ON REALIZABILITY OF LINES ON TROPICAL CUBIC SURFACES AND THE BRUNDU-LOGAR NORMAL FORM

ALHEYDIS GEIGER

We present results on the relative realizability of infinite families of lines on general smooth tropical cubic surfaces. Inspired by the problem of relative realizability of lines on surfaces, we investigate the information we can derive tropically from the Brundu-Logar normal form of smooth cubic surfaces. In particular, we prove that for a residue field of characteristic $\neq 2$ the tropicalization of the Brundu-Logar normal form is not smooth. We also take first steps in investigating the behavior of the tropicalized lines.

1. Introduction

Since 1849 it has been a well-known fact that every classical smooth cubic surface contains exactly 27 lines [4, 16]. With the rise of tropical geometry the question about an analogous result about tropical cubic surfaces was inevitable. In 2007 Vigeland proved in [19] that on smooth tropical cubic surfaces, there can be infinitely many tropical lines. Accordingly, the following questions arose:

Question 1.1 (Relative Realizability)**.** For a given pair (L, X) of a tropical line L on a tropical cubic surface X, is there a pair (ℓ, C) of a classical line ℓ on a classical cubic surface C such that $\mathrm{trop}(\ell) = L$ and $\mathrm{trop}(C) = X$?

Question 1.2 (Relative Realizability, Lifting Multiplicities)**.** For a pair
(L,X) and a fixed lift C of X, how many lines $\ell \subset$ C tropicalize to L, i.e.
what is the *lifting multiplicity* of L with respect to C?

The question of the geometric behavior of the tropical lines and the
problem of their relative realizability has been worked on for more than
a decade and is not yet completely solved [1, 2, 5, 8, 11, 14, 15, 19, 20].
With respect to the question of lifting multiplicities, there has been a re-
cent new development: in [14] an octanomial model for cubic surfaces
is presented, which satisfies that all 27 lines on a tropically smooth cu-
bic in this form have distinct tropicalizations [14, Theorem 3.4], i.e. all
lifting multiplicities are one or zero. Also, the conjecture is posed that
the 27 lines on a tropically smooth cubic surface have distinct tropical-
izations [14, Conjecture 4.1]. Moreover, according to [14], Kristin Shaw
has announced a proof that every tropically smooth family of complex
cubic surfaces contains 27 lines with distinct tropicalizations.

By tools introduced in [15, 20] it is possible to divide tropical lines
into isolated lines and infinite families. On general smooth tropical cu-
bics there can only occur two types of families; see Proposition 2.2 and
[8, 15]. The lifting behavior of one type of family over characteristic 0
has been investigated in [1, 2]. Our first main result, Theorem 2.6, is
a generalization of this result to fields of arbitrary characteristics, ob-
tained by complementary methods. Also, we present two trend-setting
new examples, one for the second family type in Example 4.1 and one
on the unsolved case to Theorem 2.6 in Example 4.2.

Unfortunately, the computational expenditure to compute the 27
lines on a generic classical smooth surface is very high, impeding an
easy access to examples of more complex lifts over arbitrary character-
istic. The existence of a normal form of classical cubic surfaces which
allows the direct computation of all 27 lines from the parameters of the
cubic polynomial by simple formulas seems to present a new perspec-
tive to solving this problem. Therefore, we ask

Question 1.3 ([18, Question 26])**.** How to compute the Brundu-Logar
normal form in practice? What does it tell us tropically?

The first part of this question has been computationally answered
for generic cubics over the p-adic fields by Avinash Kulkarni [12]. As
a smooth cubic is transformed in the Brundu-Logar normal form by
projective linear transformations, this question is closely related to

Question 1.4 ([18, Question 12])**.** How can we decide if a given polyno-
mial defines a smooth tropical surface after a linear transformation of

$\mathbb{P}^3$?

Unfortunately, the Brundu-Logar normal form of a lift of any chosen smooth tropical cubic is itself no longer tropically smooth. More precisely, our second main result is the following theorem:

Theorem (Theorem 3.4). *If val(2) = 0, the tropicalization of any cubic in Brundu-Logar normal form is not tropically smooth.*

However, we can still investigate the tropicalization of cubics in Brundu-Logar normal form and look into the behavior of the tropicalizations of the 27 lines. We will see in Section 3.3 that the statement of Theorem 3.4 from [14] does not hold for the Brundu-Logar normal form, as we can have higher lifting multiplicities in this setting, as illustrated in Examples 4.3 and 4.4. These examples prove that [14, Conjecture 4.1] does not hold for non-smooth tropical cubics.

This paper is organized as follows. In Section 2 we present results on the relative realizability of lines on tropical cubic surfaces. In Section 3 we study the tropical Brundu-Logar normal form and its tropicalized lines. Section 4 presents significant examples of realizable lines in tropical cubic surfaces, one of them obtained by transforming to the Brundu-Logar normal form.

2.   Relative realizability of infinite families of lines

Let $3\Delta_3 := \mathrm{conv}((0,0,0),(3,0,0),(0,3,0),(0,0,3))$. Denote the facets of $3\Delta_3$ with F_i, F_j, F_k, F_l. Using the concept of decorations introduced in [20], we can classify lines on surfaces by their way of being contained in the surface: passing through vertices or edges. Decorations allow one to distinguish isolated lines and infinite families of lines contained in the tropical surface. This concept was refined by the theory of motifs introduced in [15], where also a thorough classification of all motifs of
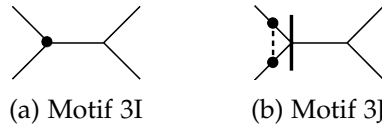
(a) Motif 3I   (b) Motif 3J

Figure 1: Infinite families on general smooth tropical cubic surfaces

lines on general smooth surfaces of varied degree can be found, completing the started classification from [20]. The concept of generality used in this section is the one introduced by [20].

Definition 2.1 ([15, Definition 6]). A motif of a tropical line L on a tropical surface X is a pair $(G, \mathcal{R})$, where the primal motif G is a decoration of the underlying graph of L with a finite number of dots and vertical line segments. Every dot stands for a vertex of X contained in that cell of L corresponding to the decorated edge or vertex of the graph, while a vertical line segment stands for an edge of X that L intersects in its relative interior with the cell whose corresponding part in the graph is decorated. The dual motif $\mathcal{R}$ is the subcomplex in the dual subdivision of X dual to the union of all cells of X that L passes through: $\mathcal{R} = \bigcup_{p \in L} (\min\{C \subset X \text{ cell of } X \mid p \in C\})^{\vee}$.

Proposition 2.2 ([15, Proposition 23], [8, Section 3.2]). *Only the motifs 3I and 3J allow infinite families on general smooth tropical cubic surfaces; see Figure 1.*

Remark 2.3. The dual motif to 3I is a unimodular tetrahedron, which has one edge in $F_i \cap F_j$ and at least a one-dimensional intersection with both F_k and F_l. This is called a *4-exit tetrahedron*. The property of relative realizability of families of motif 3I depends on how the dual motif, a 4-exit tetrahedron, is contained in the subdivision of the Newton polytope $3\Delta_3$ of the cubic surface. For smooth cubic surfaces, there are two ways up to coordinate changes to include a 4-exit tetrahedron in $3\Delta_3$.

Definition 2.4. A family of lines $(L_a)_{a \geq 0}$ on a smooth cubic surface is *of type 3I-1* if it has motif 3I with the dual motif contained in $3\Delta_3$ in a position equivalent to $\{(0,0,0),(0,0,1),(2,1,0),(1,0,2)\}$ up to coordinate changes; see Figure 2a. It is *of type 3I-2* if it is of motif 3I and the dual motif is contained in $3\Delta_3$ in a position equivalent by coordinate changes to $\{(0,0,0),(0,0,1),(2,1,0),(1,1,1)\}$; see Figure 2b.

A tropical line is called *degenerate,* if the bounded middle line segment has length zero.
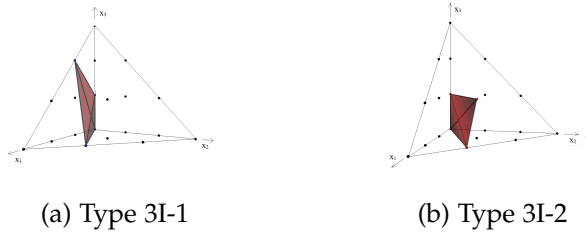
(a) Type 3I-1   (b) Type 3I-2

Figure 2: Position of the 4-exit tetrahedron in both types

Proposition 2.5. *Over* $K = \mathbb{C}$ *a line* L *of motif 3I on a tropical cubic surface* X *lifts if and only if it is degenerate and of type 3I-1.*

Proof. This is a corollary of Theorem 7.2 in [2] which states, that for an algebraic surface $\mathcal{S}$ over $(\mathbb{C}^*)^3$ with Newton polytope a unimodular 4-exit tetrahedron, the tropical line $\mathrm{star}_{(0,0,0)}(L_l) \subset \mathcal{S}$ is approximable by a complex algebraic line $\mathcal{L}_l \subset \mathcal{S}$ if and only if $l = 0$ and the Newton polytope is of the form $\{(0,0,0), (0,0,1), (2,1,0), (1,0,2)\}$. We can assume the vertex of X contained in L to be $(0,0,0)$ and apply this locally to $\mathrm{star}_{(0,0,0)}(X)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 2.6 ([8, Theorems 4.1.9 and 4.1.16]). *Let* (K, val) *be an algebraically closed field with valuation and let* $\Bbbk$ *be its residue field. Let* X *be a general smooth tropical cubic surface containing a family of lines* $(L_a)_{a>0}$. *If the family* $(L_a)_{a>0}$ *is of type 3I-1 with* $char(\Bbbk) \neq 2$ *or the family is of type 3I-2, then for any* $f \in K[x_0, \ldots, x_3]$ *homogeneous with* $X = V(trop(f))$ *the family cannot be realized as lines on* $V(f)$, *i.e. the lifting multiplicity of any nondegenerate member of the family is zero.*

Proof. Since we consider our cubic $f \in K[x_0, \ldots, x_3]$ homogeneous, we consider everything over $\mathbb{R}^4/\mathbb{R}\mathbf{1}$. Without loss of generality we can assume that the monomials corresponding to the vertices of the dual motif of the family have coefficients of valuation zero, while the coefficients of all other monomials in f have strictly positive valuation. We write $\mathbf{0} := (0,0,0,0)$.

For $a > 0$ let $I_a \subset K[x_0, ..., x_3]$ be a linear homogeneous ideal, such that $trop(V(I_a)) = L_a$. If the lines are realizable, we have $0 \neq in_{\mathbf{0}}(f) \in in_{\mathbf{0}}(I_a)$. The initial ideal of I_a in $\mathbf{0}$ is given as $in_{\mathbf{0}}(I_a) = \langle x_0 - x_1 + x_3, x_1 + x_2 \rangle$. This basis is a Gröbner basis with the lexicographic order with $x_3 \succ \ldots \succ x_0$. If $(L_a)_{a>0}$ has type 3I-1, we obtain $\sigma := supp(in_{\mathbf{0}}(f)) = \{x_0^3, x_0^2 x_3, x_1 x_3^2, x_1^2 x_2\}$. By Gröbner basis theory, we know that there exits a polynomial $\alpha x_0^3 + \beta x_0^2 x_3 + \gamma x_1 x_3^2 + \delta x_1^2 x_2 \in in_{\mathbf{0}}(I_a)$ of support σ if and only if its polynomial

division with the Gröbner basis of $\text{in}_0(I_a)$ has remainder zero. Carrying out the polynomial division, we find the remainder

$$(\alpha - \beta)x_0^3 + (\beta + \gamma)x_0^2 x_1 - 2\gamma x_0 x_1^2 + (\gamma - \delta)x_1^3 \in \Bbbk[x_0, ..., x_3].$$

If $\text{char}(\Bbbk) \neq 2$, this polynomial is zero if and only if $\alpha = \beta = \gamma = \delta = 0$. For $(L_a)_{a>0}$ of type 3I-2, $\text{supp}(\text{in}_0(f)) = \{x_0^3, x_0^2 x_3, x_1^2 x_2, x_1 x_2 x_3\}$. The remainder of $\alpha x_0^3 + \beta x_0^2 x_3 + \gamma x_1^2 x_2 + \delta x_1 x_2 x_3$ by polynomial division is

$$(\alpha - \beta)x_0^3 + \beta x_0^2 x_1 - \delta x_0 x_1^2 + (\gamma + \delta)x_1^3 \in \Bbbk[x_0, ..., x_3].$$

This polynomial is zero if and only if $\alpha = \beta = \gamma = \delta = 0$. $\square$

Theorem 2.6 deals with the non-degenerate lines of motif 3I. We can use the same techniques to prove the following theorem.

Theorem 2.7. *Let* (K, val) *be an algebraically closed field with valuation and let* $\Bbbk$ *be its residue field. Let* X *be a general smooth tropical cubic surface containing a family of lines of type 3I-2. The degenerate line of this family cannot be realized on any lift* $V(f)$ *of* X, $f \in K[x_0, \ldots, x_3]$ *homogeneous.*

Proof. As before we consider everything over $\mathbb{R}^4/\mathbb{R}\mathbf{1}$ and assume that the monomials corresponding to the vertices of the dual motif of the family have coefficients of valuation zero, while all other coefficients in f have strictly positive valuation. Again we denote $\mathbf{0} := (0,0,0,0)$.
Let L be the degenerate line of the family of type 3I-2 on X and let $I \subset K[x_0, ..., x_3]$ be a linear homogeneous ideal, such that $\text{trop}(V(I)) = L$. We use the same argument as in the proof of Theorem 2.6: if the line is realizable, it follows that $0 \neq \text{in}_0(f) \in \text{in}_0(I)$. The initial ideal of I in $\mathbf{0}$ can only be determined up to a $\lambda \neq 0, -1$: $\text{in}_0(I) = \langle x_0 + \lambda x_1 - x_3, x_0 - x_1 - x_2 \rangle$, given by a Gröbner basis with the lexicographic order with $x_3 \succ \ldots \succ x_0$. If L is of type 3I-2, we know $\sigma := \text{supp}(\text{in}_0(f)) = \{x_0^3, x_0^2 x_3, x_1^2 x_2, x_1 x_2 x_3\}$. By polynomial division of a polynomial $\alpha x_0^3 + \beta x_0^2 x_3 + \gamma x_1^2 x_2 + \delta x_1 x_2 x_3$ of support σ with the Gröbner basis, we obtain the remainder:

$$(-\gamma - \lambda\delta)x_1^3 + (\lambda\delta + \gamma - \delta)x_1^2 x_0 + (\lambda\beta + \delta)x_1 x_0^2 + (\alpha + \beta)x_0^3 \in \Bbbk[x_0, ..., x_3].$$

Since $\lambda \neq 0, -1$, this polynomial is zero if and only if $\alpha = \ldots = \delta = 0$. $\square$

Remark 2.8. The same techniques do not reveal similar results for type 3I-1. Indeed the lifting behavior of the two types is quite different.

Example 4.2 shows that an analogous statement to Theorem 2.7 for the degenerate lines of type 3I-1 over characteristic p is in general not true, suggesting an extension of the validity of Proposition 2.5.

Also recall that for families of type 3I-1 in $\mathrm{char}(\Bbbk) = 2$, there is a non-zero solution for the coefficients $\alpha, \beta, \gamma, \delta$ in the proof of Theorem 2.6. Example 4.2 shows a surface on which a non-degenerate representative of a family of type 3I-1 lifts, giving rise to the following question.

Question 2.9. For any member L of a family of type 3I-1 on a smooth tropical surface X, is there a lift of X defined over a field K with residue field $\Bbbk$ of $\mathrm{char}(\Bbbk) = 2$ containing a lift of L?

It remains to investigate families of motif 3J. Similar to lines of type 3I-1, we can find a lift of a smooth tropical cubic surface defined over $\mathbb{Q}_p$, for $p = 2$ and $p = 5$ each, where a non-degenerate member of the family of motif 3J is realizable. The details are shown in Example 4.1. The arising question is

Question 2.10. For any member L of a family of motif 3J on a smooth tropical surface X, is there a lift of X over any algebraically closed field K containing a lift of L?

3. The Brundu-Logar normal form and its tropicalization

The fact that the Plücker coordinates of the lines on cubics in Brundu-Logar normal form are known explicitly motivates our study of their tropicalizations. In this section let K be algebraically closed, $\mathrm{char}(K) = 0$.

3.1. The classical Brundu-Logar normal form

Definition 3.1. A homogeneous cubic polynomial $F \in K[x_1, ..., x_4]$ is in *Brundu-Logar normal form* if

$$
\begin{aligned}
F = {} & 2ax_1^2x_2 + (b-g)x_1^2x_3 + (-2a)x_1x_2^2 + (d+g)x_1x_2x_3 + (a-c-d)x_1x_3^2 + \\
& (b+g)x_1x_2x_4 + (-a-b-c)x_1x_3x_4 + (d-g)x_2^2x_4 + (a+c-d)x_2x_3x_4 \\
& + (-a-b+c)x_2x_4^2,
\end{aligned}
$$

where $(a, b, c, d, g) \notin \Sigma$ and $\Sigma := V(\sigma) \subset \mathbb{P}^4$ is the hypersurface defined by

$$\begin{aligned}
\sigma := &c(a+b-c)(2a+b-d)(a-c-d)(a+c+g)(a+c-g) \\
&(4ac-g^2)(a^2+ac-2ad+ag+d^2-dg) \\
&(a^2+2ab+ac-ag+b^2-bg) \\
&(4a^2+3ab-4ac-3ad-bc-2bd+bg+cd+dg) \\
&(4a^3+4a^2b+8a^2c-4a^2d+ab^2-4abc-2abd \\
&+2abg+4ac^2+4acd+ad^2+2adg+b^2c+b^2g+2bcd \\
&-2bcg+cd^2-2cdg-d^2g).
\end{aligned}$$

Theorem 3.2 ([3, Theorem 1.4 and 1.7])**.** *Every smooth homogeneous cubic polynomial* $f \in K[x_1, x_2, x_3, x_4]$ *can be brought into Brundu-Logar normal form by applying projective transformations.*
Every cubic in Brundu-Logar normal form is smooth and thus contains exactly 27 lines. In particular, the points in $\mathbb{P}^4 \setminus \Sigma$ *parametrize all smooth cubics.*

The question how to obtain the Brundu-Logar normal form for a given cubic is answered by a Magma script written by Avinash Kulkarni for generic cubics over p-adic fields [12].

3.2. Tropicalizing the Brundu-Logar normal form

Let $P := (a, b, c, d, g) \in \mathbb{P}^4 \setminus \Sigma$ and f_P the cubic in Brundu-Logar normal form defined by P. We denote by S_P the corresponding cubic surface. The tropicalization of f_P in Brundu-Logar normal form is:

$$\begin{aligned}
\text{trop}(f_P) = \min\{&\text{val}(2a)+2x_1+x_2, \text{val}(b-g)+2x_1+x_3, \text{val}(-2a)+x_1+2x_2, \\
&\text{val}(d+g)+x_1+x_2+x_3, \text{val}(a-c-d)+x_1+2x_3, \\
&\text{val}(b+g)+x_1+x_2+x_4, \text{val}(-a-b-c)+x_1+x_3+x_4, \\
&\text{val}(d-g)+2x_2+x_4, \text{val}(a+c-d)+x_2+x_3+x_4, \\
&\text{val}(-a-b+c)+x_2+2x_4\}.
\end{aligned}$$

If all parameters of f_P are not zero, the Newton polytope is given by

$$\begin{aligned}
\text{Newt}(f_P) = \text{conv}(&(0,1,0,2), (1,1,0,1), (1,0,1,1), (0,2,0,1), (0,1,1,1), \\
&(2,1,0,0), (2,0,1,0), (1,2,0,0), (1,1,1,0), (1,0,2,0)).
\end{aligned}$$

Dehomogenized with respect to x_4 and embedded in the tetrahedron $3\Delta_3$ this polytope is shown in Figure 3.
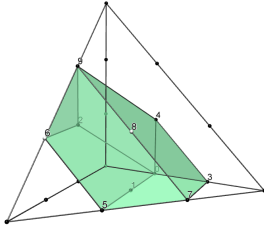
0	(0,1,0,2)	$\text{val}(-a-b+c)$
1	(1,1,0,1)	$\text{val}(b+g)$
2	(1,0,1,1)	$\text{val}(-a-b-c)$
3	(0,2,0,1)	$\text{val}(d-g)$
4	(0,1,1,1)	$\text{val}(a+c-d)$
5	(2,1,0,0)	$\text{val}(2a)$
6	(2,0,1,0)	$\text{val}(b-g)$
7	(1,2,0,0)	$\text{val}(-2a)$
8	(1,1,1,0)	$\text{val}(d+g)$
9	(1,0,2,0)	$\text{val}(a-c-d)$

Figure 3: Newton polytope of the tropicalized Brundu-Logar normal form

Table 1: Weights and vertices for a cubic in Brundu-Logar normal form

The weight vector induced by a cubic in Brundu-Logar normal form with $P = (a,b,c,d,g) \in \mathbb{P}^4 \setminus \Sigma$ is

$$\omega_P = (\text{val}(-a-b+c), \text{val}(b+g), \text{val}(-a-b-c), \text{val}(d-g), \text{val}(a+c-d),$$
$$\text{val}(2a), \text{val}(b-g), \text{val}(-2a), \text{val}(d+g), \text{val}(a-c-d)).$$

Remark 3.3. By definition, a tropical variety $V(\text{trop}(f))$ is smooth if and only if the dual subdivision of $\text{Newt}(f)$ is unimodular [13, Section 4.5]. For all coefficients of $P \in \mathbb{P}^4 \setminus \Sigma$ not zero, we have $\text{vol}(\text{Newt}(f_P)) = \frac{5}{3}$, so a unimodular triangulation contains exactly 10 tetrahedra.

By [3, Theorem 1.4], every classical Brundu-Logar normal form is smooth. However, the conditions on the weights to a cubic in Brundu-Logar normal form do not allow unimodular triangulations.

Theorem 3.4. *If $\text{val}(2) = 0$, the tropicalization of any cubic in Brundu-Logar normal form is not tropically smooth.*

In particular, this statement holds for most choices of fields with a valuation (K, val).

Proof. The idea is to make a case distinction of which combinations of the parameters in $P = (a,b,c,d,g) \in \mathbb{P}^4 \setminus \Sigma$ have minimal valuation. In every case, we obtain either that too many vertices have minimal valuation, that 4 vertices contained in a plane have minimal valuation or that the 4 vertices of minimal valuation form a tetrahedron of size larger than $\frac{1}{6}$. These arguments are valid independent of P containing zeros or coordinates of ω_P being ∞ and thus of the exact volume of $\text{Newt}(f_P)$.

In the proof we classify all possible cases. For each case we find obstructions to unimodular triangulations. For brevity, we abstain from listing all the cases; rather we show the concept by an exemplary case:
Let $\text{val}(a) = \text{val}(c) < \text{val}(b), \text{val}(d), \text{val}(g)$.
For the numeration of the vertices see Table 1. Since on the level of minimal valuations we can only have cancellations in either $a+c$ or $a-c$, at least one of $\text{val}(-a-b+c)$ and $\text{val}(-a-b-c)$ has value $\text{val}(a)$, similar with $\text{val}(a-d+c)$ and $\text{val}(a-d-c)$. If a and c have no cancellation, we obtain that the six vertices 0, 2, 4, 5, 7, 9 have the same minimal valuation. But a subdivision of a polytope in $\mathbb{R}^3$ can only be unimodular if a maximum of 4 vertices have minimal valuation. We need to have cancellations:

1. Cancellation in $a-c$.
   It follows that $\text{val}(a) = \text{val}(c) = \text{val}(a+c)$. Therefore, the vertices with numbers 2, 4, 5, 7 have same minimal valuation. However, these 4 vertices are contained in a plane and therefore induce subpolytopes in the subdivision which are not unimodular.

2. Cancellation in $a+c$.
   We obtain $\text{val}(a) = \text{val}(c) = \text{val}(a-c)$. So the vertices with numbers 0, 5, 7, 9 form a polytope in the triangulation. However, this tetrahedron is too big, since the edge from vertex number 0 to 5 contains another lattice point in its interior.

$\square$

In contrast to the octanomial model presented in [14], the tropicalized Brundu-Logar normal form is not smooth for most choices of valuated fields. Additionally in Example 4.4 we will see that an analogous statement to [14, Theorem 3.4] on the bound 1 for lifting multiplicities does not hold for the tropicalized Brundu-Logar normal form.

3.3. Tropicalizing the 27 lines in the Brundu-Logar normal form

For a cubic f_P in Brundu-Logar normal form, [3, Table 2] gives the Plücker coordinates of the 27 lines in terms of (a, b, c, d, e, f), where e and f satisfy $g = e + f$, $ac = ef$.

This offers a new perspective on working the problem of relative realizability: tropicalizing the coordinates we can conclude which of the tropical lines in the tropicalized cubic surface in Brundu-Logar normal

form are realizable. Further, for any cubic we can determine the transformation bringing it into normal form, apply its inverse to the 27 lines and obtain analogous information.

Table 2 shows the tropicalized Plücker coordinates. We can obtain the vertices of tropical lines from their Plücker coordinates; see [7], [13, Example 4.3.19]. For shorter notation, we write v_q instead of val(q).

$$\text{trop}(E_1) = [\infty, \infty, 0, \infty, \infty, \infty]$$
$$\text{trop}(G_4) = [\infty, \infty, \infty, \infty, \infty, 0]$$
$$\text{trop}(E_2) = [\infty, \infty, \infty, 0, \infty, \infty]$$
$$\text{trop}(G_3) = [0, 0, \infty, \infty, 0, 0]$$
$$\text{trop}(E_3) = [\infty, 0, 0, 0, 0, \infty]$$
$$\text{trop}(G_5) = [2v_c, v_c + v_e, \infty, \infty, v_c + v_e, 2v_e]$$
$$\text{trop}(G_6) = [2v_c, v_c + v_f, \infty, \infty, v_c + v_f, 2v_f]$$
$$\text{trop}(F_{24}) = [\infty, \infty, \infty, v_{bc-c^2+ef}, v_{-cd+ef+c^2}, v_c + v_{e+f-d}]$$
$$\text{trop}(F_{14}) = [\infty, v_{bc+c^2+ef}, v_{-c^2-cd+ef}, \infty, \infty, v_{-b+e+f} + v_c]$$
$$\text{trop}(F_{34}) = [\infty, v_{-bc+c^2-ef}, v_{c^2+cd-ef}, v_{-bc+c^2-ef}, v_{c^2+cd-ef}, v_c + v_{d+b}]$$
$$\text{trop}(F_{13}) = [v_{bc-c^2+ef}, v_{bc-c^2+ef}, v_{c-e} + v_{c-f}, \infty, v_{bc-ce-cf+2ef}, v_{bc-ce-cf+2ef}]$$
$$\text{trop}(F_{15}) = [v_c + v_{bc-c^2+ef}, v_e + v_{bc-c^2+ef}, v_{c-e} + v_{cd-cf-ef}, \infty, 2v_c + v_{e-f-b},$$
$$v_c + v_e + v_{e-f-b}]$$
$$\text{trop}(F_{16}) = [v_c + v_{bc-c^2+ef}, v_f + v_{bc-c^2+ef}, v_{c-f} + v_{cd-ce-ef}, \infty, 2v_c + v_{f-e-b},$$
$$v_c + v_f + v_{f-e-b}]$$
$$\text{trop}(F_{25}) = [v_{c^2+cd-ef} + v_c, v_{d-e+f} + 2v_c, \infty, v_{c+e} + v_{bc-cf+ef}, v_{c^2+cd-ef} + v_e,$$
$$v_{d-e+f} + v_c + v_e]$$
$$\text{trop}(F_{26}) = [v_{c^2+cd-ef} + v_c, v_{d-f+e} + 2v_c, \infty, v_{c+f} + v_{bc-ce+ef}, v_{c^2+cd-ef} + v_f,$$
$$v_{d-f+e} + v_c + v_f]$$
$$\text{trop}(F_{23}) = [v_{c^2+cd-ef}, v_{-cd+ce+cf+2ef}, \infty, v_{c+e} + v_{c+f}, v_{-c^2-cd+ef},$$
$$v_{cd-ce-cf-2ef}]$$
$$\text{trop}(F_{35}) = [v_c + v_{bc-cd+2ef}, v_{c^2f-c^2d+bce+e^2f}, v_{c-e} + v_{cd-cf-ef},$$
$$v_{c+e} + v_{bc-cf+ef}, v_{c^2f-bc^2-cde+e^2f}, v_c + v_e + v_{2f-d-b}]$$
$$\text{trop}(F_{36}) = [v_c + v_{bc-cd+2ef}, v_{c^2e-c^2d+bcf+ef^2}, v_{c-f} + v_{cd-ce-ef},$$
$$v_{c+f} + v_{bc-ce+ef}, v_{c^2e-bc^2-cdf+ef^2}, v_c + v_f + v_{2e-b-d}]$$
$$\text{trop}(E_5) = [\infty, v_{c-f} + v_{bc-cf+ef} + v_{cd-cf-ef}, v_{f-c} + 2v_{cd-cf-ef}, v_{c+f} + 2v_{bc-cf+ef},$$
$$v_{c+f} + v_{bc-cf+ef} + v_{cd-cf-ef}, v_2 + v_{bc-cf+ef} + v_{cd-cf-ef} + v_f]$$
$$\text{trop}(E_6) = [\infty, v_{c-e} + v_{bc-ce+ef} + v_{cd-ce-ef}, v_{e-c} + 2v_{cd-ce-ef}, v_{c+e} + 2v_{bc-ce+ef},$$
$$v_{c+e} + v_{bc-ce+ef} + v_{cd-ce-ef}, v_2 + v_{bc-ce+ef} + v_{cd-ce-ef} + v_e]$$
$$\text{trop}(F_{56}) = [2v_c + v_{bc-cd+2ef}, v_{-bc^2d+bc^2f+bc^2e-c^2ef+fe^2c-cdef+bcef+f^2ec+e^2f^2},$$
$$v_{cd-ce-ef} + v_{cd-cf-ef}, v_{bc-cf+ef} + v_{bc-ce+ef},$$
$$v_{c^2ef-c^2df-c^2de+bc^2d+fe^2c+f^2ec+cdef-bcef-e^2f^2}, v_{bc-cd+2ef} + v_e + v_f]$$
$$\text{trop}(E_4) = [v_2 + v_{c^2+cd-ef} + v_{bc-c^2+ef} + 2v_c, v_{c^2+cd-ef} + v_{bc-c^2+ef}$$
$$+ v_{c^2+ce+cf-ef}, 2v_{c^2+cd-ef} + v_{c-e} + v_{c-f}, v_{c+e} + v_{c+f} + 2v_{bc-c^2+ef},$$
$$v_{c^2+cd-ef} + v_{c^2-ce-cf-ef} + v_{bc-c^2+ef}, v_2 + v_{c^2+cd-ef} + v_{bc-c^2+ef} + v_e + v_f]$$
$$\text{trop}(F_{45}) = [\infty, v_{c+f} + v_{c-f} + v_{cd-cf-ef} + v_{bc-c^2+ef}, 2v_{c-f} + v_{c^2+cd-ef} + v_{cd-cf-ef},$$
$$2v_{c+f} + v_{bc-cf+ef} + v_{-bc+c^2-ef}, v_{c+f} + v_{c-f} + v_{c^2+cd-ef} + v_{bc-cf+ef},$$
$$v_{c+f} + v_{c-f} + v_{bc^2+bcf+c^2d-2c^2f-cdf+2ef^2} + v_e]$$
$$\text{trop}(F_{46}) = [\infty, v_{c+e} + v_{c-e} + v_{cd-ce-ef} + v_{bc-c^2+ef}, 2v_{c-e} + v_{c^2+cd-ef} + v_{cd-ce-ef},$$
$$2v_{c+e} + v_{bc-ce+ef} + v_{-bc+c^2-ef}, v_{c+e} + v_{c-e} + v_{c^2+cd-ef} + v_{bc-ce+ef},$$
$$v_{c+e} + v_{c-e} + v_{bc^2+bce+c^2d-2c^2e-cde+2e^2f} + v_f]$$

$$
\begin{aligned}
\mathrm{trop}(G_1) \;=\; [\;&2v_{bc-c^2+ef}+v_{bc-cd+2ef}, v_{bc-c^2+ef}+v_{bc-ce-cf+2ef}+v_{bc-cd+2ef}, \infty, \\
&v_2+v_{bc-c^2+ef}+v_{bc-cf+ef}+v_{bc-ce+ef}, \\
&v_{bc-c^2+ef}+v_{b^2c^2+bc^2d-bc^2e-bc^2f-c^2de-c^2df+2c^2ef+2cdef-2e^2f^2}, \\
&v_{bc-ce-cf+2ef}+v_{b^2c^2+bc^2d-bc^2e-bc^2f-c^2de-c^2df+2c^2ef+2cdef-2e^2f^2}\;] \\[4pt]
\mathrm{trop}(F_{12}) \;=\; [\;&2v_{bc-cd+2ef}+v_{c^2+cd-ef}+v_{-bc+c^2-ef}, v_{bc-cd+2ef} \\
&+v_{-2bcef-2e^2f^2+(-b-d+2e)c^2f+(b+d)(d-e)c^2}+v_{bc-c^2+ef}, \infty, \infty, \\
&v_{c^2+cd-ef}+v_{bc-cd+2ef}+v_{2cdef-2e^2f^2-(b+d-2e)c^2f-(-b+e)(b+d)c^2}, \\
&v_{bc^2d-bc^2e-bc^2f-2bcef+c^2d^2-c^2de-c^2df+2c^2ef-2e^2f^2} \\
&+v_{b^2c^2+bc^2d-bc^2e-bc^2f-c^2de-c^2df+2c^2ef+2cdef-2e^2f^2}\;] \\[4pt]
\mathrm{trop}(G_2) \;=\; [\;&2v_{c^2+cd-ef}+v_{bc-cd+2ef}, v_{c^2+cd-ef} \\
&+v_{bc^2d-bc^2e-bc^2f-2bcef+c^2d^2-c^2de-c^2df+2c^2ef-2e^2f^2}, \\
&v_2+v_{c^2+cd-ef}+v_{cd-ce-ef}+v_{cd-cf-ef}, \infty, v_{c^2+cd-ef} \\
&+v_{cd-ce-cf-2ef}+v_{bc-cd+2ef}, v_{cd-ce-cf-2ef} \\
&+v_{bc^2d-bc^2e-bc^2f-2bcef+c^2d^2-c^2de-c^2df+2c^2ef-2e^2f^2}\;]
\end{aligned}
$$

Table 2: The tropical Plücker coordinates of the 27 tropicalized lines on the tropicalized Brundu-Logar normal form

Remark 3.5. Note that the tropicalizations of the first five lines E_1, G_4, E_2, G_3 and E_3 are always distinct. These are depicted in Figures 4a, 4c.

As 23 of the tropicalized lines on a tropicalized surface in Brundu-Logar normal form contain at least one tropical Plücker coordinate of value ∞, here is a short overview how to visualize these lines: with five Plücker coordinates of value ∞ the tropical line is „an edge" of the tetrahedron T modelling the tropical projective space [13, Chapter 6], as shown in Figure 4a. The case of four infinite Plücker coordinates does not occur naturally in Table 2; it would lead to a line segment in one of the facets of T. Three Plücker coordinates of value ∞ lead to a tropical half line visible in one of the facets of T as in Figure 4b. Two Plücker coordinates of value ∞ lead to a classical line in the interior of T, corresponding to the middle line segment of a tropical line of infinite length in the tropical surface; see for example Figure 4c. One Plücker coordinate of value ∞ leads to a tropical half line visible in the finite parts of the surface. This is depicted for Example 4.3 in Figure 6.

Since at least 23 of the tropicalized lines have Plücker coordinates of value ∞, at most 4 classical lines on S_p can have tropicalizations fully visible on $\mathrm{trop}(S_P)$. In comparison, of the 27 lines on a cubic in octanomial form, only one classical line has finite tropical Plücker coordinates [14].

For a given surface S_P in Brundu-Logar normal form we can compute all completely visible tropical lines on $\mathrm{trop}(S_P)$ using the `polymake` extension `a-tint` by Simon Hampe [6, 9], and compare which come from a classical line on S_P.

(a) Lines in ∞
blue: $\mathrm{trop}(E_1)$,
green: $\mathrm{trop}(E_2)$,
magenta: $\mathrm{trop}(G_4)$

(b) Half line in ∞,
green: $\mathrm{trop}(F_{14})$

(c) Vertices in ∞,
red: $\mathrm{trop}(G_3)$,
black: $\mathrm{trop}(E_3)$
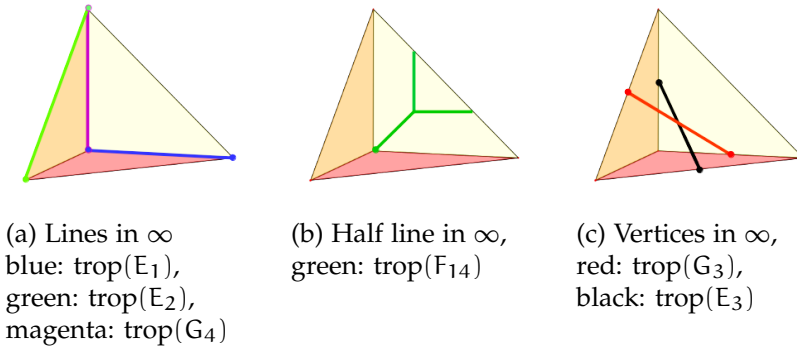
Figure 4: The tropical projective space. The facets are as follows: orange $x = \infty$, yellow $y = \infty$, red $z = \infty$.

This is carried out for a sample cubic in Example 4.3. We obtain higher lifting multiplicities in this setting than for the octanomial form [14]. Example 4.4 shows that for the Brundu-Logar normal form even isolated lines can have higher lifting multiplicities.

4.   Examples

In this section we show some examples on the theory presented in Sections 2 and 3. We start with two examples on the relative realizability of families of motif 3J and 3I-1 indicating positive answers to Questions 2.9 and 2.10. We conclude with examples investigating the behavior of lines on the tropicalized Brundu-Logar normal form.

4.1.   Relative Realizability

Example 4.1 (Lines of motif 3J)**.** This example indicates that Question 2.10 might be answerable with yes. In fact it shows a cubic surface on which a member of a family of motif 3J is realizable.

We investigate the cubic surface detected by Hampe and Joswig in [10] containing two families of motif 3I and one of motif 3J. We choose the weight vector ω as below, which induces a surface containing no lines of non-general motifs and can therefore be considered as generic:

$$\omega := (143, 0, 64, 122, 0, 2, 0, 15, 55, 107, 36, 23, 39, 16, 14, 48, 12, 12, 49, 95).$$

Using the Magma script [17] by Emre Sertoz written for [14] we can compute the 27 lines on a simple lift of the tropical cubic over an p-adic field. A simple lift is a cubic where the coefficients c_i are given as t^{ω_i},

where t is either a prime if we are over the p-adics, or $t \in \mathbb{C}\{\{t\}\}$ over the Puiseux series. The following table shows the tropicalized Plücker coordinates of the computed 27 lines for $p = 2$ matching those for $p = 5$, indicating that $p = 2$ does not have a special role in the lifting behavior of this case. The table also shows the Plücker coordinates of the tropical lines contained in the tropical surface sorted into isolated lines and families as computed by the `polymake` extension `a-tint` [9, 10].

	tropicalized lines							tropical lines						motif
1	0	−40	−86	37	3	−49		0	−40	−86	37	3	−49	3B
2	0	−3	−49	0	−37	−49		0	−3	−49	0	−37	−49	3A
3	0	22	0	−12	−12	−12		0	22	0	−12	−12	−12	3G
4	0	22	0	0	0	0		0	22	0	0	0	0	3D
5	0	2	0	−12	−12	−12		0	2	0	−12	−12	−12	3G
6	0	2	0	0	0	0		0	2	0	0	0	0	3D
7	0	−49	−86	37	3	−49		0	−49	−86	37	3	−49	3H
8	0	2	−30	−2	−48	−46		0	2	−30	−2	−48	−46	3F
9	0	2	−30	−11	−48	−46		0	2	−30	−11	−48	−46	3F
10	0	−12	−12	46	0	−12	isolated	0	−12	−12	46	0	−12	3G
11	0	−12	−12	37	0	−12		0	−12	−12	37	0	−12	3G
12	0	−12	−49	9	−37	−49		0	−12	−49	9	−37	−49	3A
13	0	0	−36	−2	−48	−48		0	0	−36	−2	−48	−48	3D
14	0	0	−36	−11	−48	−48		0	0	−36	−11	−48	−48	3D
15	0	22	−12	−2	−48	−26		0	22	−12	−2	−48	−26	3D
16	0	22	−12	−11	−48	−26		0	22	−12	−11	−48	−26	3D
17	0	22	0	46	0	22		0	22	0	46	0	22	3D
18	0	22	0	37	0	22		0	22	0	37	0	22	3D
19	0	0	0	46	0	0		0	0	0	46	0	0	3D
20	0	0	0	37	0	0		0	0	0	37	0	0	3D
21	0	2	0	46	0	2		0	2	0	46	0	2	3D
22	0	2	0	37	0	2		0	2	0	37	0	2	3D
23	0	20	2	−16	−16	−14		0	20	2	−16	−16	−14	3H
24	0	−61	−107	52	9	−55		0	−61	−107	52	9	−55	3H
25	0	−55	−55	49	3	−52		0	−55	−55	49	3	−52	3H
26	0	−55	−92	49	3	−52		0	−55	−92	49	3	−52	3H
27	0	0	0	0	0	2		0	0	0	0	0	t	3J, t ≥ 0
							fam.	12+t	0	0	12	12	0	3I-2, t ≥ 0
								12+t	12	12	0	0	0	3I-2, t ≥ 0

Table 3: The tropicalized lines compared with the tropical lines on the tropicalization of a simple lift over $\mathbb{Q}_2$ to the weight vector ω

Investigating the dual subdivision of the surface, we see that the two families of motif 3I have type 3I-2. As proven in Theorems 2.6 and 2.7 no member of these families lifts. In particular, all the 27 lines on this simple lift tropicalize to non-degenerate lines. It is notable that all the 26 isolated lines are realizable on our chosen lift of the surface. Furthermore, for this example the statement from [14, Conjecture 4.1]

holds: the tropicalizations of the 27 lines are distinct.

Notice that in this example the member for $t = 2$ of the family of motif 3J lifts, supporting the idea that we have a different lifting behavior for families of motif 3J than for motif 3I; see Question 2.10.

Example 4.2 (Lines of type 3I-1 over char($\mathbb{k}) = 2$)**.** We consider an example of a smooth tropical cubic surface containing a family of type 3I-1 and its lifting behavior over $\mathbb{Q}_2$ and $\mathbb{Q}_3$. The results suggest Question 2.9 might be answered positively and Proposition 2.5 might be extendable to characteristic p. We also compare the lifting of the lines with the corresponding Brundu-Logar normal form to the chosen surface.

Using the Magma script from [12], we can calculate the Brundu-Logar normal form to any generic cubic over the p-adics and investigate the behavior of the tropical lines on both sides. We consider the tropical cubic surface given by the weight vector v below. This cubic is of interest as it contains a family of type 3I-1. We investigate a simple lift S_v of this surface over $\mathbb{Q}_2$, giving an example corroborating a positive answer to Question 2.9.

$$v = (0, 12, 30, 0, 30, 12, 3, 145, 54, 10, 51, 0, 18, 123, 30, 0, 265, 150, 80, 21).$$

The tropicalized Brundu-Logar normal form, $\text{trop}(S_v^{nf})$, to this cubic contains 4 families, numbered F_i, $i = 0, ..., 3$. We have the weight vector

$$v^{nf} = (-10, -6, -6, -10, -10, -10, -8, -10, -10, -9).$$
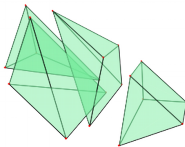
Figure 5:
Subdiv($\text{trop}(S_v^{nf})$)

	tropical Plücker coordinates						
F_0	0	0	λ	1	0	0	$\lambda \geq 0$
F_1	0	1	0	1	0	$1+\lambda$	$\lambda \geq 0$
F_2	0	$1+\lambda$	0	1	0	1	$\lambda \geq 0$
F_3	$1-\lambda$	0	0	$2-\lambda$	$1-\lambda$	0	$\lambda \in [0,1]$
	0	λ	0	1	0	λ	$\lambda \in [0,1]$

Table 4: Tropical lines on $\text{trop}(S_v^{nf})$

Computing the tropicalizations of the lines in the Brundu-Logar normal form S_v^{nf} (right columns in Table 5) and their inverse images under the transformation leading to the normal form (left columns), we can investigate the lifting behaviour of the tropical lines on both surfaces.

Investigating the Plücker coordinates of the tropicalized lines on $\text{trop}(S_v^{nf})$, as demonstrated in Table 5, we obtain two tropicalized lines of higher lifting multiplicity on $\text{trop}(S_v^{nf})$: a tropical half line of lifting

multiplicity 3 and a middle line segment of infinite length of lifting multiplicity 4. The other 20 lines on S_v^{nf} have distinct tropicalizations. Comparing Table 4 with the middle right column of Table 5, we obtain that three inner members of the family F_0 lift and that one inner member of family F_1 lifts, while no members of F_2 and F_3 lift. Additionally, F_0 approximates the tropicalized line of lifting multiplicity 3 for $\lambda \to \infty$.

lines on trop(S_v)						lifting?	lines on trop(S_v^{nf})						
0	0	0	t	0	0	(3I-1), yes t = 1	∞	∞	∞	∞	∞	0	(e)
0	−40	−20	−30	−10	−40	yes	∞	∞	0	∞	∞	∞	(e)
0	−123	−21	−111	−9	−123	yes	∞	∞	∞	0	∞	∞	(e)
0	−41	−21	−31	−11	−41	yes	∞	−1	−4	∞	∞	−3	(d)
0	−123	−22	−111	−9	−133	yes	0	0	82	∞	0	0	(b)
0	−45	−22	−34	−10	−56	yes	0	1	∞	1	0	1	(b)
0	−45	−22	−34	−4	−56	yes	0	1	0	1	0	7	(a), F_1: $\lambda = 6$
0	−48	−23	−35	−10	−48	yes	∞	0	0	0	0	∞	(c)
0	−48	−23	−36	−11	−58	yes	0	0	∞	∞	0	0	(c)
0	−123	−23	−111	−9	−134	yes	0	0	81	1	0	0	(a), F_0: $\lambda = 81$
0	−48	−23	−36	−11	−48	yes	∞	0	1	0	1	0	(b)
0	−48	−23	−35	−5	−58	yes	0	0	1	∞	0	0	(b)
0	−127	−27	−109	−11	−138	yes	0	0	∞	1	0	0	(b), F_0: $\lambda \to \infty$
0	−128	−27	−110	−10	−138	yes	0	0	∞	∞	0	0	(c)
0	−129	−27	−111	−9	−129	yes	∞	∞	∞	0	0	0	(d)
0	−75	−30	−63	−18	−86	yes	0	0	∞	1	0	0	(b), F_0: $\lambda \to \infty$
0	−75	−30	−63	−18	−75	yes	∞	20	41	0	21	20	(b)
0	−75	−30	−63	−18	−85	yes	0	0	∞	∞	0	0	(c)
0	−114	−30	−102	−18	−114	yes	∞	62	122	3	63	62	(b)
0	−114	−30	−102	−18	−124	yes	0	0	∞	∞	0	0	(c)
0	−114	−30	−102	−18	−125	yes	0	0	∞	1	0	0	(b), F_0: $\lambda \to \infty$
0	−120	−36	−102	−18	−131	yes	0	0	61	1	0	0	(a), F_0: $\lambda = 61$
0	−81	−36	−63	−18	−91	yes	0	0	21	∞	0	0	(b)
0	−81	−36	−63	−18	−81	yes	∞	20	20	0	0	0	(b)
0	−81	−36	−63	−18	−92	yes	0	0	22	1	0	0	(a), F_0: $\lambda = 22$
0	−120	−36	−102	−18	−130	yes	0	0	60	∞	0	0	(b)
0	−120	−36	−102	−18	−120	yes	∞	59	59	0	0	0	(b)
0	−48	−23	−36+t	−11	−59	(3I-2), no							

Table 5: On the last column, cf. Remark 3.5:
(a) „visible" tropical line,
(b) tropical half line (see Figure 6),
(c) middle line segment with vertices in infinity (see Figure 4c),
(d) tropical half line in infinite boundary (see Figure 4b),
(e) tropical line in the infinite boundary (see Figure 4a).

Using polymake, we obtain that the tropical cubic surface trop(S_v) contains 26 isolated lines. They are all non-degenerate and lift onto the chosen simple lift of the surface. Additionally, the surface contains

two families, both of motif 3I. However, an investigation of the dual subdivision shows that one family is of type 3I-1, while the other is of type 3I-2. As proven in Theorems 2.6 and 2.7 the family of type 3I-2 does not lift. Nevertheless, we are in the case that the characteristic of the residue field is equal to 2, and we observe that for $t = 1$ a member of the family of type 3I-1 does lift in this setting.

Comparing these results with a simple lift over $\mathbb{Q}_3$, we obtain that in this case in accordance with Theorem 2.6 the non-degenerate lines of type 3I-1 do not lift. However, the degenerate line of type 3I-1 lifts, suggesting Proposition 2.5 might hold in different characteristics; see Remark 2.8. The isolated lines have the same lifting behavior for $p = 2$ and $p = 3$.

4.2.   Lines on the tropicalized Brundu-Logar normal form

In the following let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series over $\mathbb{C}$.

Example 4.3. This example is tropically very degenerate. Most of the tropicalized lines vanish completely or partly into the infinite boundary, but we observe that some of them can be approximated by families. We also obtain higher lifting multiplicities.

Choose $P = (a, b, c, d, g) \in \mathbb{P}^4$ as $a = \frac{1}{2}(1 + t + t^2)$, $b = t^5$, $c = 2 + 2t^2$, $d = t^{10}$, $e = 1$, $f = 1 + t + 2t^2 + t^3 + t^4$, where $g = e + f$. This choice satisfies $ac = ef$ and $\sigma(P) \neq 0$, so P encodes the Brundu-Logar normal form of some smooth cubic surface. These parameters lead to the trivial subdivision of $\mathrm{Newt}(\mathrm{trop}(S_P))$; the dual surface is shown in Figure 6a. In this case we identify the tropicalizations of all 27 lines. The tropical surface $\mathrm{trop}(S_P)$ contains no isolated lines and 7 families, whose tropical Plücker coordinates are as follows with $\mu, \lambda \geq 0$:

		Plücker	coordinates			
F_0	μ	0	0	0	0	λ
F_1	0	μ	0	0	λ	0
F_2	0	0	μ	λ	0	0
	$\lambda + \mu$	0	0	λ	λ	0
F_3	λ	0	0	$\lambda + \mu$	λ	0
	λ	0	0	λ	$\lambda + \mu$	0
	$\lambda + \mu$	λ	λ	0	0	0
F_4	λ	$\lambda + \mu$	λ	0	0	0
	λ	λ	$\lambda + \mu$	0	0	0
	0	0	$\lambda + \mu$	0	λ	λ
F_5	0	0	λ	0	$\lambda + \mu$	λ
	0	0	λ	0	λ	$\lambda + \mu$

	Plücker coordinates					
F_6	0	$\lambda+\mu$	0	λ	0	λ
	0	λ	0	$\lambda+\mu$	0	λ
	0	λ	0	λ	0	$\lambda+\mu$

Table 6: The tropical Plücker coordinates of the families on $\text{trop}(S_P)$

The tropical Plücker coordinates of the 27 lines are shown in Table 7. We see that the 27 lines tropicalize to 14 distinct lines, of which only one has finite Plücker coordinates. The maximal lifting multiplicity observed in this example is 4; see Table 7.

By taking the limit of λ and/or $\mu \to \infty$ while fixing the other parameter accordingly, we can approximate some tropicalized lines as an infinite border point of some of the families; see last column of Table 7. The lines E_1, G_4, E_2, which are completely in infinity, as shown in Figure 4, are no limit of any of the families. The tropical lines $\text{trop}(F_{15}) = \text{trop}(F_{16})$ and $\text{trop}(F_{25}) = \text{trop}(F_{26})$ have one finite vertex; the other one is in infinity, cf. Remark 3.5. So only a tropical half line is visible; see Figure 6b-6c. This also does not appear as a limit of the families in this case.

	Plücker coordinates						
$\text{trop}(E_1)$	∞	∞	0	∞	∞	∞	
$\text{trop}(G_4)$	∞	∞	∞	∞	0	∞	not approximable
$\text{trop}(E_2)$	∞	∞	∞	0	∞	∞	
$\text{trop}(E_3)$	∞	0	0	0	0	∞	F_0 with $\lambda, \mu \to \infty$
$\text{trop}(F_{24})$	∞	∞	∞	0	0	0	F_4 with $\lambda \to \infty$
$\text{trop}(F_{14})$	∞	0	0	∞	∞	0	F_3 with $\lambda \to \infty$
$\text{trop}(F_{34})$	∞	0	0	0	0	5	F_0 with $\mu \to \infty, \lambda = 5$
$\text{trop}(F_{13})$	0	0	0	∞	0	0	F_2 with $\lambda \to \infty, \mu = 0$
$\text{trop}(G_2)$							F_3 (b) and F_6 (b) with $\lambda = 0, \mu \to \infty$
$\text{trop}(F_{15})$	0	0	0	∞	1	1	
$\text{trop}(F_{16})$							not approximable
$\text{trop}(F_{25})$	0	1	∞	0	0	1	
$\text{trop}(F_{26})$							
$\text{trop}(F_{23})$	0	0	∞	0	0	0	F_2 with $\mu \to \infty, \lambda = 0$
$\text{trop}(G_1)$							F_4 (c) and F_5 (a) with $\lambda = 0, \mu \to \infty$
$\text{trop}(G_3)$							
$\text{trop}(G_5)$	0	0	∞	∞	0	0	F_2 with $\lambda, \mu \to \infty$
$\text{trop}(G_6)$							
$\text{trop}(F_{12})$							
$\text{trop}(E_5)$							
$\text{trop}(E_6)$	∞	0	0	0	0	0	F_0 with $\mu \to \infty, \lambda = 0$
$\text{trop}(F_{45})$							F_3 (a) and F_4 (a) with $\lambda = 0, \mu \to \infty$
$\text{trop}(F_{46})$							

	Plücker coordinates	
trop(F_{35}) trop(F_{36}) trop(F_{56}) trop(E_4)	0 0 0 0 0 0	all families with $\lambda, \mu = 0$

Table 7: The tropical Plücker coordinates of the tropicalizations of the lines on S_P and their connection to the tropical lines on trop(S_P)

Note, that the `a-tint` output for the computation of families has to be checked for redundancies.

(a) Surface dual to the trivial subdivision

(b) The tropical half line trop(F_{15})
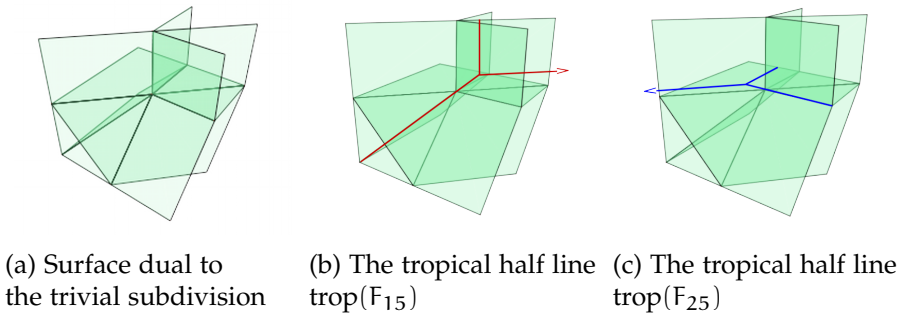
(c) The tropical half line trop(F_{25})

Figure 6: The realizable tropical half lines to Example 4.3

Example 4.4. The concluding example is a subdivision relatively close to a unimodular triangulation: it contains only two polytopes of the too large volume $\frac{1}{3}$. However, we can still observe lifting multiplicities higher than one, but we can no longer approximate lines partly contained in the infinite boundary by families.

We choose the parameters $a = t^{20}$, $b = 1 + t - t^6$, $c = -1 - t + t^6$, $d = -1 - t + t^6 + t^8$, $g = -1 - t + t^6 + t^8 + t^{15}$, and obtain the weight vector $w = (0, 8, 6, 15, 8, 20, 0, 20, 0, 0)$, which induces a subdivision with 8 maximal cells; see Figure 8a. The corresponding surface has 1 isolated line L_{iso} and 2 families F_0 and F_1, computed using `a-tint` in `polymake`:

$$F_0 = [0, 0, 8, 0, 8, 12 + \lambda], \lambda \geq 0$$
$$F_1 = [0, 0, 8, 0, 8 + \lambda, 8], \lambda \geq 0$$
$$L_{iso} = [0, 15, 15, 0, 0, 20].$$

The primal motifs of the two families can be seen in Figure 7. We obtain motifs that are not in the classification of [15], since our surface

is not tropically smooth. Only the motif of the isolated line, motif 3A, is one that can occur on general smooth tropical cubics.

Computing the tropicalized lines with finite Plücker coordinates using Table 2, we obtain that $\text{trop}(F_{36}) = \text{trop}(F_{56}) = L_{iso}$, while E_4 tropicalizes to an inner member of F_0 given by $\lambda = 8$ and F_{35} tropicalizes to the degenerate representative of F_1 with $\lambda = 0$. In this case, we cannot obtain any of the other 23 lines as limits of the families as $\lambda \to \infty$.

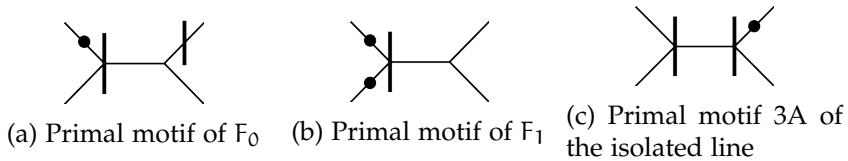(a) Primal motif of F_0 (b) Primal motif of F_1 (c) Primal motif 3A of the isolated line

Figure 7: Primal motifs of the lines from Example 4.4

(a) Dual subdivision   (b) The tropicalized surface with its isolated line and two families. F_0 in red, F_1 in green.
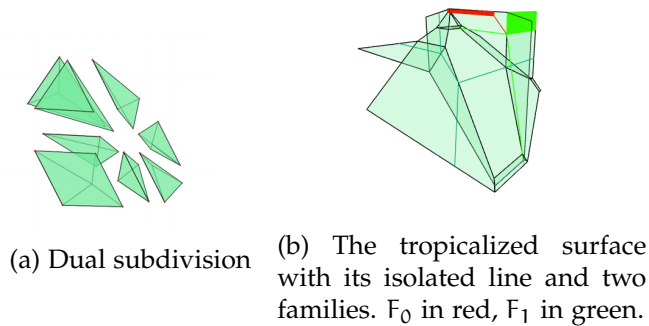
Figure 8: Dual subdivision and surface to Example 4.4

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ALHEYDIS GEIGER
Department of Mathematics, University of Tübingen, Germany
e-mail: alheydis.geiger@math.uni-tuebingen.de