \cap I \cap \{x \in \mathbb{R} : |h(x) - \beta| \ge \varepsilon\}$ is a set of zero measure.

We will make use of the Vitali Covering Lemma.

Let any $\eta > 0$ be fixed. Then for each $x \in L \cap I$ there exists a $\delta_x > 0$ such that for every $\delta \in (0, \delta_x]$ we have

$$I_\delta(x) \subset I$$

and also

$$\frac{\mu^*\big(I_\delta(x)\cap h^{-1}(\Delta_\varepsilon(\beta))\big)}{\delta} < \eta,$$

that is,

$$\mu^*\big(I_\delta(x)\cap h^{-1}(\Delta_\varepsilon(\beta))\big) < \frac{1}{2}\eta\mu(I_\delta(x)).$$

Now it is apparent that the family $\mathcal{V} = \{I_\delta(x) : x \in L\cap I,\ 0 < \delta \le \delta_x\}$ covers $L \cap I$ in the sense of Vitali, thus there is a countable subfamily $\{I_n\} \subset \mathcal{V}$, with $I_{n_1} \cap I_{n_2} = \emptyset$ for $n_1 \ne n_2$, such that

$$\mu^*\Big(L\cap I \setminus \Big(\bigcup_n I_n\Big)\Big) = 0.$$

It follows that

$$
\begin{aligned}
\mu^*\big(L \cap I \cap &\{x \in \mathbb{R} : |h(x) - \beta| \geq \varepsilon\}\big) \\
&= \mu^*\Big(L \cap I \cap \{x \in \mathbb{R} : |h(x) - \beta| \geq \varepsilon\} \cap \Big(\bigcup_n I_n\Big)\Big) \\
&= \mu^*\Big(L \cap \{x \in \mathbb{R} : |h(x) - \beta| \geq \varepsilon\} \cap \Big(\bigcup_n I_n\Big)\Big) \\
&\leq \sum_n \mu^*\big(I_n \cap L \cap \{x \in \mathbb{R} : |h(x) - \beta| \geq \varepsilon\}\big) \\
&\leq \sum_n \mu^*\big(I_n \cap h^{-1}(\Delta_\varepsilon(\beta))\big) \\
&< \frac{\eta}{2} \sum_n \mu(I_n) \leq \frac{\eta}{2}\mu(I),
\end{aligned}
$$

thus

$$
\mu^*\big(L \cap I \cap \{x \in \mathbb{R} : |h(x) - \beta| \geq \varepsilon\}\big) = 0
$$

since $\eta > 0$ is arbitrary. □

Proof of Theorem 2.4. We first show that $f = Mf$ almost everywhere. To this aim we use Sierpinski's theorem ([3], Theorem 2.6.2), already quoted in the introduction. According to that theorem, there exists a nullset N such that

$$
x \in N^C \quad \Longrightarrow \quad \lim_{\delta \to 0} \frac{\mu^*\big(I_\delta(x) \cap f^{-1}(I_\varepsilon(f(x)))\big)}{2\delta} = 1 \qquad \forall\ \varepsilon > 0.
$$

It is easily seen that $\{x \in \mathcal{D}_f : f(x) \neq Mf(x)\} \subset N$. Indeed, if we assume by contradiction the existence of a point $\bar{x} \in \{x \in \mathcal{D}_f : f(x) \neq Mf(x)\} \setminus N$, then, denoting $l = Mf(\bar{x})$, for $0 < \varepsilon < \frac{1}{2}|l - f(\bar{x})|$, since $I_\varepsilon(f(\bar{x})) \subset \Delta_\varepsilon(l)$, we get

$$
\lim_{\delta \to 0} \frac{\mu^*(I_\delta(x) \cap f^{-1}(I_\varepsilon(f(x))))}{2\delta} = 0,
$$

contrary to the fact that $\bar{x} \in N^C$. Thus also $\{x \in \mathcal{D}_f : f(x) \neq Mf(x)\}$ is a nullset, that is, $f = Mf$ almost everywhere.

Next, we prove that the measurability of the restriction $f|_{\mathcal{D}_f}$ is a direct consequence of the measurability of $\mathcal{D}_f$. It is sufficient to consider the case $f > 0$. Indeed, the general case will follow from this by considering the function e^f, taking into account that $\mathcal{D}_f \subset \mathcal{D}_{e^f}$ and $\mu(\mathcal{D}_{e^f} \setminus \mathcal{D}_f) = 0$, by virtue of the previous lemmas and of the obvious remark that $Me^{f(x)} \geq 0\ \forall x \in \mathbb{R}$, and making use of the subsequent argument:

$$\begin{aligned}&\mathcal{D}_f \text{ measurable} \quad\Longrightarrow\quad \mathcal{D}_{e^f} \text{ measurable} \quad\Longrightarrow\quad e^f|_{\mathcal{D}_{e^f}} \text{ measurable}\\ &\Longrightarrow\quad f|_{\mathcal{D}_{e^f}} \text{ measurable} \quad\Longrightarrow\quad f|_{\mathcal{D}_f} \text{ measurable.}\end{aligned}$$

Thus, assume $f > 0$ and define $g = f\chi_{\mathcal{D}_f} - \chi_{\mathcal{D}_f^C}$. We will prove that the function g is M-continuous almost everywhere. By Theorem 2.2 this implies that g is measurable, hence also the restriction $g|_{\mathcal{D}_f}$, namely, $f|_{\mathcal{D}_f}$, is measurable as well. Given $x_0 \in \mathcal{D}_f^C \cap \Psi(\mathcal{D}_f^C)$ and $0 < \varepsilon < 1$, we have

$$g^{-1}(\Delta_\varepsilon(g(x_0))) = g^{-1}(\Delta_\varepsilon(-1)) = \mathcal{D}_f$$

and consequently

$$\lim_{\delta\to 0} \frac{\mu^*\big(I_\delta(x_0) \cap g^{-1}(\Delta_\varepsilon(g(x_0)))\big)}{\delta} = \lim_{\delta\to 0} \frac{\mu^*\big(I_\delta(x_0) \cap \mathcal{D}_f\big)}{\delta} = 0.$$

On the other hand, for $x_0 \in \{x \in \mathcal{D}_f : f(x) = Mf(x)\} \cap \Psi(\mathcal{D}_f)$ and $0 < \varepsilon < 1$, we have the set-theoretical inclusion

$$g^{-1}(\Delta_\varepsilon(g(x_0))) = g^{-1}(\Delta_\varepsilon(f(x_0))) \subset f^{-1}(\Delta_\varepsilon(f(x_0))) \cup \mathcal{D}_f^C$$

and since

$$\lim_{\delta\to 0} \frac{\mu^*\big(I_\delta(x_0) \cap f^{-1}(\Delta_\varepsilon(f(x_0)))\big)}{\delta} = 0 \quad\text{and}\quad \lim_{\delta\to 0} \frac{\mu^*\big(I_\delta(x_0) \cap \mathcal{D}_f^C\big)}{\delta} = 0,$$

it follows that also in this case we have

$$\lim_{\delta\to 0} \frac{\mu^*\big(I_\delta(x_0) \cap g^{-1}(\Delta_\varepsilon(g(x_0)))\big)}{\delta} = 0.$$

In conclusion, the above limit holds for each $\varepsilon \in (0, 1)$ (hence for each $\varepsilon > 0$) and each point x_0 belonging to the set

$$G = \Big(\mathcal{D}_f^C \cap \Psi(\mathcal{D}_f^C)\Big) \cup \Big(\big\{x \in \mathcal{D}_f : f(x) = Mf(x)\big\} \cap \Psi(\mathcal{D}_f)\Big).$$

The complement of this set, that is,

$$\begin{aligned}G^C &= \big(\mathcal{D}_f^C \setminus G\big) \cup \big(\mathcal{D}_f \setminus G\big)\\ &= \Big(\mathcal{D}_f^C \setminus \big(\mathcal{D}_f^C \cap \Psi(\mathcal{D}_f^C)\big)\Big) \cup \Big(\mathcal{D}_f \setminus \big(\{x \in \mathcal{D}_f : f(x) = Mf(x)\} \cap \Psi(\mathcal{D}_f)\big)\Big)\\ &= \Big(\mathcal{D}_f^C \setminus \Psi(\mathcal{D}_f^C)\Big) \cup \Big(\{x \in \mathcal{D}_f : f(x) \neq Mf(x)\} \cup (\mathcal{D}_f \setminus \Psi(\mathcal{D}_f))\Big),\end{aligned}$$

is a set of zero measure, because we are assuming that $\mathcal{D}_f$ is a measurable set and we already proved that $f = Mf$ almost everywhere. It follows that g is M-continuous almost everywhere, so this step of the proof is concluded.

We are left to show the measurability of the set $\mathcal{D}_f$.

We first consider the case of a function $f : \mathbb{R} \to \mathbb{R}$ whose range is a closed discrete set:

$$f(\mathbb{R}) = B = \{\alpha_j : j \in J\},$$

so that

$$f = \sum_{j \in J} \alpha_j \chi_{A_j}$$

having denoted $A_j = f^{-1}(\{\alpha_j\})$, $j \in J$. It is easily seen that if $\alpha \in B^C$ then it is impossible that $Mf(x) = \alpha$ for some $x \in \mathcal{D}_f$. It follows that

$$\mathcal{D}_f = \bigcup_{j \in J} \mathcal{E}_j \ ,$$

where

$$\mathcal{E}_j = \left\{ x \in \mathcal{D}_f : Mf(x) = \alpha_j \right\} \qquad \forall \ j \in J.$$

Then it is sufficient to show that every set $\mathcal{E}_j$, $j \in J$, is measurable. Indeed, if $\varepsilon > 0$ is small enough (to be precise, less than the distance of the point α_j from the set $B \setminus \{\alpha_j\}$), we have the equality

$$f^{-1}(\Delta_\varepsilon(\alpha_j)) = f^{-1}(\{\alpha_j\}^C) = A_j^C \ ,$$

from which the equivalence

$$\lim_{\delta \to 0} \frac{\mu^*\big(I_\delta(x) \cap f^{-1}(\Delta_\varepsilon(\alpha_j))\big)}{\delta} = 0 \quad \iff \quad \lim_{\delta \to 0} \frac{\mu^*\big(I_\delta(x) \cap A_j^C\big)}{\delta} = 0$$

follows. This implies that $\mathcal{E}_j = \Psi(A_j)$, hence $\mathcal{E}_j$ is measurable.

To complete the proof we consider an arbitrary function $f : \mathbb{R} \to \mathbb{R}$. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence such that, for every $n \in \mathbb{N}$,

$$\frac{n}{n+1} < a_n \leq 1$$

and a_n / a_{n+1} is irrational. Set then, for every $n \in \mathbb{N}$ and $j \in \mathbb{Z}$,

$$\alpha_j^n = \frac{a_n(2j-1)}{2n} \qquad \text{and} \qquad U_j^n = (\alpha_j^n, \alpha_{j+1}^n).$$

Finally, introduce the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ as follows: for all $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} \frac{ja_n}{n} & \text{if } x \in f^{-1}(U_j^n), \quad j \in \mathbb{Z} \\ \alpha_j^n & \text{if } x \in f^{-1}(\{\alpha_j^n\}), \quad j \in \mathbb{Z}. \end{cases}$$

Since f_n is of the form considered before, we have that $\mathcal{D}_{f_n}$ is measurable. Consider now the measurable set

$$\mathcal{D} = \bigcap_n \big(\mathcal{D}_{f_n} \cup \mathcal{D}_{f_{n+1}}\big).$$

We first show that $\mathcal{D} \supset \mathcal{D}_f$. Let

$$W_n = \big\{x \in \mathcal{D}_f : Mf(x) = \alpha_j^n \text{ for some } j \in \mathbb{Z}\big\}.$$

Then $\mathcal{D}_{f_n} \supset \mathcal{D}_f \setminus W_n$ for all $n \in \mathbb{N}$. Indeed, if $x \in \mathcal{D}_f \setminus W_n$, then $Mf(x) \in U_j^n$ for some $j \in \mathbb{Z}$. Choosing $\varepsilon > 0$ so small that $I_\varepsilon(Mf(x)) \subset U_j^n$, it is clear that

$$f^{-1}\big(\Delta_\varepsilon(Mf(x))\big) \supset f_n^{-1}(\{ja_n/n\}^C) \supset f_n^{-1}\big(\Delta_\eta(ja_n/n)\big) \qquad \forall \eta > 0,$$

which implies at once that $x \in \mathcal{D}_{f_n}$ and $Mf_n(x) = ja_n/n$. Thus,

$$\mathcal{D}_{f_n} \cup \mathcal{D}_{f_{n+1}} \supset \mathcal{D}_f \cap \big(W_n \cap W_{n+1}\big)^C = D_f \qquad \forall n \in \mathbb{N}.$$

Last equality comes from the fact that $W_n \cap W_{n+1} = \emptyset$. Indeed, if the intersection were not empty, there would exist $i,\ l \in \mathbb{Z}$ such that $\alpha_i^n = \alpha_l^{n+1}$, i.e.,

$$\frac{a_n}{a_{n+1}} = \frac{n}{n+1}\ \frac{2l-1}{2i-1},$$

which is impossible since the left-hand side of the above equality is irrational. Hence, taking the intersection over n,

$$\mathcal{D} = \bigcap_n \big(\mathcal{D}_{f_n} \cup \mathcal{D}_{f_{n+1}}\big) \supset \mathcal{D}_f.$$

Finally denote

$$\mathcal{D}' = \mathcal{D} \setminus \big\{x \in D_{f_n} \text{ for some } n \in \mathbb{N} : f_n(x) \neq Mf_n(x)\big\}.$$

Recalling the first part of the proof, $\mu(\mathcal{D} \setminus \mathcal{D}') = 0$. We prove the inclusion $\mathcal{D}' \subset \mathcal{D}_f$. Let $x \in \mathcal{D}'$. Then there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$, such that $k_n = n$

or $k_n = n + 1$ for any $n \in \mathbb{N}$, $x \in \mathcal{D}_{f_{k_n}}$, and $f_{k_n}(x) = Mf_{k_n}(x)$. Select $\varepsilon > 0$. Since $f_{k_n} \to f$ uniformly, choose n large enough such that f and f_{k_n} differ less than $\varepsilon/3$. If $y \in f_{k_n}^{-1}(I_{\varepsilon/3}(f_{k_n}(x)))$, it follows that

$$|f(y) - f(x)| \leq |f(y) - f_{k_n}(y)| + |f_{k_n}(y) - f_{k_n}(x)| + |f_{k_n}(x) - f(x)| < \varepsilon,$$

which yields the inclusion

$$f_{k_n}^{-1}\big(\Delta_{\varepsilon/3}(f_{k_n}(x))\big) \supset f^{-1}\big(\Delta_\varepsilon(f(x))\big),$$

and therefore $\mathcal{M}_\varepsilon[f, f(x), x] = 0$. We conclude that $\mathcal{D}_f$ is measurable, and this finishes the proof. $\square$

A straightforward consequence of Theorem 2.4 is a sufficient condition for a function $f : \mathbb{R} \to \mathbb{R}$ in order to be measurable.

Corollary 2.7. *If $\mu(D_f^C) = 0$ then f is measurable.*

Notice that, if f is approximately continuous almost everywhere, then $\mu(D_f^C) = 0$; so the above corollary is a little bit stronger than the “ if ” implication of Theorem 2.2.

Finally we show that the set $\mathcal{D}_f$ is the maximal measurable set (with respect to the relation “inclusion except for a nullset”) where f is measurable. Thus the set $\mathcal{D}_f$ gives an estimate of the measurability degree of f. Of course $\mathcal{D}_f$ might be an emptyset. In this case the function f is completely nonmeasurable.

Theorem 2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be given. For any measurable set $A \subset \mathbb{R}$ such that $f_{|A}$ is measurable, we have that $\mu(A \setminus \mathcal{D}_f) = 0$.*

Proof. If A is measurable and $f_{|A}$ is measurable, then also the function $e^f{}_{|A}$ is measurable. Moreover, we have that $\mathcal{D}_f = \mathcal{D}_{e^f} \setminus L$, where L is a nullset, as we already pointed out in the proof of Theorem 2.4. Thus, if the theorem is true for e^f, it is true for f as well. So we assume without loss of generality $f \geq 0$. Introduce now $h = (f + 1)\chi_A$. Then h is measurable, and from Theorem 2.2, $\mu(\mathcal{D}_h^C) = 0$ and $h = Mh$ almost everywhere. Set $C = \{x \in A \cap \mathcal{D}_h : Mh(x) \geq 1\}$. Observe that $\mu(A \setminus C) = 0$. We finish the proof by proving that $C \subset \mathcal{D}_f$. Indeed, let $x \in C$, and select $\varepsilon < 1$. Then

$$h^{-1}\big(\Delta_\varepsilon(Mh(x))\big) \supset f^{-1}\big(\Delta_\varepsilon(Mh(x) - 1)\big)$$

which bears

$$\mathcal{M}_\varepsilon[f, Mh(x) - 1, x] \leq \mathcal{M}_\varepsilon[h, Mh(x), x] = 0$$

that is, f admits M-limit at point x and $Mf(x) = Mh(x) - 1$. $\square$

Remark 2.9. The results established in this section for real functions defined on the whole real line $\mathbb{R}$ actually extend to any real function f, whose domain is an arbitrary subset of $\mathbb{R}$, not necessarily measurable.

To see this extension we first need the appropriate notion of M-limit and the definitions of $\mathcal{D}_f$ and of Mf in this more general setting.

Let $f : E \to \mathbb{R}$ be any function, with $E \subset \mathbb{R}$. Given $x,\ l \in \mathbb{R}$, we say that f has M-limit l at the point x provided that

$$\lim_{\delta \to 0} \frac{\mu^*\big(I_\delta(x) \setminus f^{-1}(I_\varepsilon(l))\big)}{\delta} = 0$$

for every $\varepsilon > 0$ (see [1], p. 158). Also, we denote by $\mathcal{D}_f$ the set of all points $x \in \mathbb{R}$ where the M-limit of f does exist and by Mf the real function on $\mathcal{D}_f \cup E$ defined according to the following rule:

$$Mf(x) = \begin{cases} M\text{-}\lim\limits_{y \to x} f(y) & \text{if } x \in \mathcal{D}_f \\ f(x) & \text{if } x \in E \setminus \mathcal{D}_f. \end{cases}$$

It is apparent that these definitions generalize the ones already introduced when $E = \mathbb{R}$.

Now, we can state the above mentioned general result.

Theorem 2.10. *For any function $f : E \to \mathbb{R}$, $E \subset \mathbb{R}$, the following statements hold true:*

i) *the sets $\mathcal{D}_f$ and $\mathcal{D}_f \cap E$ are measurable and $\mu(\mathcal{D}_f \setminus E) = 0$;*

ii) *the restriction of f to $\mathcal{D}_f \cap E$ is a measurable function;*

iii) *for any measurable set $A \subset E$ having the property that $f_{|A}$ is a measurable function, we have that $\mu(A \setminus (\mathcal{D}_f \cap E)) = 0$;*

iv) *$f = Mf$ almost everywhere, that is $\{x \in E : f(x) \neq Mf(x)\}$ is a set of zero measure.*

Proof. We first assume that f satisfies $f(x) \geq 1$ for every $x \in E$. Then, it is an obvious remark that also Mf satisfies $Mf(x) \geq 1$ for every $x \in \mathcal{D}_f \cup E$.

Let $g : \mathbb{R} \to \mathbb{R}$ be the following extension of f to the whole $\mathbb{R}$:

$$g(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{if } x \in E^C. \end{cases}$$

Then, it is apparent that the implication

$$M\text{-}\lim_{y \to x} f(y) = l \quad \Longrightarrow \quad M\text{-}\lim_{y \to x} g(y) = l$$

holds true. As a consequence of this fact and of the previous remark we get the set-theoretical inclusion $\mathcal{D}_f \subset \mathcal{D}_g \cap \{x \in \mathbb{R} : Mg(x) \geq 1\}$. Furthermore, since for $l \geq 1$ and $0 < \varepsilon < 1$ we have $f^{-1}(I_\varepsilon(l)) = g^{-1}(I_\varepsilon(l))$, it is clear that also the reverse inclusion holds, so

$$\mathcal{D}_f = \mathcal{D}_g \cap \left\{x \in \mathbb{R} : Mg(x) \geq 1\right\} .$$

Now, by Theorem 2.4, $\mathcal{D}_g$ is a measurable set, $g_{|\mathcal{D}_g}$ is a measurable function and $g = Mg$ almost everywhere. Having this in mind, we immediately deduce from the above equality that $\mathcal{D}_f$ is measurable. Moreover, we have that $\mathcal{D}_f \setminus E$ is a set of zero measure, since

$$\begin{aligned} \mathcal{D}_f \setminus E &= \mathcal{D}_g \cap \left\{x \in \mathbb{R} : Mg(x) \geq 1\right\} \setminus \left\{x \in \mathbb{R} : g(x) \geq 1\right\} \\ &\subset \left\{x \in \mathbb{R} : Mg(x) \neq g(x)\right\} . \end{aligned}$$

It follows that also $\mathcal{D}_f \cap E$ is a measurable set and consequently we have that the restriction $f_{|\mathcal{D}_f \cap E}$ is a measurable function, since $f_{|\mathcal{D}_f \cap E} = g_{|\mathcal{D}_f \cap E}$ and $\mathcal{D}_f \cap E \subset \mathcal{D}_g$. Thus, we have shown facts i) and ii).

To prove iii), notice that if $A \subset E$ is measurable and $f_{|A}$ is measurable, then also $g_{|A}$ is measurable, hence $\mu(A \setminus \mathcal{D}_g) = 0$ by Theorem 2.8, that is $A \subset \mathcal{D}_g \cup N$, where N is a nullset. It follows that

$$\begin{aligned} A &\subset \big(\mathcal{D}_g \cup N\big) \cap E \\ &\subset \Big[\mathcal{D}_g \cap \left\{x \in \mathbb{R} : Mg(x) = g(x)\right\} \cap E\Big] \cup \left\{x \in \mathbb{R} : Mg(x) \neq g(x)\right\} \cup N , \end{aligned}$$

hence $\mu(A \setminus (\mathcal{D}_f \cap E)) = 0$, since

$$\begin{aligned} &\mathcal{D}_g \cap \left\{x \in \mathbb{R} : Mg(x) = g(x)\right\} \cap E \\ &\subset \mathcal{D}_g \cap \left\{x \in \mathbb{R} : Mg(x) \geq 1\right\} \cap E = \mathcal{D}_f \cap E \end{aligned}$$

and since $\{x \in \mathbb{R} : Mg(x) \neq g(x)\} \cup N$ is a set of zero measure.

Finally, to show iv), it is enough to observe that by virtue of the implication

$$M\text{-}\lim_{y \to x} f(y) = l \quad \Longrightarrow \quad M\text{-}\lim_{y \to x} g(y) = l ,$$

we have

$$\left\{x \in E : Mf(x) \neq f(x)\right\} \subset \left\{x \in \mathbb{R} : Mg(x) \neq g(x)\right\} .$$

Next, we prove the theorem in general. Given any $f : E \to \mathbb{R}$, we consider the function $h = e^f + 1$. By the preceding part of the proof all statements i)–iv) are true for such a function h. Also, it is obvious that for every measurable set $A \subset E$ we have the equivalence:

$$f|_A \text{ measurable} \iff h|_A \text{ measurable} .$$

We first show the measurability of $\mathcal{D}_f$. To this aim, we notice that Lemma 2.5 is still true, just by the same proof, even if the domain of the function f, there considered, is assumed to be a subset of $\mathbb{R}$. Thus, coming back to our functions f and h, we get the following expression for $\mathcal{D}_f$:

$$\mathcal{D}_f = \left\{x \in \mathcal{D}_h : Mh(x) > 1\right\} ,$$

which implies

$$\begin{aligned}\mathcal{D}_f = &\Big[\left\{x \in \mathcal{D}_h : Mh(x) > 1\right\} \cap \left\{x \in E : Mh(x) = h(x)\right\}\Big] \\ &\cup \Big[\left\{x \in \mathcal{D}_h : Mh(x) > 1\right\} \setminus \left\{x \in E : Mh(x) = h(x)\right\}\Big] ,\end{aligned}$$

whence the measurability of $\mathcal{D}_f$ follows, since both members of the above written union are measurable sets by the properties of h. In fact, the first set can be written as

$$\left\{x \in \mathcal{D}_h \cap E : h(x) > 1\right\} \setminus \left\{x \in E : Mh(x) \neq h(x)\right\} ,$$

while the second one is contained in the nullset

$$\big(\mathcal{D}_h \setminus E\big) \cup \left\{x \in E : Mh(x) \neq h(x)\right\} .$$

The above expression of $\mathcal{D}_f$ also implies that $\mathcal{D}_f \setminus E \subset \mathcal{D}_h \setminus E$, thus $\mathcal{D}_f \setminus E$ is a nullset and $\mathcal{D}_f \cap E$ is measurable, and that $\mathcal{D}_f \cap E \subset \mathcal{D}_h \cap E$, thus $h|_{\mathcal{D}_f \cap E}$ is a measurable function, hence $f|_{\mathcal{D}_f \cap E}$ is measurable too.

Now, we prove iii). We observe that also Lemma 2.6 is true, by the same argument, for functions defined on subsets of $\mathbb{R}$, thus we have that $L = \{x \in E : Mh(x) = 1\}$ is a nullset. Let A be any measurable subset of E such that $f|_A$ is measurable. Then $h|_A$ is measurable too, hence $\mu\big(A \setminus (\mathcal{D}_h \cap E)\big) = 0$. On the other hand, we have $\mathcal{D}_f \cap E = \mathcal{D}_h \cap E \setminus L$, hence

$$A \setminus \big(\mathcal{D}_f \cap E\big) = A \setminus \big(\mathcal{D}_h \cap E \setminus L\big) \subset \Big[A \setminus \big(\mathcal{D}_h \cap E\big)\Big] \cup L ,$$

thus also $A \setminus \big(\mathcal{D}_f \cap E\big)$ is a nullset.

Finally, to prove iv), we observe that by Lemma 2.5, generalized, we have the implication

$$M\text{-}\lim_{y\to x} f(y) = l \quad \Longrightarrow \quad M\text{-}\lim_{y\to x} h(y) = e^l + 1$$

and hence

$$\{x \in E : Mf(x) \neq f(x)\} \subset \{x \in E : Mh(x) \neq h(x)\},$$

from which the result follows. □

3. Continuity of Real Functions.

Given a function $f : \mathbb{R} \to \mathbb{R}$, it is interesting to see if it is equal almost everywhere to a continuous function. The problem is not trivial, since, as everybody knows, strange things may happen. For instance, the Dirichlet function is nowhere continuous, but it is in the same equivalence class of the null function. On the other hand, the Heaviside step function is continuous everywhere except in zero, but no representatives of its equivalence class exhibits continuity at zero. If one knows from the beginning that f is equal almost everywhere to a continuous function, then a continuous representative of $[f]$ is given by Lf. The converse, however, is not true, namely, as we will show in the following example, there are equivalence classes not containing any continuous representative, for which Lf is defined everywhere. It is then a natural question to ask whether, given f, it is possible to find the "most" (if any) continuous representative of the class $[f]$.

Example 3.1. Define

$$f(x) = \begin{cases} 1 & \text{if } x \in (-\infty, 0] \\ 2^{2n}(x - 2^{-n}) & \text{if } x \in \big[2^{-n}, 2^{-n}(1 + 2^{-n})\big),\ n \in \mathbb{N},\ n \geq 2 \\ 1 & \text{if } x \in \big[2^{-n}(1 + 2^{-n}), 2^{-n+1}(1 - 2^{-n})\big), \\ & \qquad n \in \mathbb{N},\ n \geq 2 \\ 2^{2n-1}(2^{-n+1} - x) & \text{if } x \in \big[2^{-n+1}(1 - 2^{-n}), 2^{-n+1}\big),\ n \in \mathbb{N},\ n \geq 2 \\ 0 & \text{if } x \in [1/2, \infty). \end{cases}$$

Notice that f is continuous except in zero. Therefore Lf is defined in every point except at most zero. We show that $Lf(0) = 1$. Set $\delta > 0$, and let $n = n(\delta)$

the smallest $n \in \mathbb{N}$ such that $2^{-n-1} < \delta$. Select $\varepsilon > 0$. Then, for $\delta \leq 1/2$ we have

$$\begin{aligned} \frac{\mu\big(I_\delta(0) \cap f^{-1}(\Delta_\varepsilon(1))\big)}{\delta} &\leq 2^{n+1}\mu\big((0, 2^{-n}) \cap f^{-1}(\{1\}^C)\big) \\ &< 2^{n+3} \sum_{j=n+1}^{\infty} 4^{-j} \\ &< 2^{-n+2}. \end{aligned}$$

Since $n \to \infty$ as $\delta \to 0$, we conclude that $Mf(0) = 1$, and thus from Proposition 2.1 $Lf(0) = 1$.

We remarked in the introduction that no point of continuity of $g \in [f]$ can be in $\mathcal{C}_f^C$. Here we show the converse, namely, we exhibit a function $g \in [f]$ whose set of points of continuity is exactly $\mathcal{C}_f$. We need two preliminary lemmas.

Lemma 3.2. *Given $f : \mathbb{R} \to \mathbb{R}$, the function $Mf_{|\mathcal{D}_f}$ is continuous on $\mathcal{C}_f$.*

Proof. Suppose the lemma is not true. Then there exist $x \in \mathcal{C}_f$, $\varepsilon > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of $\mathcal{D}_f$ converging to x such that $Mf(x_n) \in \Delta_\varepsilon(l)$, having set $l = Mf(x)$. Since f has C-limit l at x, there exists $\delta > 0$ such that

$$\mu\big(I_\delta x \cap f^{-1}(\Delta_{\varepsilon/2}(l))\big) = 0.$$

Choose n large enough so that $x_n \in I_\delta(x)$. Denote $l_n = Mf(x_n)$. Then there exists $\delta_n > 0$ such that $I_{\delta_n}(x_n) \subset I_\delta(x)$, and

$$\mu^*\big(I_{\delta_n}(x_n) \cap f^{-1}(\Delta_{\varepsilon/2}(l_n))\big) < \delta_n.$$

Hence

$$\mu^*\big(I_{\delta_n}(x_n) \cap f^{-1}(I_{\varepsilon/2}(l_n))\big) \geq 2\delta_n - \mu^*\big(I_{\delta_n}(x_n) \cap f^{-1}(\Delta_{\varepsilon/2}(l_n))\big) > \delta_n.$$

On the other hand, $I_{\varepsilon/2}(l_n) \subset \Delta_{\varepsilon/2}(l)$, thus

$$\mu^*\big(I_\delta(x) \cap f^{-1}(\Delta_{\varepsilon/2}(l))\big) \geq \mu^*\big(I_{\delta_n}(x_n) \cap f^{-1}(I_{\varepsilon/2}(l_n))\big) > \delta_n,$$

which leads to a contradiction. $\square$

Lemma 3.3. *Given $f : \mathbb{R} \to \mathbb{R}$, there exists $\mathcal{D} \subset \mathbb{R}$ such that $\mathcal{D} \supset \mathcal{C}_f$, $\mu(\mathcal{D}^C) = 0$ and $Mf_{|\mathcal{D}}$ is continuous on $\mathcal{C}_f$.*

Proof. Since $\mathcal{C}_f$ depends only on $[f]$, in force of Theorem 2.4 we can (and do) assume that $f = Mf$. Fix $n \in \mathbb{N}$. Then for every $x \in \mathcal{C}_f$ there exists $\delta_{x,n} \leq 1/n$ such that

$$\mu\big(I_\delta(x) \cap f^{-1}(\Delta_{1/n}(f(x)))\big) = 0 \qquad \forall \delta \leq \delta_{x,n}.$$

The family $\mathcal{V}_n = \{I_{\delta_{x,n}}(x) : x \in \mathcal{C}_f\}$ is an open cover of $\mathcal{C}_f$. Since $\mathbb{R}$ is second countable, any open cover of a subset of $\mathbb{R}$ admits a countable subcover. Thus there exists a countable subset $\{x_j^n\}_{j\in J_n}$, with $J_n \subset \mathbb{N}$, of $\mathcal{C}_f$ such that the countable family $\{I_{\delta_{j,n}}(x_j^n)\}_{j\in J_n}$ (where we write for simplicity $\delta_{j,n}$ in place of $\delta_{x_j^n,n}$), of (not necessarily disjoint) elements of $\mathcal{V}_n$, is a cover of $\mathcal{C}_f$. Further, denote

$$P_n = \bigcup_{j\in J_n} \big(I_{\delta_{j,n}}(x_j^n) \cap f^{-1}(\Delta_{1/n}(f(x_j^n)))\big) \quad \textit{and} \quad P = \bigcup_n P_n,$$

and set

$$\mathcal{D} = \mathcal{D}_f \cup P^C.$$

We claim that $f_{|\mathcal{D}}$ is continuous on $\mathcal{C}_f$. Thus let $x \in \mathcal{C}_f$ and select $\varepsilon > 0$. Then by Lemma 3.2 there exists $\delta > 0$ such that

$$f\big(I_\delta(x) \cap \mathcal{D}_f\big) \subset I_{\varepsilon/2}(f(x)).$$

Choose

$$n > \max\left\{\frac{2}{\varepsilon}, \frac{2}{\delta}\right\}.$$

Then there exists x_j^n (which may coincide with x) such that

$$x \in I_{\delta_{j,n}}(x_j^n) \subset I_\delta(x).$$

Let $y \in I_{\delta_{j,n}}(x_j^n) \cap \mathcal{D}$. If $y \in \mathcal{D}_f$ then $f(y) \in I_\varepsilon(f(x))$. If $y \in \mathcal{D}_f^C \cap P^C$, then in particular we have $y \in P_n^C$, and since $y \in I_{\delta_{j,n}}(x_j^n)$, it follows that

$$|f(y) - f(x_j^n)| < \frac{1}{n} < \frac{\varepsilon}{2}.$$

Therefore

$$|f(y) - f(x)| \leq |f(y) - f(x_j^n)| + |f(x_j^n) - f(x)| < \varepsilon.$$

Thus, if we choose $\eta > 0$ such that $I_\eta(x) \subset I_{\delta_{j,n}}(x_j^n)$, we conclude that

$$f(I_\eta(x) \cap \mathcal{D}) \subset I_\varepsilon(f(x)),$$

as claimed. □

Notice that, in order to prove Lemma 3.3, it would have been enough to prove a weaker version of Lemma 3.2, namely, to show that $Mf_{|\mathcal{C}_f}$ is continuous on $\mathcal{C}_f$. However, if f is measurable, then $\mu(\mathcal{D}_f^C) = 0$ and Lemma 3.2, as stated, immediately implies Lemma 3.3.

Theorem 3.4. *Given $f : \mathbb{R} \to \mathbb{R}$, there exists $g \in [f]$ whose set of points of continuity is exactly $\mathcal{C}_f$.*

Proof. Again, we assume that $f = Mf$. Then from previous Lemma 3.3, there exists $\mathcal{D} \subset \mathbb{R}$ such that $\mathcal{D} \supset \mathcal{C}_f$, $\mu(\mathcal{D}^C) = 0$ and $f_{|\mathcal{D}}$ is continuous on $\mathcal{C}_f$. For every $n \in \mathbb{N}$ and every $j \in \mathbb{Z}$ define the half-open interval $O_j^n = [j/n, (j+1)/n)$. Then, for every $n \in \mathbb{N}$, $\mathbb{R} = \bigcup_j O_j^n$. Furthermore, select $x_j^n \in O_j^n \cap \mathcal{D}$ (which always exists since $\mu(\mathcal{D}^C) = 0$). Finally define

$$f_n(x) = \begin{cases} f(x) & \text{if } x \in \mathcal{D} \\ f(x_j^n) & \text{if } x \in \mathcal{D}^C \cap O_j^n, \end{cases}$$

and let

$$g(x) = \begin{cases} \limsup\limits_{n\to\infty} f_n(x) & \text{if } \limsup\limits_{n\to\infty} f_n(x) \in \mathbb{R} \\ 0 & \text{otherwise}. \end{cases}$$

It is apparent that $g = f$ on $\mathcal{D}$ (so $g = f$ almost everywhere) and $g_{|\mathcal{D}}$ is continuous on $\mathcal{C}_f$. We claim that g is continuous on $\mathcal{C}_f$. Indeed, let $x \in \mathcal{C}_f$ and select $\varepsilon > 0$. Then there exists $\delta > 0$ such that, if $y \in I_\delta(x) \cap \mathcal{D}$, it follows that $|g(y) - g(x)| < \varepsilon/2$. On the other hand, if $y \in I_\delta(x) \cap \mathcal{D}^C$ then there is a sequence $\{j_n\}_{n\in\mathbb{N}} \subset \mathbb{Z}$ such that $f_n(y) = f(x_{j_n}^n) = g(x_{j_n}^n)$ for all $n \in \mathbb{N}$. Since $y \in I_\delta(x)$, we have also $x_{j_n}^n \in I_\delta(x)$, and hence

$$|f_n(y) - g(x)| = |g(x_{j_n}^n) - g(x)| < \frac{\varepsilon}{2},$$

for all n large enough. This implies that $\limsup_{n\to\infty} f_n(y) \in \mathbb{R}$ and that

$$|g(y) - g(x)| = |\limsup_{n\to\infty} f_n(y) - g(x)| \le \limsup_{n\to\infty} |f_n(y) - g(x)| \le \frac{\varepsilon}{2} < \varepsilon.$$

We have then proved that $g(I_\delta(x)) \subset I_\varepsilon(g(x))$, that is, g is continuous at x. □

Since $\mathcal{C}_f$ is the set of points of continuity of a function g, we can also conclude that it is a G_δ-set (see, for instance, [4]).

It is worth observing that Mf may not be continuous on $\mathcal{C}_f$. This justifies the rather indirect proof of Theorem 3.4.

Example 3.5. Define

$$f(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0] \cup [1, \infty) \\ 1/n & \text{if } x \in (1/(n+1), 1/n), \quad n \in \mathbb{N} \\ 1 & \text{if } x = 1/n, \quad n \in \mathbb{N}. \end{cases}$$

It is clear that f has C-limit 0 at $x = 0$ (so $0 \in \mathcal{C}_f$). On the other hand, since f does not have M-limit at $1/n$, it follows that $Mf(1/n) = f(1/n) = 1$. Thus Mf is not continuous at $x = 0$.

In force of Theorem 3.4 we can provide a characterization of classes of functions containing a continuous representative.

Corollary 3.6. *A function $f : \mathbb{R} \to \mathbb{R}$, is equal almost everywhere to a continuous function if and only if $\mathcal{C}_f = \mathbb{R}$.*

Remark 3.7. The result expressed by Theorem 3.4 actually holds for a function $f : E \subset \mathbb{R} \to \mathbb{R}$, provided that $\partial E \cap E$ is a closed discrete set.

In this case we still define $\mathcal{C}_f$ as the subset of E consisting of all points x which are continuity points for some $g : E \to \mathbb{R}$, $g = f$ almost everywhere. Equivalently, a point $x \in E$ belongs to $\mathcal{C}_f$ if there exists a (possibly nonunique) $l \in \mathbb{R}$ such that for every $\varepsilon > 0$ it results $\mu\big(I_\delta(x) \cap f^{-1}(\Delta_\varepsilon(l))\big) = 0$ for some $\delta > 0$.

To get the above claimed extension of Theorem 3.4 one has simply to consider any function $h : \mathbb{R} \to \mathbb{R}$, continuous at every point $x \in \partial E \cap E \cap \mathcal{C}_f$ and such that $h_{|E} = f$ almost everywhere, and apply Theorem 3.4 to h. Notice that the construction of such a function h is possible since the set $\partial E \cap E$ is cluster-point free.

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