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VARIETIES OF SIMULTANEOUS SUMS OF POWER FOR BINARY FORMS

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The problem of simultaneous decomposition of binary forms as sums of powers of linear forms is studied. For generic forms the minimal number of linear forms needed is found and the space parametrizing all the possible decompositions is described. More generally the variety parametrizing the 0-dimensional schemes apolar to a set of generic binary forms is described. These results are applied to the study of particular secant spaces of rational curves.

1. Introduction.

Let K be an algebraically closed field. Consider the polynomial ring $S = K[x_0, \dots, x_n]$ and a form $f \in S_d$. A well known problem deals with the possible decompositions of f as a sum of powers of linear forms, that is

$$f = c_1 l_1^d + \dots + c_k l_k^d$$

$$l_i \in S_1,\ c_i \in K.$$

In geometric terms the problem reads as follow: given a point $[f] \in \mathbb{P}S_d$ find points $[l_1^d], \dots, [l_k^d]$ on the Veronese variety $\nu_d(\mathbb{P}S_1)$ such that the k-secant space they span contains $[f]$.

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We can ask for the minimal number of linear forms needed to decompose a given $f \in S_d$. Define this number as

$$k_{\min}(f) = \min\left\{k : \exists\exists l_1, \ldots, l_k \in S_1, \quad f = \sum_{i=1}^{k} c_i \, l_i^d\right\}.$$

When a decomposition exists this is usually not unique, so it is interesting to study all possible ways of decomposing a form $f \in S_d$ using exactly k linear forms. With this in mind we set

$$VSP(f;k) = \overline{\left\{\{L_1, \ldots, L_k\} \in \mathrm{Hilb}_k\check{\mathbb{P}S_1} : L_i \neq L_j, \quad f = \sum_{i=1}^{k} c_i l_i^d\right\}}$$

where L_i is the hyperplane of $\mathbb{P}S_1$ defined by $l_i = 0$ and $\mathrm{Hilb}_k\check{\mathbb{P}S_1}$ is the Hilbert scheme of length k subschemes of $\check{\mathbb{P}S_1}$.

Both $k_{\min}(f)$ and $VSP(f,k)$ were classically studied, but much remains unknown about them. An expected value for $k_{\min}$ was obtained by a naive parameters count, but only recently it was proved that this value is exact for a generic form f, see [1]. The study of the variety $VSP(f,k)$ is still challenging and only few results are known in general. For more on this see [7].

A straightforward generalization is the study of the simultaneous decompositions of a set of forms $f_1, \ldots, f_r \in S_d$, that is

$$f_i = c_{i1} l_1^d + \ldots c_{ik} l_k^d, \quad i = 1, \ldots, r$$

involving the same linear forms l_j.

In this case also we have a geometric interpretation: given points $[f_1], \ldots, [f_r] \in \mathbb{P}S_d$ find points $[l_1^d], \ldots, [l_k^d]$ on the Veronese variety $\nu_d(\mathbb{P}S_1)$ such that the k-secant space they span contains the linear space $< [f_1], \ldots, [f_r] >$.

This problem was classically studied by means of polar polyhedra, e.g. see [6] for $n = 2$ and $d = 3$, and in a more general setting by Terracini. In [8] a solution for the case $n = 2$, $r = 2$ is claimed and a general criterion is stated. For a rigorous proof and a generalization of Terracini's result see [3]. For an exposition in modern terms and an interesting interpretation of Terracini's criterion see [2].

As in the case of one form, there are two main objects of interest.

Definition. Let $f_1, \ldots, f_r \in S_d$. We define

$$k_{\min}(f_1, \ldots, f_r) = \min\left\{k : \exists\exists l_1, \ldots, l_k \in S_1, \ \ f_i = \sum_{j=1}^{k} c_{ij} \, l_j^d \ \ i = 1, \ldots, r\right\}.$$

Definition. Let $f_1, \ldots, f_r \in S_d$. We define the Variety of Simultaneous Sums of Powers of the f_i's with respect to k to be

$$VSSP(f_1, \ldots, f_r; k) =$$

$$= \overline{\left\{\{L_1, \ldots, L_k\} \in \mathrm{Hilb}_k \mathbb{P}\check{S}_1 : L_i \neq L_j,\, f_i = \sum_{j=1}^{k} c_{ij} l_j^d \quad i = 1, \ldots, r\right\}},$$

where L_i is the hyperplane of $\mathbb{P}S_1$ defined by $l_i = 0$ and $\mathrm{Hilb}_k \mathbb{P}\check{S}_1$ is the Hilbert scheme of length k subschemes of $\mathbb{P}\check{S}_1$.

We notice that if $f_1, \ldots, f_r$ are linearly dependent, then the problem reduces to the study of r' independent forms, $r' < r$. Therefore we may assume the f_i's to be linearly independent.

As in the $r = 1$ case, there is an expected value for $k_{\min}$ obtained by a parameters count: consider the incidence correspondence Γ

$$G(r, S_d) \times X^k \supset \Gamma = \{(\Lambda, P_1 \ldots P_k) : < P_1 \ldots P_k > \supseteq \Lambda\}$$

where $G(r, S_d)$ is the Grassmannian of r dimensional subspaces of S_d and X is the d-th Veronese embedding of $\mathbb{P}S_1$. The expected value for $k_{\min}$ is the minimal k such that $\dim \Gamma \geq \dim G(r, S_d)$, i.e.

$$\left\lceil \frac{r}{r+n} \binom{n+d}{d} \right\rceil.$$

When $r > 1$ only few values of (n, d, r, k) are known for which $k_{\min}$ is not the predicted value for a generic choice of forms, see [2]. The existence of these exceptions can be proved by ad hoc methods but there are few general results. The most general result about $k_{\min}$ asserts that, in the binary case $n = 1$, the actual and the expected value of $k_{\min}(f_1, \ldots, f_r)$ are equal for generic forms. This is proved in [2]. As far as we know there are almost no results about the variety $VSSP$ in the case $r > 1$.

In this paper we restrict our attention to the binary case $S = K[x_0, x_1]$. Our main result is a complete description, for any k, of the variety parametrizing the 0-dimensional length k schemes apolar to generic forms $f_1, \ldots, f_r$. In particular, when the schemes are smooth, we describe $VSSP(f_1, \ldots, f_r; k)$. As a byproduct we get a formula for $k_{\min}(f_1, \ldots, f_r)$ in a more direct way than it was done in [2] (we don't use the notion of *grove* and the splitting of line bundles). In the last section we show an application of these results to the study of particular secant spaces of rational curves.

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2. Apolarity and inverse systems.

Set $S = K[x_0, \dots, x_n]$ and $T = K[y_0, \dots, y_n]$ with $K = \overline{K}$ a field of characteristic 0. We make S into a T-module via differentiation: given monomials y^α, x^β we define

$$y^\alpha \circ x^\beta = \begin{cases} 0 & \text{if } \alpha_i > \beta_i \text{ for some } i \\ \alpha!\binom{\beta}{\alpha} x^{\beta-\alpha} & \text{otherwise} \end{cases},$$

where the computations on the multi-indices are made componentwise, e.g. $\alpha! = \alpha_0! \cdot \ldots \cdot \alpha_n!$.

We recall that:

- given $f \in S$ we set $f^\perp = \{D \in T : D \circ f = 0\}$, this is an ideal in T and it is called the *orthogonal ideal of* f;
- given $D \in T$ we set $D^{-1} = \{f \in S : D \circ f = 0\}$, this is a graded T-submodule of S and it is called the *inverse system of* D.

We will need some basic properties about orthogonal ideals and inverse systems:

Properties. (see [4], pp. 11–19)

1. if $f \in S_d$, then $f^\perp$ is a Gorenstein artinian ideal with socle degree d;
2. if $D \in T_k$, $d \geq k$, then $\dim_K(D^{-1})_d = \binom{n+d}{n} - \binom{d-h+n}{n}$;
3. if $D, G \in T_k$, $d \geq k$, such that $(D^{-1})_d = (G^{-1})_d$, then $[D] = [G]$ in $\mathbb{P}T_k$;
4. if $f_1, \dots, f_r \in S_d$, then

$$\left(\bigcap_{i=1}^{r} f_i^\perp \right)_k = \{D \in T_k : (D^{-1})_d \supset < f_1, \dots, f_r >\};$$

5. via apolarity we have a natural identification $\mathbb{P}\check{S}_k = \mathbb{P}T_k$.

Apolarity is a powerful tool in studying the decomposition of forms as sums of powers because of the following (see [7], 1.3)

Lemma 2.1. (Apolarity Lemma) *Let $f_1, \dots, f_r \in S_d$, then the following facts are equivalent:*

1. $\exists\exists\ c_{ij} \in K$, $l_1, \dots, l_k \in S_1$, $[l_a] \neq [l_b]$ *in* $\mathbb{P}S_1$ *for* $a \neq b$, *such that*

$$f_i = \sum_{j=1}^{k} c_{ij} l_j^d, \quad i = 1, \dots, r;$$

2. $\exists\exists\ L_1, \dots, L_k \in \mathbb{P}T_1$, $L_i \neq L_j$ *for* $i \neq j$, *such that*

$$I_\Gamma \subset \bigcap_{i=1}^{r} f_i^\perp$$

where I_Γ *is the ideal of the set of points* $\Gamma = \{L_1, \dots, L_k\}$.

The lemma leads our attention to the ideals contained in $\bigcap_{i=1}^r f_i^\perp$, so we give the following definitions.

Definition. Given $f \in S_d$ and $\Gamma \in \mathrm{Hilb}_k \mathbb{P}\check{S}_1$, Γ is apolar to f if $I_\Gamma \subset f^\perp$.

Definition. Given $f_1, \dots, f_r \in S_d$ the Variety of Simultaneous Apolar Subschemes of length k of the f_i's is

$$VSPS(f_1, \dots, f_r; k) = \left\{ \Gamma \in \mathrm{Hilb}_k \mathbb{P}\check{S}_1 : I_\Gamma \subset \bigcap_{i=1}^{r} f_i^\perp \right\}.$$

The Apolarity Lemma shows why the binary case is easier to treat. When $n = 1$ the ideal of a set of points is a principal ideal and the generator is square free if the points are distinct. Hence there is a natural identification

$$\mathrm{Hilb}_k \mathbb{P}\check{S}_1 = \mathbb{P}\check{S}_k = \mathbb{P}T_k,$$

where the last equality comes from Property 5.

Finally, using Property 4 and Lemma 2.1, we get an useful description of $VSPS$ and of $VSSP$ in the binary case.

Let $S = K[x_0, x_1]$ and $f_1, \dots, f_r \in S_d$. Using the definition of $VSPS$ and recalling that the ideal of a set of k points in $\mathbb{P}^1$ is generated by a form of degree k, we get:

$$(*) \qquad \begin{aligned} VSPS(f_1, \dots, f_r; k) &= \left\{ [D] \in \mathbb{P}T_k : D \in (\cap_{i=1}^r f_i^\perp)_k \right\} \\ &= \mathbb{P}\Big(\bigcap_{i=1}^{r} f_i^\perp\Big)_k \\ &= \left\{ [D] \in \mathbb{P}T_k : (D^{-1})_d \supset \langle f_1, \dots, f_r \rangle \right\}. \end{aligned}$$

Using the definition of $VSSP$ and recalling that the ideal of a set of k distinct points in $\mathbb{P}^1$ is generated by a form of degree k without repeated roots, we get:

$$(**) VSSP(f_1, \dots, f_r; k) = \overline{\{[D] \in \mathbb{P}T_k : D \in (\cap_{i=1}^r f_i^\perp)_k, D \notin \Delta_k\}}$$
$$= \overline{\mathbb{P}\Big(\bigcap_{i=1}^r f_i^\perp\Big)_k \setminus \Delta_k} \subset \mathbb{P}T_k$$
$$= \overline{\{[D] \in \mathbb{P}T_k : (D^{-1})_d \supset < f_1, \dots, f_r >, D \notin \Delta_k\}}$$

where Δ_k is the locus of polynomials of degree k with at least a repeated root. We also notice that

$$k_{\min}(f_1, \dots, f_r) = \min\{k : \exists D \in T_k : (D^{-1})_d \supset < f_1, \dots, f_r >, D \notin \Delta_k\}.$$

It is useful to summarize these results.

Proposition 2.2. *If $S = K[x_0, x_1]$ and $f_1, \dots, f_r \in S_d$, then $VSPS(f_1, \dots, f_r; k)$ and $VSSP(f_1, \dots, f_r; k)$ are projective spaces for every k. Moreover*

$$VSPS(f_1, \dots, f_r; k) \supseteq VSSP(f_1, \dots, f_r; k)$$

and they are equal whenever the latter is not empty.

Given explicit binary forms $f_1, \dots, f_r$ we can actually determine $k_{\min}$, $VSSP$ and $VSPS$ using $(*)$ and $(**)$. This requires an easy algorithm involving linear algebra (orthogonal ideals) and basic Gröbner basis computations (intersection of ideals).
The really tough problem is deriving results holding for a generic choice of r forms. Part of the difficulty is related to the bad behavior of orthogonal ideals. For example it easy to show that for any binary form $f \in S_d$

$$\dim_K (f^\perp)_k \geq 2k - d,$$

but the actual value of the dimension depends on the particular form we choose.

The best result we can obtain for orthogonal ideals is an easy consequence of the previous bound and of Grassmann's formula for vector spaces.

Lemma 2.3. *Let d, r, k be natural numbers and $S = K[x_0, x_1]$. If*

$$k > \frac{r(d+1)-1}{r+1},$$

then for any choice of $f_1, \dots, f_r \in S_d$ we have

$$(f_1^\perp \cap \dots \cap f_r^\perp)_k \neq (0).$$

Proof. Let $V_i = (f_i^{\perp})_k$ and $V = S_k$, then $V_i \subseteq V$ and $\dim V = k+1$. Applying Grassmann's formula and using the inequality $\dim V_i \geq 2k - d$, we get

$$\dim V_1 \cap V_2 \geq 2(2k-d) - (k+1)$$

$$\dim (V_1 \cap V_2) \cap V_3 \geq 3(2k-d) - 2(k+1)$$

$$\vdots$$

$$\dim \bigcap_{i=1}^{r} V_i \geq r(2k-d) - (r-1)(k+1)$$

The last inequality gives the result. $\square$

3. The geometric setting.

From now on we will consider only binary forms, so that $S = K[x_0, x_1]$ and $T = K[y_0, y_1]$.

Consider the map

$$\begin{aligned} \psi_k : \quad & \mathbb{P}T_k \to G(k, S_d) \\ & [D] \mapsto (D^{-1})_d \end{aligned} ;$$

by Property 2 it is well defined when $k \leq d$. Using Plücker coordinates and Property 3 one verifies that ψ_k is an isomorphism on its image, $G_k = \psi(\mathbb{P}T_k)$, for all $k \leq d$. If we let $\Delta_k \subset T_k$ be the locus of forms with repeated roots, then we have

$$\dim G_k = \dim G_{\Delta_k} + 1,$$

where $G_{\Delta_k} = \psi_k(\mathbb{P}\Delta_k)$.

Now consider the following diagram

$$\begin{array}{ccc} G(r, S_d) \times G(k, S_d) \supset \Sigma_k = \left\{(\Lambda, \Gamma) : \Lambda \subseteq \Gamma,\ \Gamma \in G_k\right\} \supset \Sigma_{\Delta_k} = & & \\ = \left\{(\Lambda, \Gamma) : \Lambda \subseteq \Gamma, \Gamma \in G_{\Delta_k}\right\} & & \\ \downarrow \varphi_k & & \\ G(r, S_d) \supset \qquad \widetilde{\Sigma}_k & & \end{array}$$

where φ_k is the projection on the first factor and $\widetilde{\Sigma}_k = \varphi_k(\Sigma_k)$.

The study of the simultaneous decompositions of a set of forms $f_1, \ldots, f_r$ is equivalent to the study of the map φ_k , as shown by the following

Proposition 3.1. *Let $f_1, \ldots, f_r \in S_d$ be linearly independent forms and let $\Lambda =< f_1, \ldots, f_r >$. Then*

1. $k_{min}(f_1, \ldots, f_r) = \min\{k : \overline{\varphi_k^{-1}(\Lambda) \setminus \Sigma_{\Delta_k}} \neq \emptyset\}$;
2. $VSSP(f_1, \ldots, f_r; k) \simeq \overline{\varphi_k^{-1}(\Lambda) \setminus \Sigma_{\Delta_k}}$;
3. $VSPS(f_1, \ldots, f_r; k) \simeq \varphi_k^{-1}(\Lambda)$.

Proof. First we compute the fiber of φ_k on Λ:

$$\varphi_k^{-1}(\Lambda) = \{(\Lambda, (D^{-1})_d) : \Lambda \subset (D^{-1})_d, [D] \in \mathbb{P}T_k\};$$

from this we immediately get part *1*.

Now, using $(**)$, we notice that

$$\psi_k^{-1}\overline{(\varphi_k^{-1}(\Lambda) \setminus \Sigma_{\Delta_k})} = \overline{\{[D] \in \mathbb{P}T_k : \Lambda \subset (D^{-1})_d, D \notin \Delta_k\}} = \overline{\mathbb{P}(\cap_i f_i^{\perp})_k \setminus \Delta_k}.$$

Because ψ_k is an isomorphism we get part 2. The same argument and $(*)$ give part 3. □

The map φ_k is useful in solving our problem also because of the properties of the varieties Σ_k and Σ_{Δ_k}. In fact we have

Lemma 3.2. *Σ_k and Σ_{Δ_k} are Grassmannian bundles on irreducible varieties. In particular they are irreducible and we have*

$$\dim \Sigma_k = \dim \Sigma_{\Delta_k} + 1 = k + r(k - r).$$

Proof. It is enough to consider the projection maps

$$\Sigma_k \longrightarrow G_k$$

$$\Sigma_{\Delta_k} \longrightarrow G_{\Delta_k}$$

and to notice that their fibers are the Grassmannians $G(r, k)$. □

Finally we can state our main result.

Theorem 3.3. *Let $S = K[x_0, x_1]$. Given natural numbers d, r set*

$$k_{\min}(d, r) = \min\left\{k : k > \frac{r(d+1) - 1}{r + 1}\right\}.$$

There exists an open non-empty subset

$$V_{d,r} \subset G(r, S_d)$$

such that, for all $f_1, \dots, f_r \in S_d$ *satisfying* $[< f_1, \dots, f_r >] \in V_{d,r}$*, the following hold:*

1. $k_{\min}(f_1, \dots, f_r) = k_{\min}(d, r)$;
2. $VSSP(f_1, \dots, f_r; k) = \begin{cases} \mathbb{P}^{k(r+1)-r(d+1)} & k \geq k_{\min}(d, r) \\ \emptyset & k < k_{\min}(d, r) \end{cases}$;
3. $VSPS(f_1, \dots, f_r; k) = \emptyset$ *if* $k < k_{\min}(d, r)$.

Moreover

$$VSSP(f_1, \dots, f_r; k_{\min}(d, r)) = \begin{cases} \mathbb{P}^{r+1-\varepsilon} & \varepsilon \neq 0 \\ \mathbb{P}^0 & \varepsilon = 0 \end{cases}.$$

where $\varepsilon \equiv r(d+1) \bmod (r+1)$.

Proof. Set $Z_k = \{\Lambda \in \widetilde{\Sigma}_k : \varphi_k^{-1}(\Lambda) \subset \Sigma_{\Delta_k}\}$ and consider the following diagram

$$\begin{array}{ccc} \Sigma_k & \supset \Sigma_{\Delta_k} \supset & \varphi_k^{-1}(Z_k) \\ \Big\downarrow{\scriptstyle \varphi_k} & & \\ \widetilde{\Sigma}_k & \supset Z_k & \end{array}$$

Set $\lambda = \min\{\dim \varphi_k^{-1}(\Lambda) : \Lambda \in \widetilde{\Sigma}_k\}$. Using Lemma 3.2 and the Fiber Dimension Theorem (see [5], lecture 11) we get

$$\dim \Sigma_k = \dim \widetilde{\Sigma}_k + \lambda,$$

$$\dim \Sigma_{\Delta_k} \geq \dim Z_k + \lambda.$$

Hence $\dim \widetilde{\Sigma}_k - \dim Z_k \geq 1$.

If $\widetilde{\Sigma}_k$ is dense, then $\dim \Sigma_k \geq \dim G(r, S_d)$. This gives the condition

$$k \geq k_1 = \min\left\{k : k \geq \frac{r(d+1)}{r+1}\right\}.$$

By Lemma 2.3 we know that if $k \geq k_2 = \min\{k : k > \frac{r(d+1)-1}{r+1}\}$ then φ_k is dominant. It is easy to check that $k_1 = k_2 = k_{\min}(d, r)$. For the sake of simplicity set $\bar{k} = k_{\min}(d, r)$.

Finally we set

$$U = \widetilde{\Sigma}_{\bar{k}} \setminus \overline{(Z_{\bar{k}} \cup \widetilde{\Sigma}_{\bar{k}-1})}.$$

By the preceding consideration $U \subset G(r, S_d)$ is open and non-empty. Moreover, if $\Lambda \in U$ then

$$\varphi_{\bar{k}}^{-1}(\Lambda) \setminus \Sigma_{\Delta_{\bar{k}}} \neq \emptyset,$$

$$\varphi_k^{-1}(\Lambda) = \emptyset, k < \bar{k}.$$

Using Proposition 3.1 we conclude that U satisfies part 1.

Proposition 2.2 yields

$$VSSP(f_1, \dots, f_r; k) = VSPS(f_1, \dots, f_r; k) = \emptyset, \quad k < \bar{k} = k_{\min}(d, r)$$

for $f_1, \dots, f_r \in S_d$ such that $< f_1, \dots, f_r > \in U$. This proves part 3.

By Propositions 2.2 and 3.1 we know that $VSSP(f_1, \dots, f_r; k)$, $k \geq k_{\min}(d, r)$, is a projective space of dimension $\dim \varphi_k^{-1}(< f_1, \dots, f_r >)$. As the fiber dimension is an upper semicontinuous function, there exists an open non-empty subset $U' \subset U$ such that

$$\dim VSSP(f_1, \dots, f_r; k) = \dim \Sigma_k - \dim G(r, S_d) = k(r+1) - r(d+1)$$

for $< f_1, \dots, f_r > \in U'$. This completes the proof of part *2*.

To get the expression for $\dim VSSP(f_1, \dots, f_r; k_{\min}(d, r))$ we only have to use part *2* and to study the congruence class of $r(d+1) - 1 \bmod r+1$.

Letting $V_{d,r} = U'$ completes the proof. □

Example. Using Theorem 3.3 we can recover a classical result of Sylvester (1851). Given a generic binary form $f \in S_d$, i.e. $f \in V_{d,1}$, the minimal number of linear forms needed to decompose it is

$$k_{min}(f) = \min\{k : k > \frac{d}{2}\} = \left[\frac{d}{2}\right] + 1$$

and the possible decompositions are parametrized by

$$VSSP(f, k_{min}(f)) = \begin{cases} \mathbb{P}^1 & d \text{ even} \\ \mathbb{P}^0 & d \text{ odd} \end{cases}.$$

4. Rational curves.

Definition. A rational curve $C \subset \mathbb{P}^n$ is the image of a rational map $\alpha : \mathbb{P}^1 \to \mathbb{P}^n$. We say that the curve is non-degenerate if it is not contained in a hyperplane.

Let $S = k[x_0, x_1]$ and fix the standard lex ordered monomial basis, e.g. with respect to $x_0 > x_1$, in each of the homogeneous pieces S_n, so we have the identifications $\mathbb{P}^n = \mathbb{P}S_n$ for all n.

Let $C \subset \mathbb{P}^n$ be a rational curve of degree d, with $d > n$. There exists a unique $\Lambda_C \in \mathbb{G}(d-n-1, \mathbb{P}^d) = \{\Lambda \subset \mathbb{P}^d : \Lambda \simeq \mathbb{P}^{d-n-1}\}$ such that the following diagram commutes

$$\begin{array}{ccccc} & & \mathbb{P}^d & \supset & C_d \\ & \overset{\nu_d}{\nearrow} & \downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^1 & \longrightarrow & \mathbb{P}^n & \supset & C \end{array}$$

where ν_d is the d-uple embedding of $\mathbb{P}^1$, C_d is the rational normal curve of degree d and π is the projection from Λ_C. In particular $\pi(C_d) = C$.

Definition. Let $C \subset \mathbb{P}^n$ be a rational curve. We define

$$S_b^a(C) = \{\Gamma \in \mathbb{G}(a, \mathbb{P}^n) : \alpha^{-1}(\Gamma \cap C) \text{ has length } b\}$$

$$S_b^a(C)_{\neq} = \{\Gamma \in \mathbb{G}(a, \mathbb{P}^n) : \alpha^{-1}(\Gamma \cap C) \text{ is smooth of length } b\}.$$

We notice that $S_b^0(C)$ is the set of b-uple points of C and $S_b^0(C)_{\neq}$ is the set of b-uple points of C having b distinct tangent lines, e.g. if C is a plane cubic with a node then $S_2^0(C)_{\neq} = \mathbb{P}^0$ while $S_2^0(C')_{\neq} = \emptyset$ if C' is a cusp.

If C is a smooth curve, then $S_b^a(C)_{\neq}$ has a nicer geometric description:

$$S_b^a(C)_{\neq} = \{\Gamma \in \mathbb{G}(a, \mathbb{P}^n) : \Gamma \cap C \text{ is a set of } b \text{ distinct points }\}.$$

Let $C \subset \mathbb{P}^n$ be a rational non-degenerate curve of degree d. It is immediate to verify the following:

- $S_d^{n-1}(C) = \check{\mathbb{P}}^n$ as, taking multiplicities into account, *any* hyperplane intersects C in d points;
- $S_d^{n-1}(C)_{\neq}$ is dense in $\check{\mathbb{P}}^n$ as a *generic* hyperplane intersects C in d distinct points;
- $S_{d'}^{n-1}(C) = S_{d'}^{n-1}(C)_{\neq} = \emptyset$ if $d' \neq d$.

We notice that $S_b^a(C)$, $S_b^a(C)_{\neq}$ are not interesting for all the values of a and b, as shown by the following

Lemma 4.1. *Let $C \subset \mathbb{P}^n$ be a rational non-degenerate curve of degree d, $d > n$. If $b - a > d - n + 1$, then $S^a_b(C) = S^a_b(C)_{\neq} = \emptyset$.*

Proof. Let $\Gamma \in S^a_b(C)$; then choosing $P \in C \setminus \Gamma$ we build $\Gamma' =< \Gamma, P > \in S^{a+1}_{b'}(C)$, $b' \geq b + 1$. Repeating the construction we get

$$\overline{\Gamma} \in S^{n-1}_{\overline{b}}(C)$$

where $\overline{b} \geq b + n - a - 1$. As $S^{n-1}_{\overline{b}}(C) \neq \emptyset$ we must have $d - n + 1 \geq b - a$. We have shown that $S^a_b(C) \neq \emptyset$ implies $d - n + 1 \geq b - a$ and by negation we get the thesis. □

We will describe $S^a_b(C)$, $S^a_b(C)_{\neq}$ in the extremal case $b - a = d - n + 1$.

Proposition 4.2. *Let $C \subset \mathbb{P}^n$ be a rational non-degenerate curve of degree d, $d > n$, and a, b natural numbers such that $b - a = d - n + 1$. Then the following isomorphisms hold*

$$S^a_b(C) \simeq VSPS(f_1, \ldots, f_r; b)$$

$$S^a_b(C)_{\neq} \simeq V \subset VSSP(f_1, \ldots, f_r; b)$$

where V is dense and $r = d - n$.

Proof. Let $\Lambda_C =< f_1, \ldots, f_r >$, $f_1, \ldots, f_r \in S_d$. As C is the projection of C_d from Λ_C we have

$$\begin{aligned} S^a_b(C) &= \{\Gamma \in \mathbb{G}(a, \mathbb{P}^n) : \alpha^{-1}(\Gamma \cap C) \text{ has length } b\} \\ &\simeq \{\Gamma \in \mathbb{G}(a + d - n, \mathbb{P}^d) : \Gamma \supset \Lambda_C, \nu_d^{-1}(\Gamma \cap C_d) \text{ has length } b\} = A \end{aligned}$$

and

$$\begin{aligned} S^a_b(C)_{\neq} &= \{\Gamma \in \mathbb{G}(a, \mathbb{P}^n) : \alpha^{-1}(\Gamma \cap C) \text{ is smooth of length } b\} \\ &\simeq \{\Gamma \in \mathbb{G}(a+d-n, \mathbb{P}^d) : \Gamma \supset \Lambda_C, \nu_d^{-1}(\Gamma \cap C_d) \text{ is smooth of length } b\} \\ &= B \end{aligned}$$

First we construct the isomorphism $A \simeq VSPS(f_1, \ldots, f_r; b)$.
Given $\Gamma \in A$ consider the subscheme $Y = \nu_d^{-1}(\Gamma \cap C_d) \subset \mathbb{P}^1$: it is defined by the ideal

$$I_Y = (g_1, \ldots, g_{n-a})$$

where the g_i's are the pullbacks of the linear equations defining Γ, as the scheme has length b they have a common factor of degree b. The g_i's are independent so that the numerical condition on a and b implies that the saturation of I_Y is the principal ideal (g), $g = GCD(g_i) \in S_b$. Let $G_i(y_0, y_1) = g_i(y_0, y_1) \in T_b$

and $G(y_0, y_1) = g(y_0, y_1) \in T_b$. By apolarity the hyperplanes $(G_i^{-1})_d$ define Γ. It is easy to verify that $(G^{-1})_d \supset \Gamma$, i.e. $G \in VSPS(f_1, \dots, f_r; b)$, so we have a map $A \to VSPS(f_1, \dots, f_r; b)$. The inverse map is defined by sending $G \in VSPS(f_1, \dots, f_r; b)$ to the linear space $(G^{-1})_d \in A$.

Now we construct the isomorphism $B \simeq V$, V a dense subset of $VSSP(f_1, \dots, f_r; b)$. Let V be the dense subset of $VSSP(f_1, \dots, f_r; b)$ consisting of square free polynomials. Then we define the map $V \to B$ by sending $G \in V$ to the linear space $(G^{-1})_d$. To define the inverse map, given $\Gamma \in B$, we consider the subscheme $\nu_d^{-1}(\Gamma \cap C_d)$ which is defined by the principal ideal (g), $g \in S_b$: $G(y_0, y_1) = g(y_0, y_1)$ is a square free polynomial and $G \in VSSP(f_1, \dots, f_r; b)$. □

Example. Let $f_1 = -2x_0^5 + 2x_1^5 + (x_0 - x_1)^5$, $f_2 = -6x_0^5 + 3x_1^5 + 2(x_0 - x_1)^5$ and $\Lambda_C =< f_1, f_2 >$. We want to study the rational curve $C = \pi(C_5) \subset \mathbb{P}^3$ obtained as projection of the rational normal curve of $\mathbb{P}^5$ from Λ_C.

We know that

$$S_b^a(C) = S_b^a(C)_{\neq} = \emptyset$$

when $b - a > 3$. Hence the interesting cases are:

$$S_{a+3}^a(C), S_{a+3}^a(C)_{\neq} \quad a = 0, 1, 2.$$

To apply Proposition 4.2 we need to determine $I = f_1^{\perp} \cap f_2^{\perp}$. By direct computation we get

$$I = (y_0y_1(y_0 + y_1), y_0^4 + y_1^4) \cap (y_0y_1(y_0 + y_1), y_0^4 + 2y_1^4),$$

in particular

$$\dim_K I_5 = 4, \quad \dim_K I_4 = 2, \quad \dim_K I_3 = 1, \quad \dim_K I_d = 0 \quad \text{for } d < 3$$

It is possible to verify that each homogeneous piece of I is generated by forms without common roots. Moreover the generator of I_3 is square free. Hence, using $(**)$ and Proposition 4.2, we obtain

$$S_3^0(C)_{\neq} = \mathbb{P}I_3 = \mathbb{P}^0,$$
$$S_4^1(C)_{\neq} = \mathbb{P}I_4 = \mathbb{P}^1,$$
$$S_5^2(C)_{\neq} = \mathbb{P}I_5 = \mathbb{P}^3.$$

We also have that $S_{a+3}^a(C)_{\neq}$ is dense in $S_{a+3}^a(C)$ for $a = 0, 1, 2$.

In particular

$$S_3^0(C)_{\neq} = \mathbb{P}^0$$

means that C has a unique triple point.

We have shown how to use $VSSP$ to study curves that are projection of the rational normal curve: given the center of projection $\Lambda =< f_1, \dots, f_r >$ we have to investigate the decompositions of the $f_1, \dots, f_r$ as sums of powers of linear forms, so that each case has to be treated separately. To find general results we have to exclude curves with a pathological behavior and this can be done using Theorem 3.3.

Definition. A rational curve $C \in \mathbb{P}^n$ of degree d, $d > n$, is said to be generic if $\Lambda_C \in V_{d,r}$, where $r = \dim_K \Lambda_C$ and $V_{d,r}$ is as given by Theorem 3.3.

We notice that, because $V_{d,r}$ is open and non-empty, almost all the rational curves $C \subset \mathbb{P}^n$ of degree d, $d > n$, are generic.

An easy consequence of Theorem 3.3 and of Proposition 4.2 is the following:

Corollary 4.3. *Let $C \subset \mathbb{P}^n$ be a generic rational curve of degree d, $d > n$. If $b - a = d - n + 1$ then, defining $k_{\min}(d, d-n)$ as in Theorem 3.3, we have*

1. $S_b^a(C) = S_b^a(C)_{\neq} = \emptyset$ *for* $b < k_{\min}(d, d-n)$;
2. $S_b^a(C) = \mathbb{P}^{b(d-n+1)-(d-n)(d+1)}$ *for* $b \geq k_{\min}(d, d-n)$;
3. $S_b^a(C)_{\neq}$ *is dense in* $\mathbb{P}^{b(d-n+1)-(d-n)(d+1)}$ *for* $b \geq k_{\min}(d, d-n)$.

Example. Let $C \subset \mathbb{P}^3$ be a rational non-degenerate curve of degree 5. By Lemma 4.1 we know that $S_b^a(C) = S_b^a(C)_{\neq} = \emptyset$ for $b - a > 3$.

If C is generic, then using Corollary 4.3 and the fact that $k_{\min}(5, 2) = 4$ we get

$$S_3^0(C)_{\neq} = \emptyset$$

$$S_4^1(C)_{\neq} = \mathbb{P}^0$$

$$S_5^2(C)_{\neq} = \mathbb{P}^3.$$

In particular, because $S_3^0(C) = \emptyset$, we conclude that C has no triple points. This shows that the curve of the previous example is not generic.

If C is also smooth then the equality

$$S_4^1(C)_{\neq} = \mathbb{P}^0$$

means that there exists a unique 4-secant line to C.

Example. Let $C \subset \mathbb{P}^{16}$ be a smooth generic rational curve of degree 19. Because the curve is smooth we can get interesting geometric properties from studying $S_a^b(C)_{\neq}$.

C is the projection of the rational normal curve C_{19} from $\Lambda_C =< f_1, f_2, f_3 >$ and, because the curve is generic, we know that $k_{\min}(f_1, f_2, f_3) = 15$. This means that

$$S_{a+4}^a(C)_{\neq} = VSSP(f_1, f_2, f_3; a+4) = \emptyset, \text{ for } a = 0, \ldots, 10.$$

We get that

$$S_{15}^{11}(C)_{\neq} = \mathbb{P}^0$$

so that there exists a unique 15-secant $\mathbb{P}^{11}$ to C. We also have

$$S_{16}^{12}(C)_{\neq} = \mathbb{P}^4, S_{17}^{13}(C)_{\neq} = \mathbb{P}^8, S_{18}^{14}(C)_{\neq} = \mathbb{P}^{12}.$$

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