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COMPUTING INVARIANTS OF CUBIC SURFACES

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We report on the computation of invariants, covariants, and contravariants of cubic surfaces. The approach is based on the Clebsch transfer principle and transvection. All algorithms are implemented in the computer algebra system `magma`. The code can be used to efficiently compute invariants of surfaces definied over number fields and function fields.

1. Introduction

Given two hypersurfaces of the same degree in projective space over an algebraically closed field, one may ask for the existence of an automorphism of the projective space that maps one of the hypersurfaces to the other. It turns out that if the hypersurfaces are stable [11, Def. 1.7] in the sense of geometric invariant theory, such an automorphism exists if and only if all the invariants of the hypersurfaces coincide [10, Prop. 1.3.i)].

Aside from cubic curves in $\mathbf{P}^2$ and quartic surfaces in $\mathbf{P}^3$, an isomorphism between smooth hypersurfaces of degree $d \geq 3$ always extends to an automorphism of the ambient projective space [8, Th. 2]. Thus, the invariants may be used to test abstract isomorphy.

If the base field is not algebraically closed, two varieties with equal invariants can differ by a twist. A necessary condition for the existence of a non-trivial twist is that the variety has a non-trivial automorphism.

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In this article, we focus on the case of cubic surfaces. For them, it was proven by Clebsch [3] that the ring of invariants of even weight is generated by five fundamental invariants of degrees 8, 16, 24, 32, and 40. Later, Salmon [14] worked out explicit formulas for these invariants based on the pentahedral representation of the cubic surface, introduced by Sylvester [18].

We describe an approach to compute the Clebsch-Salmon invariants, linear covariants, and some contravariants of cubic surfaces, that does not rely on a calculation of the pentahedron. Instead, it is based on the Clebsch transfer principle. The algorithm works for any cubic surface with coefficients in a field of characteristic not equal to 2, 3, or 5. Using this, we also compute an invariant of degree 100 [5, Sec. 9.4.5] and odd weight that vanishes if and only if the cubic surface has a non-trivial automorphism. The square of this invariant is a polynomial expression in Clebsch's invariants.

This can be used as an isomorphy test for all stable cubic surfaces over algebraically closed fields and for all surfaces over non-closed fields, for which the degree 100 invariant does not vanish.

All algorithms are available since December 2012 [2, Sec. 6.4] in the computer algebra system magma [1]. We illustrate the computation of the invariants and the discriminant by some examples:

```
> r<x,y,z,w> := PolynomialRing(Rationals(),4);
> ClebschSalmonInvariants(x^3+y^3+z^3+w^3);
[ 1, 0, 0, 0, 0 ]
-7625597484987
> Factorization(7625597484987);
[ <3, 27> ]
```

The function ClebschSalmonInvariants computes the five fundamental invariants and the discriminant of a cubic surface. As the discriminant is -3^{27}, the diagonal surface has bad reduction only for $p = 3$. The function can handle surfaces with coefficients in any field, as long as the characteristic is different from 2, 3, and 5. Families of surfaces can be dealt with working over the function field:

```
> r<t> := PolynomialRing(Rationals());
> r4<x,y,z,w> := PolynomialRing(FieldOfFractions(r),4);
> S := x^3+y^3+z^3+w^3+t*x*y*z;
> inv,disc := ClebschSalmonInvariants(S);
> Factorization(Numerator(disc));
[
    <t + 3, 6>,
    <t^2 - 3*t + 9, 6>
```

```
]
```

This shows that the degenerated fibers of the family $x^3+y^3+z^3+txyz+w^3=0$ are located at $t=-3$ and $t=\frac{3\pm3\sqrt{-3}}{2}$. A more complex example is given by the octanomial family [13]. The invariants of this family can be computed in the same way as above within about 2 seconds of CPU time. They are polynomials consisting of 49, 364, 1302, 3709, and 7689 terms, resprectively, in the eight coefficients of the family.

Further, the functions

```
LinearCovariantsOfCubicSurface, SkewInvariant100,
ContravariantsOfCubicSurface, CubicSurfaceFromClebschSalmon
```

are available. They can be used to compute the covariants, the degree 100 invariant of odd weight, and the contravariants, described in this article. The last function computes a cubic surface with prescribed invariants, as long as the last invariant is not zero. It can be used as follows

```
r4<x,y,z,w> := PolynomialRing(Rationals(),4);
S := CubicSurfaceFromClebschSalmon([1,2,3,4,5]);
r4!MinimizeReduceCubicSurface(S);
```

to compute the model

$$\begin{aligned}-125x^3+320x^2y+100x^2z+94x^2w-64xy^2-470xyz+492xyw\\+530xz^2-886xzw+390xw^2-79y^3+228y^2z-197y^2w-180yz^2\\-94yzw-242yw^2-235z^3+526z^2w-825zw^2+279w^3=0\end{aligned}$$

of a cubic surface with invariants 1, 2, 3, 4, and 5.

Earlier works

We would not be surprised if others implemented the computation of invariants of cubic surfaces before. By personal communication we heard about an `maple` implementation written by Andrew du Plessis. But, we are not aware of any publication.

2. The Clebsch-Salmon invariants

The first part of this section presents the concepts of invariants, co- and contravariants. The second part describes the invariants and linear covariants of cubic surfaces as introduced by Salmon [14].

Let K be a field of characteristic zero and $K[X_1,\dots,X_n]^{(d)}$ the K-vector space of all homogeneous forms of degree d. Further, we fix the left group action

$$\mathrm{GL}_n(K)\times K[X_1,\dots,X_n]\to K[X_1,\dots,X_n],\quad (M,f)\mapsto M\cdot f,$$

with $(M\cdot f)(X_1,\dots,X_n):=f((X_1,\dots,X_n)M)$.

Finally, on the polynomial ring $K[Y_1,\dots,Y_n]$, we choose the action

$$\mathrm{GL}_n(K)\times K[Y_1,\dots,Y_n]\to K[Y_1,\dots,Y_n],\quad (M,f)\mapsto M\cdot f,$$

given by $(M\cdot f)(Y_1,\dots,Y_n):=f((Y_1,\dots,Y_n)\left(M^{-1}\right)^{\top})$.

Remark 2.1. In the actions above, the operation of a matrix M can be viewed as a composition of maps, as follows,

$$(X\mapsto (M\cdot f)(X))=(X\mapsto f(X))\circ(X\mapsto XM)$$

and

$$(Y\mapsto (M\cdot f)(Y))=(Y\mapsto f(Y))\circ\left(Y\mapsto Y\left(M^{-1}\right)^{\top}\right).$$

As we work with row vectors, in both cases, we get $M_1\cdot(M_2\cdot f)=(M_1M_2)\cdot f$, and indeed have group actions.

Definition 2.2. An *invariant I of degree D and weight w* is a map

$$K[X_1,\dots,X_n]^{(d)}\to K$$

that may be given by a homogeneous polynomial of degree D in the coefficients of f and satisfies

$$I(M\cdot f)=\det(M)^w\cdot I(f),$$

for all $M\in\mathrm{GL}_n(K)$ and all forms $f\in K[X_1,\dots,X_n]^{(d)}$.

Definition 2.3. A *covariant C of degree D, order p, and weight w* is a map

$$K[X_1,\dots,X_n]^{(d)}\to K[X_1,\dots,X_n]^{(p)}$$

such that each coefficient of $C(f)$ is a homogeneous degree D polynomial in the coefficients of f and that satisfies

$$C(M\cdot f)=\det(M)^w\cdot M\cdot(C(f)),$$

for all $M\in\mathrm{GL}_n(K)$ and all forms $f\in K[X_1,\dots,X_n]^{(d)}$.

Definition 2.4. A *contravariant c of degree D, order p, and weight w* is a map

$$K[X_1,\dots,X_n]^{(d)} \to K[Y_1,\dots,Y_n]^{(p)}$$

such that each coefficient of $c(f)$ is a homogeneous degree D polynomial in the coefficients of f and that satisfies

$$c(M\cdot f) = \det(M)^w \cdot M\cdot c(f),$$

for all $M \in \mathrm{GL}_n(K)$ and all forms $f \in K[X_1,\dots,X_n]^{(d)}$. Note that the right hand side uses the action on $K[Y_1,\dots,Y_n]$.

Remark 2.5. The set of all invariants is a commutative ring and an algebra over the base field. The set of all covariants (resp. contravariants) is a commutative ring and a module over the ring of invariants.

Remark 2.6. Geometrically, the vanishing locus of f or a covariant $C(f)$ is a subset of the projective space whereas the vanishing locus of a contravariant $c(f)$ is a subset of the dual projective space. Replacing the matrix by the transpose inverse matrix gives the action on the dual space in a naive way.

Example 2.7. The discriminant of binary forms of degree d is an invariant of degree $2d-2$ and weight $d(d-1)$ [12, Chap. 2].

Example 2.8. Let f be a form of degree $d>2$ in n variables. Then the *Hessian H*, defined by

$$H(f) := \det\left(\frac{\partial^2 f}{\partial X_i\,\partial X_j}\right)_{i,j=1,\dots,n},$$

is a covariant of degree n, order $(d-2)n$, and weight 2.

Example 2.9. Let a smooth plane curve $V\subset \mathbf{P}^2$ be given by a ternary form f of degree d. Mapping f to the form that defines the dual curve [5, Sec. 1.2.2] of V is an example of a contravariant of degree $2d-2$ and order $d(d-1)$.

Salmon's formulas

Definition 2.10. A cubic surface given by a system of equations of the shape

$$a_0X_0^3 + a_1X_1^3 + a_2X_2^3 + a_3X_3^3 + a_4X_4^3 = 0, \quad X_0+X_1+X_2+X_3+X_4 = 0$$

is said to be in *pentahedral form*. The coefficients $a_0,\dots,a_4$ are called the *pentahedral coefficients* of the surface. The planes $X_i=0$ are called the *faces* and the intersection point of any three faces is called a *vertex of the pentahedron*.

The set of all cubic surfaces that have a pentahedral form is Zariski open in the Hilbert scheme of all cubic surfaces. Thus, it suffices to describe the invariants for these surfaces. For this, we denote by $\sigma_1, \ldots, \sigma_5$ the elementary symmetric functions in the pentahedral coefficients. Then the Clebsch-Salmon invariants (as mentioned in the introduction) of the cubic surface are given by [14, § 467],

$$I_8 = \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} = \sigma_1\sigma_5^3, \quad I_{24} = \sigma_4\sigma_5^4, \quad I_{32} = \sigma_2\sigma_5^6, \quad I_{40} = \sigma_5^8\,.$$

Further, Salmon lists four linear covariants of degrees 11, 19, 27, and 43 [14, § 468],

$$L_{11} = \sigma_5^2 \sum_{i=0}^{4} a_i x_i, \qquad L_{19} = \sigma_5^4 \sum_{i=0}^{4} \frac{1}{a_i} x_i,$$
$$L_{27} = \sigma_5^5 \sum_{i=0}^{4} a_i^2 x_i, \qquad L_{43} = \sigma_5^8 \sum_{i=0}^{4} a_i^3 x_i\,.$$

Finally, the 4×4 determinant of the matrix formed by the coefficients of these linear covariants of a cubic surface in $\mathbf{P}^3$ is an invariant I_{100} of degree 100. It vanishes if and only if the surface has Eckardt points or equivalently a non-trivial automorphism group [5, Sec. 9.4.5, Table 9.6]. The square of I_{100} can be expressed in terms of the other invariants above. For a modern view on these invariants, we refer to [5, Sec. 9.4.5].

For a general cubic surface, the vertices of the pentahedron coincide with the 10 singular points of its Hessian. Thus, using modern computer algebra, one can compute the singular points of the Hessian and deduce the faces of the pentahedron. Once the faces are found, the pentahedral coefficients can be computed by solving a linear system of equations. Having done this, one can use Salmon's formulas to compute the invariants [6, Algo. A.4]. A second method to compute the pentahedron, which is based on syzygies, was described in [9, Algo. 3.1].

3. Transvection

One classical approach to write down invariants is to use the transvection (called Überschiebung in German). This is part of the so called symbolic method [19, Chap. 8, §2], [7, App. B.2]. We illustrate it in the case of ternary forms.

Definition 3.1. Let $K[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z_1, \ldots, Z_n]$ be the polynomial ring in

$3n$ variables. For $i, j, k \in \{1, \ldots, n\}$, we denote by $(i\,j\,k)$ the differential operator

$$(i\,j\,k) := \det \begin{pmatrix} \frac{\partial}{\partial X_i} & \frac{\partial}{\partial X_j} & \frac{\partial}{\partial X_k} \\ \frac{\partial}{\partial Y_i} & \frac{\partial}{\partial Y_j} & \frac{\partial}{\partial Y_k} \\ \frac{\partial}{\partial Z_i} & \frac{\partial}{\partial Z_j} & \frac{\partial}{\partial Z_k} \end{pmatrix}.$$

Example 3.2. Using this notation, the *Aronhold invariants* S and T of the ternary cubic form f are given by

$$S(f) := (1\,2\,3)(2\,3\,4)(3\,4\,1)(4\,1\,2) f(X_1, Y_1, Z_1) \cdots f(X_4, Y_4, Z_4),$$
$$T(f) := (1\,2\,3)(1\,2\,4)(2\,3\,5)(3\,1\,6)(4\,5\,6)^2 f(X_1, Y_1, Z_1) \cdots f(X_6, Y_6, Z_6).$$

The first one is of degree and weight 4, the second one of degree and weight 6.

Using S and T, one can write down the discriminant of a ternary cubic as $\Delta := S^3 - 6T^2$. The discriminant vanishes if and only if the corresponding cubic curve is singular.

See [15, Sec. V] for a historical and [5, Sec. 3.4.1] for modern references concerning invariants of ternary cubic forms.

Remark 3.3. One can use the transvection to write down invariants of quaternary forms, as well. For example, if f is a quartic form in four variables then

$$(1\,2\,3\,4)^4 f(X_1, Y_1, Z_1, W_1) \cdots f(X_4, Y_4, Z_4, W_4)$$

is an invariant of degree 4. Here, $(1\,2\,3\,4)$ denotes the differential operator

$$(1\,2\,3\,4) := \det \begin{pmatrix} \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_3} & \frac{\partial}{\partial X_4} \\ \frac{\partial}{\partial Y_1} & \frac{\partial}{\partial Y_2} & \frac{\partial}{\partial Y_3} & \frac{\partial}{\partial Y_4} \\ \frac{\partial}{\partial Z_1} & \frac{\partial}{\partial Z_2} & \frac{\partial}{\partial Z_3} & \frac{\partial}{\partial Z_4} \\ \frac{\partial}{\partial W_1} & \frac{\partial}{\partial W_2} & \frac{\partial}{\partial W_3} & \frac{\partial}{\partial W_4} \end{pmatrix}.$$

For a quaternary cubic form, one can apply this to its Hessian to get an invariant of degree 16. However, a direct evaluation of such formulas for forms in four variables is too slow in practice. The reason is that both the differential operators and the product $f(X_1, Y_1, Z_1, W_1) \cdots f(X_4, Y_4, Z_4, W_4)$ usually have many terms.

4. The Clebsch transfer principle

We refer to [5, Sec. 3.4.2] for a detailed and modern description of the Clebsch transfer principle. The basic idea is to compute a contravariant of a form of degree d in n variables out of an invariant of a form of degree d in $(n-1)$ variables. In the case of cubic surfaces, this can be explained geometrically as follows. First, we intersect the cubic surface with a variable plane to get a cubic curve. Then we evaluate an invariant of cubic curves on this intersection. This construction gives a form on the dual projective space which turns out to be a contravariant.

Definition 4.1. We consider the vector space $V = K^n$ and choose the volume form given by the determinant. We have the following isomorphism

$$\Phi\colon \Lambda^{n-1}V \to V^*, \quad v_1 \wedge \cdots \wedge v_{n-1} \mapsto (v \mapsto \det(v, v_1, \ldots, v_{n-1}))\,.$$

Definition 4.2. Let I be a degree D, weight w invariant on $K[U_1, \ldots, U_{n-1}]^{(d)}$. Then the *Clebsch transfer* of I is the contravariant $\tilde{I}$ of degree D and order w

$$\tilde{I}\colon K[X_1, \ldots, X_n]^{(d)} \to K[Y_1, \ldots, Y_n]^{(w)},$$

given by

$$\tilde{I}(f)\colon (K^n)^* \to K, \quad l \mapsto I(f(U_1 v_1 + \cdots + U_{n-1} v_{n-1}))\,.$$

Here, $v_1, \ldots, v_{n-1}$ are given by $v_1 \wedge \ldots \wedge v_{n-1} = \Phi^{-1}(l)$. Note that $\tilde{I}(f)$, as defined, is indeed a polynomial mapping and homogeneous of degree w.

Example 4.3. Denote by S and T the invariants of ternary cubic forms, introduced above. Then $\tilde{S}$ is a degree 4, order 4 contravariant of quaternary cubic forms. Further, $\tilde{T}$ is a contravariant of degree 6 and order 6.

The discriminant of a cubic curve is given by $\Delta = S^3 - 6T^2$. It vanishes if and only if the cubic curve is singular. Thus, the dual surface of the smooth cubic surface $V(f)$ is given by $\tilde{\Delta}(f) = \tilde{S}(f)^3 - 6\tilde{T}(f)^2 = 0$.

By definition, the dual surface of a smooth surface $V(f) \subset \mathbf{P}^3$ is the set of all tangent planes of $V(f)$. A plane $P \in (\mathbf{P}^3)^*$ is tangent if and only it the intersection $V(f) \cap P$ is singular. Thus, P is a point on the dual surface if and only if $\tilde{\Delta}(f)(P) = 0$. Here, Δ is the discriminant of ternary forms of the same degree as f.

For a given cubic form $f \in K[X, Y, Z, W]$, we compute $\tilde{S}(f)$ by interpolation as follows:

1. Choose 35 vectors $p_1, \ldots, p_{35} \in \left(K^4\right)^*$ in general position.
2. Compute $\Phi^{-1}(p_i)$, for $i = 1, \ldots, 35$.
3. Compute $s_i := S(f(U_1 v_1 + U_2 v_2 + U_3 v_3))$, for $v_1 \wedge v_2 \wedge v_3 = \Phi^{-1}(p_i)$ and all $i = 1, \ldots, 35$.
4. Compute the degree 4 form $\tilde{S}(f)$ by interpolating the arguments p_i and the values s_i.

We can compute $\tilde{T}(f)$ in the same way. The only modification necessary is to increase the number of vectors, as the space of sextic forms is of dimension 84.

5. Action of contravariants on covariants and vice versa

Is was already known in the 19th century [16, Lesson XIV] that there is a connection of co- and contravariants with differential operators. Here, we list what will be used about it.

1. Recall that the rings $K[X_1,\dots,X_n]$ and $K[Y_1,\dots,Y_n]$ are equipped with $\mathrm{GL}_n(K)$-actions, as introduced at the beginning of Section 2.

2. The ring of differential operators
$$K\left[\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_n}\right]$$
acts on $K[X_1,\dots,X_n]$.

3. The $\mathrm{GL}_n(K)$-action on $K\left[\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_n}\right]$ given by
$$M\cdot\left(\frac{\partial}{\partial v}\right):=\frac{\partial}{\partial(v\cdot M^{-1})}\text{ for all } v\in K^n$$
results in the equality
$$M\cdot\left(\frac{\partial f}{\partial v}\right)=\left(M\cdot\frac{\partial}{\partial v}\right)(M\cdot f),$$
for all $f\in K[X_1,\dots,X_n]$ and all $v\in K^n$.

4. The map
$$\psi\colon K[Y_1,\dots,Y_n]\to K\left[\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_n}\right],\quad Y_i\mapsto\frac{\partial}{\partial X_i}$$
is an isomorphism of rings. Further, for each $M\in\mathrm{GL}_n(K)$, we have the following commutative diagram
$$\begin{tikzcd}
K[Y_1,\dots,Y_n] \arrow[r, "\psi"] \arrow[d, "M"'] & K\left[\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_n}\right] \arrow[d, "M"] \\
K[Y_1,\dots,Y_n] \arrow[r, "\psi"] & K\left[\frac{\partial}{\partial X_1},\dots,\frac{\partial}{\partial X_n}\right]
\end{tikzcd}.$$
In other words, ψ is an isomorphism of $\mathrm{GL}_n(K)$-modules.

5. Let C be a covariant and c a contravariant on $K[X_1,\dots,X_n]^{(d)}$. Denote the order of C by P and the order of c by p. For $P \geq p$, we define

$$c \vdash C\colon K[X_1,\dots,X_n]^{(d)} \to K[X_1,\dots,X_n]^{(P-p)}, \quad f \mapsto \psi(c(f))\,(C(f))\,.$$

The notation $\vdash$ follows [7, p. 304].

6. Assume $c \vdash C$ not to be zero. If $p < P$ then $c \vdash C$ is a covariant of order $P - p$. If $p = P$ then $c \vdash C$ is an invariant. In both cases, the degree of $c \vdash C$ is the sum of the degrees of c and C.

7. Similarly to ψ, one can introduce a map

$$\widehat{\psi}\colon K[X_1,\dots,X_n] \to K\left[\frac{\partial}{\partial Y_1},\dots,\frac{\partial}{\partial Y_n}\right], \quad X_i \mapsto \frac{\partial}{\partial Y_i}\,.$$

As above, $\widehat{\psi}$ is an isomorphism of rings and $\mathrm{GL}_n(K)$-modules. Let C a covariant and c a contravariant on $K[X_1,\dots,X_n]^{(d)}$. We define $C \vdash c$ by

$$(C \vdash c)(f) := \widehat{\psi}(C(f))\,(c(f))\,.$$

8. Assume $c \vdash C$ not to be zero. If $p > P$ then $C \vdash c$ is a contravariant of order $p - P$. If $p = P$ then $C \vdash c$ is an invariant. In both cases, the degree of $C \vdash c$ is the sum of the degrees of C and c.

6. Explicit invariants of cubic surfaces

We are now fully prepared for the computation of invariants of cubic surfaces. In fact, this approach can be used to compute invariants of hypersurfaces of any degree and dimension in projective space.

Remark 6.1. It is well known that the ring of invariants of quaternary cubic forms is generated by the six invariants of degrees 8, 16, 24, 32, 40, and 100 [5, Sec. 9.4.5]. The first five generators are primary invariants [4, Def. 2.4.6]. Thus, the vector spaces of all invariants of degrees 8, 16, 24, 32 and 40 are of dimensions 1, 2, 3, 5, and 7. In general, these dimensions are encoded in the Molien series, which can be computed efficiently using character theory [4, Ch. 4.6].

In the lucky case that one is able to write down a basis of the vector space of all invariants of a given degree d, one can find an expression of a given invariant of degree d by linear algebra. This requires that the invariant is known for sufficiently many surfaces. For cubic surfaces, this is provided by the pentahedral equation.

Applying the methods above, we can write down many invariants for quaternary cubic forms. We start with the form f, its Hessian covariant $H(f)$, and the contravariant $\tilde{S}(f)$. Then we apply known covariants to contravariants and vice versa. Further, one can multiply two covariants or contravariants to get a new one. For efficiency, it is useful to keep the orders of the covariants and contravariants as small as possible. This way, they will not consist of too many terms.

Proposition 6.2. *Let f be a quarternary cubic form. With*

$$
\begin{aligned}
C_{4,0,4} &:= \tilde{S}(f), & C_{4,4} &:= H(f),\\
C_{6,2} &:= C_{4,0,4} \vdash f^2, & C_{9,3} &:= C_{4,0,4} \vdash (f \cdot C_{4,4}),\\
C_{10,0,2} &:= C_{6,2} \vdash C_{4,0,4}, & C_{11,1a} &:= C_{10,0,2} \vdash f,\\
C_{13,0,1} &:= C_{9,3} \vdash C_{4,0,4}, & C_{14,2} &:= C_{10,0,2} \vdash C_{4,4},\\
C_{14,2a} &:= C_{13,0,1} \vdash f, & C_{19,1a} &:= C_{13,0,1} \vdash C_{6,2},
\end{aligned}
$$

the following expressions

$$
\begin{aligned}
I_8 &:= \frac{1}{2^{11} \cdot 3^9} C_{4,0,4} \vdash C_{4,4},\\
I_{16} &:= \frac{1}{2^{30} \cdot 3^{22}} C_{6,2} \vdash C_{10,0,2},\\
I_{24} &:= \frac{1}{2^{41} \cdot 3^{33}} C_{10,0,2} \vdash C_{14,2},\\
I_{32a} &:= C_{10,0,2} \vdash C_{11,1a}^2,\\
I_{32} &:= \frac{2}{5}\left(I_{16}^2 - \frac{1}{2^{60} \cdot 3^{44}} \cdot I_{32a}\right),\\
I_{40a} &:= C_{4,0,4} \vdash (C_{11,1a}^2 \cdot C_{14,2}),\\
I_{40} &:= \frac{-1}{100} \cdot I_8 \cdot I_{32} - \frac{1}{50} \cdot I_{16} \cdot I_{24} - \frac{1}{2^{72} \cdot 3^{53} \cdot 5^2} I_{40a},
\end{aligned}
$$

give the Clebsch-Salmon invariants I_8, I_{16}, I_{24}, I_{32}, and I_{40}. Further, with

$$\begin{aligned}
C_{11,1} :=& \frac{1}{2^{20}3^{15}} C_{11,1a}, \\
C_{19,1} :=& \frac{1}{2^{33}\cdot 3^{24}\cdot 5}(C_{19,1a} + 2^{32}\cdot 3^{24}\cdot I_8 \cdot C_{11,1a}), \\
C_{27,1a} :=& \frac{1}{2^{42}3^{33}} C_{13,0,1} \vdash C_{14,2a}, \\
C_{27,1} :=& I_{16}\cdot C_{11,1} + \frac{1}{200}(C_{27,1a} - 2\cdot I_8^2 \cdot C_{11,1} - 10\cdot I_8 \cdot C_{19,1}), \\
C_{43,1a} :=& \frac{1}{2^{68}\cdot 3^{53}} C_{13,0,1} \vdash (C_{13,0,1} \vdash (C_{13,0,1} \vdash C_{4,4})), \\
C_{43,1} :=& \frac{-1}{1000} C_{43,1a} - \frac{1}{200}\cdot I_8^2 \cdot C_{27,1} + I_{16}\cdot C_{27,1} \\
&+ \frac{1}{1000}\cdot I_8^3 \cdot C_{19,1} - \frac{1}{10}\cdot I_8 \cdot I_{16} \cdot C_{19,1} - I_{24}\cdot C_{19,1} \\
&+ \frac{1}{200}\cdot I_8^2 \cdot I_{16}\cdot C_{11,1} + \frac{3}{20}\cdot I_8 \cdot I_{24}\cdot C_{11,1},
\end{aligned}$$

$C_{11,1}$, $C_{19,1}$, $C_{27,1}$, *and* $C_{43,1}$ *are Salmon's linear covariants. Here, we use the first index to indicate the degree of an invariant, covariant, or contravariant. The second index is the order of a covariant, whereas the third index is the order of a contravariant. Finally, we can compute* I_{100} *as the determinant of the 4 linear covariants.*

Proof. The following magma script shows in approximately one second of CPU time that the algorithm as described above coincides with Salmon's formulas for the pentahedral family, as the last two comparisons result in true.

```
r5 := PolynomialRing(Integers(),5);
ff5<a,b,c,d,e> := FunctionField(Rationals(),5);
r4<x,y,z,w> := PolynomialRing(ff5,4);

lfl := [x,y,z,w,-x-y-z-w];
col := [ff5.i : i in [1..5]];
f := a*x^3 + b*y^3 + c*z^3 + d*w^3 + e*(-x-y-z-w)^3;

sy_f := [ElementarySymmetricPolynomial(r5,i) : i in [1..5]];
sigma := [Evaluate(sf,col) : sf in sy_f];

I_8  := sigma[4]^2 - 4 *sigma[3] * sigma[5];
I_16 := sigma[1] * sigma[5]^3;
I_24 := sigma[4] * sigma[5]^4;
```

```
I_32 := sigma[2] * sigma[5]^6;
I_40 := sigma[5]^8;

L_11 := sigma[5]^2 * &+[   col[i] * lfl[i] : i in [1..5]];
L_19 := sigma[5]^4 * &+[ 1/col[i] * lfl[i] : i in [1..5]];
L_27 := sigma[5]^5 * &+[ col[i]^2 * lfl[i] : i in [1..5]];
L_43 := sigma[5]^8 * &+[ col[i]^3 * lfl[i] : i in [1..5]];

inv := ClebschSalmonInvariants(f);
cov := LinearCovariantsOfCubicSurface(f);

inv eq [I_8, I_16, I_24, I_32, I_40];
cov eq [L_11, L_19, L_27, L_43];
```

□

7. Performance test

Computing the Clebsch-Salmon invariants, following the approach above, for 100 cubic surfaces chosen at random with two digit integer coefficients takes about 3 seconds of CPU time *in total*. Most of the time is used for the direct evaluation of the invariant S of ternary cubics by transvection. Note that computing the contravariant $\tilde{S}$ by interpolation requires 35 evaluations of the invariant S of a ternary cubic.

Computing both contravariants $\tilde{S}$ and $\tilde{T}$ and the dual surface takes about 18 seconds of CPU time for the same 100 randomly chosen surfaces. The computations are done on one core of an Intel i5-2400 processor running at 3.1GHz. For comparison, the computation of the pentahedral form by inspecting the singular points of the Hessian takes about 10 seconds per example [6, Sec. 5.11].

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