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A TROPICAL COUNT OF BINODAL CUBIC SURFACES

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There are 280 binodal cubic surfaces passing through 17 general points. For points in Mikhalkin position, we show that 214 of these give tropicalizations such that the nodes are separated on the tropical cubic surface.

1. Introduction

Given some points in general position, one can ask for the number of varieties of a fixed dimension and fixed number of nodes passing through the points. We study tropical counts of binodal cubic surfaces over $\mathbb{C}$ and $\mathbb{R}$. The space $\mathbb{P}^{19}$ parameterizes cubic surfaces by the coefficients of their defining polynomial. The singular cubic surfaces form a variety of degree 32 called the *discriminant* in $\mathbb{P}^{19}$. The surfaces passing through a particular point in $\mathbb{P}^3$ form a hyperplane in $\mathbb{P}^{19}$. Thus, through 18 generic points there are 32 nodal surfaces. The reducible singular locus of the discriminant is the union of the cuspidal cubic surfaces and the binodal cubic surfaces. Each is a codimension 2 variety in $\mathbb{P}^{19}$.

In [13, Section 7.1] Vainsencher gives formulas for the number of k-nodal degree m surfaces in a k-dimensional family in $\mathbb{P}^3$. That is, for $k = 2$ and $m = 3$ he determines the degree of the variety parameterizing

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2-nodal cubics. For $k=2$ nodes, there are $2(m-2)(4m^3-8m^2+8m-25)(m-1)^2$ such surfaces. Setting $m=3$, we have the following count.

Theorem 1.1 ([13]). *There are 280 binodal complex cubic surfaces passing through 17 general points.*

Mikhalkin pioneered the use of tropical geometry to answer questions in enumerative geometry [11]. Tropical methods have successfully counted nodal plane curves over $\mathbb{C}$ and $\mathbb{R}$ [3, 11]. In [2, 3] this technique is enriched by the concept of floor diagrams. In our setting, we ask:

Question 1.2 (Question 10 [12]). Can the number 280 of binodal cubic surfaces through 17 general points be derived tropically?

For points in Mikhalkin position, as introduced in Definition 2.1, tropical methods are useful because the dual subdivisions of the Newton polytope are very structured. This allows us to study only 39 subdivisions of $\Delta = \mathrm{Conv}\{(0,0,0),(3,0,0),(0,3,0),(0,0,3)\}$, the Newton polytope of a cubic surface. This is minuscule compared to the 344,843,867 unimodular triangulations of this polytope [5, 6].

Singular tropical surfaces and hypersurfaces are studied in [4, 8]. A surface with δ nodes as its only singularities is called δ-*nodal*. The tropicalization of a δ-nodal surface is called a δ-*nodal* tropical surface. We say a δ-nodal surface is *real* if the polynomial defining the surface is real and the surface has real singularities.

If we count with multiplicities all tropical binodal cubic surfaces through our points, we will recover the true count. We study tropical surfaces with *separated* nodes, in the sense that the topological closures of the cells in the tropical surface containing the nodes have empty intersection. To count them, we list the dual subdivisions of candidate binodal tropical cubic surfaces and count their multiplicities.

Theorem 1.3. *There are 39 tropical binodal cubic surfaces through* 17 *points in Mikhalkin position (see Definition 2.1) containing separated nodes. They give rise to* 214 *complex binodal cubic surfaces through* 17 *points.*

Proof. We distinguish five cases based on which *floors* (see Definition 2.5) of the tropical cubic surface contain the nodes and count with complex multiplicities (see Definition 2.7).

$$214 = \underbrace{20}_{\text{Proposition 2.9}} + \underbrace{24}_{\text{Proposition 3.4}} + \underbrace{90}_{\text{Proposition 3.5}} + \underbrace{72}_{\text{Proposition 4.3}} + \underbrace{8}_{\text{Proposition 4.4}}.$$

□

Theorem 1.4. *There exists a point configuration ω of 17 real points in $\mathbb{P}^3$ all with positive coordinates, such that there are at least 58 real binodal cubic surfaces passing through ω.*

Proof. We count the floor plans in Theorem 1.3 with real multiplicities. Since the real multiplicities are difficult to determine in some cases, the propositions only give us lower bounds. We obtain that there are at least 58 real binodal cubic surfaces passing through ω.

$$58 = \underbrace{16}_{\text{Proposition 2.9}} + \underbrace{4}_{\text{Proposition 3.4}} + \underbrace{34}_{\text{Proposition 3.5}} + \underbrace{4}_{\text{Proposition 4.3}} + \underbrace{0}_{\text{Proposition 4.4}} .$$

□

As we conduct the counts in Theorems 1.3 and 1.4, we encounter cases with *unseparated* nodes. Here, the two node germs (see Definition 2.3) are close together, and so the cells that would normally contain the nodes interact and their topological closures intersect. Thus, the node germs interfere with the conditions on producing nodes [9]. These cases account for the 66 surfaces missing from our count. Their dual subdivisions contain unclassified polytope complexes, which we list in Section 5.1.

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2. Tropical Floor Plans

We now give an overview of counting surfaces using tropical geometry. Let $\mathbb{K} = \cup_{m\geq 1}\mathbb{C}\{t^{1/m}\}$ and $\mathbb{K}_{\mathbb{R}} = \cup_{m\geq 1}\mathbb{R}\{t^{1/m}\}$. We assume the reader is familiar with tropical hypersurfaces and the corresponding dual subdivision of the Newton polytope as in [7, Chapter 3.1].

For any 17 generic points in $\mathbb{P}^3_{\mathbb{K}}$, the tropicalizations of the 280 binodal cubic surfaces pass through the tropicalizations of the 17 points. However, an arbitrary choice of points might lead to the tropicalizations of the points not being distinct, or not being tropically generic. Furthermore, these surfaces would be difficult to characterize in general.

Luckily, we can choose points in *Mikhalkin position* (see Definition 2.1). This is a configuration of points in generic position such that their tropicalizations are tropically generic. Additionally, tropical surfaces passing through such points have a very nice form, and the combinatorics of the dual subdivision is well understood.

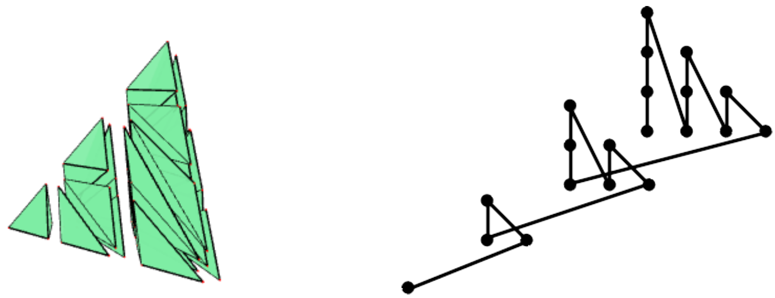
We can count tropical surfaces through points in Mikhalkin position. Since the points are in generic position, we have $280 = \sum_S \mathrm{mult}_{\mathbb{C}}(S)$, where we sum over all tropical surfaces S passing through the tropicalized points and $\mathrm{mult}_{\mathbb{C}}(S)$ denotes the lifting multiplicity of S over $\mathbb{K}$. At this time, the ways in which two nodes can appear in a tropical surface are not fully understood, so our count is incomplete. Cases we do not understand yet are listed in Section 5.1.

We now give the definition of points in Mikhalkin position.

Definition 2.1 ([9, Section 3.1]). Let $\omega = (p_1, \ldots, p_{17})$ be a configuration of 17 points in $\mathbb{P}^3_{\mathbb{K}}$ or $\mathbb{P}^3_{\mathbb{K}_{\mathbb{R}}}$. Let $q_i \in \mathbb{R}^3$ be the tropicalization of p_i for $i = 1, \ldots, 17$. We say ω is in *Mikhalkin position* if the q_i are distributed with growing distances along a line $\{\lambda \cdot (1, \eta, \eta^2) | \lambda \in \mathbb{R}\} \subset \mathbb{R}^3$, where $0 < \eta \ll 1$, and the p_i are generic.

This is possible by [11, Theorem 1]. From now on all cubic surfaces are assumed to satisfy point conditions from points in Mikhalkin position.

We now summarize the recipe for constructing binodal tropical cubic surfaces through our choice of 17 points. Given a singular tropical surface S passing through $\omega = (p_1, \ldots, p_{17})$ in Mikhalkin position, each point p_i is contained in the relative interior of its own 2-cell of S [9, Remark 3.1]. Therefore, we can encode the positions of these points by their dual edges in the Newton subdivision. Marking these edges in the subdivision leads to a path through 18 of the lattice points in the Newton polytope Δ. Thus, the path misses two lattice points in Δ. Due to our special configuration, this path is always connected for cubics [9, Section 3.4]. Moreover, satisfying point conditions in Mikhalkin position implies that the surface is floor decomposed [1], i.e., the dual subdivision can be considered as a union of the subdivided polytopes: $\mathrm{conv}\{(0,0,0),(0,3,0),(0,0,3),(1,0,0),(1,2,0),(1,0,2)\}$, $\mathrm{conv}\{(1,0,0),(1,2,0),(1,0,2),(2,0,0),(2,1,0),(2,0,1)\}$ and the tetrahedron $\mathrm{conv}\{(2,0,0),(2,1,0),(2,0,1),(3,0,0)\}$. These are the *slices*, see Figure 1a. As said before, the edges dual to the 2-cells containing the points in Mikhalkin position give rise to a path. This path leads through the triangle faces of the boundary of the slices of Δ and connects each of them by one step, see Figure 1b, [9, Section 3.2]. By looking at the triangle faces of the slices independently, we obtain subdivisions of polytopes dual to tropical curves of degrees 3, 2, and 1. These are the *floors* of our floor plans (see Definition 2.5).



(a) Floor decomposed dual subdivision

(b) The lattice path through the points of Δ corresponding to a smooth tropical cubic surface

Figure 1: Subdivision and lattice path to a smooth tropical cubic surface through points in Mikhalkin position.

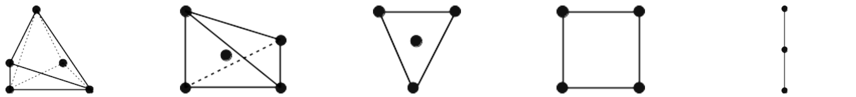
For tropical surfaces passing through points in Mikhalkin position this process is reversible. We start with a lattice path through 18 points in Δ that proceeds through the slices in the prescribed way. From this path we reconstruct the floors of the surface. Then, we extend this to a floor-decomposed subdivision of Δ by the smooth extension algorithm [9, Lemma 3.4], thus giving rise to a tropical surface passing through points in Mikhalkin position.

Tropicalizations of singularities leave a mark in the dual subdivision [8]. By [8] a tropical surface is one-nodal if it contains one of the 5 circuits shown in Figure 2. A *circuit* is a set of affinely dependent lattice points such that each proper subset is affinely independent.

Proposition 2.2. *For the tropicalization of a nodal cubic surface passing through points in Mikhalkin position only the circuits A, D and E can occur in the dual subdivision (see Figure 2).*

Proof. Since the Newton polytope to a cubic surface does not contain interior lattice points, circuit B is eliminated. The point conditions induce a lattice path in the dual subdivision, which eliminates the possibility of interior points in a triangle, so circuit C cannot occur. □

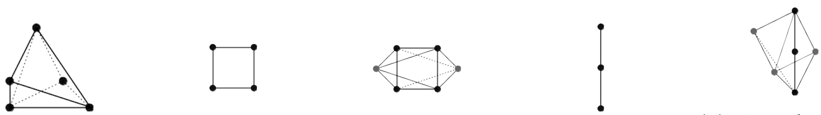
The lower dimensional circuits have to satisfy some additional conditions such that their dual cell contains a node. Circuit A is a pentatope, which is full dimensional. So its dual cell is a vertex and this



(a) Circuit A (b) Circuit B (c) Circuit C (d) Circuit D (e) Circuit E

Figure 2: Circuits in the dual subdivision inducing nodes in the surface.

vertex is the node. To encode a singularity, circuit D must be part of a bipyramid (see Figure 3c). The node is the midpoint of the edge dual to the parallelogram. Circuit E must have at least three neighboring points in special positions, forming at least two tetrahedra with the edge (see Figure 3e). The weighted barycenter of the 2-cell dual to the edge of length two is the node, where the weight is given by the choice of the three neighbors. We now introduce the definition of a node germ, which is a feature of a tropical curve appearing in a floor plan giving rise to one of these circuits in the subdivision dual to the tropical surface.



(a) Circuit A (b) Circuit D (c) Bipyramid (d) Circuit E (e) Weight two configuration

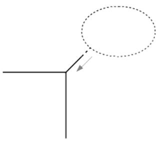
Figure 3: Circuits in the dual subdivision.

Definition 2.3 ([10], Definition 5.1)**.** Let C be a plane tropical curve of degree d passing through $\binom{d+2}{2}-2$ points in general position. A *node germ* of C of a floor plan of degree 3 is one of the following:

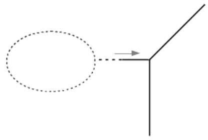
1. a vertex dual to a parallelogram,
2. a horizontal or diagonal end of weight two,
3. a right or left *string* (see below).

If the lower right (resp. left) vertex of the Newton polytope has no point conditions on the two adjacent ends, we can prolong the adjacent bounded edge in direction $(1,0)$ (resp. $(-1,-1)$) and still pass through the points. The union of the two ends is called a *right* (resp. *left*) *string*. See Figures 4a and 4b.

Remark 2.4. Now we can state the definition of separated nodes in terms of the dual subdivision: two nodes are separated if they arise



(a) Left string



(b) Right string

Figure 4: Right and left strings.

from polytope complexes of the form 3a, 3c or 3e. Any such two complexes might intersect in a unimodular face.

In [10] tropical floor plans are introduced to count surfaces satisfying point conditions, similar to the concept of floor diagrams used to count tropical curves. Their definition of tropical floor plans requires node germs to be separated by a smooth floor. This neglects surfaces where the nodes are still separated but closer together, because that is enough to count multinodal surfaces asymptotically [10, Theorem 6.1].

Definition 2.5 (Specialized from [10], Definition 5.2)**.** Let Q_i be the projection of q_i along the x-axis. A *two-nodal floor plan* F *of degree* 3 is a tuple (C_3, C_2, C_1) of plane tropical curves C_i of degree i together with a choice of indices $3 \geq i_2 \geq i_1 \geq 1$, such that $i_{j+1} > i_j + 1$ for all j, satisfying:

1. The curve C_i passes through the following points:

$$\begin{aligned} \text{if } i_\nu > i > i_{\nu-1}: &\quad Q_{\sum_{k=i+1}^{3}\binom{k+2}{2}-2+\nu}, \ldots, Q_{\sum_{k=i}^{3}\binom{k+2}{2}-4+\nu} \\ \text{if } i = i_\nu: &\quad Q_{\nu-1+\sum_{k=i+1}^{3}\binom{k+2}{2}}, \ldots, Q_{-4+\nu+\sum_{k=i}^{3}\binom{k+2}{2}}. \end{aligned}$$

2. The plane curves C_{i_j} have a node germ for each $j = 1, 2$.

3. If the node germ of C_{i_j} is a left string, then its horizontal end aligns with a horizontal bounded edge of C_{i_j+1}.

4. If the node germ of C_{i_j} is a right string, then its diagonal end aligns either with a diagonal bounded edge of C_{i_j-1} or with a vertex of C_{i_j-1} which is not adjacent to a diagonal edge.

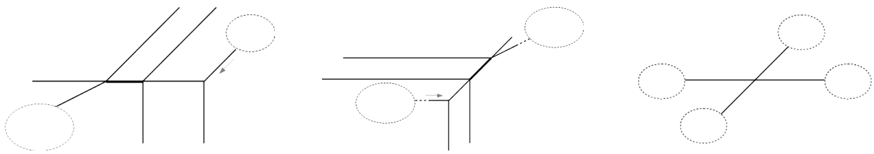
5. If $i_2 = 3$, then the node germ of C_3 is either a right string or a diagonal end of weight two.

6. If $i_1 = 1$, then the node germ of C_1 is a left string.

The information contained in a floor plan defines a unique tropical binodal cubic surface [10, Proposition 5.9].

Remark 2.6. This definition only allows node germs in floors that are separated by a smooth floor. To count all surfaces with separated singularities, we have to allow node germs in adjacent or the same floors and hence we need to extend this definition to the new cases, that cannot occur in the original setting.

As soon as adjacent floors can contain node germs, a new alignment option for the left string is possible: analogous to the second alignment option for the right string, a left string in C_i can also align with a vertex of C_{i+1} not adjacent to a horizontal edge.



(a) Left string aligning with a horizontal bounded edge

(b) Right string aligning with a diagonal bounded edge

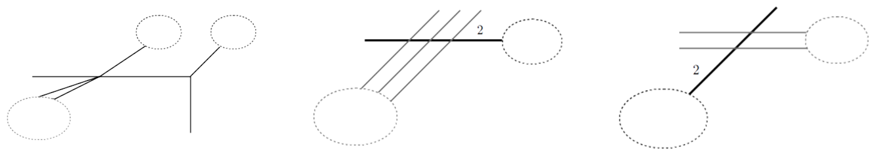
(c) Parallelogram in subdivision dual to floor

Figure 5: Node germs giving a circuit of type D.

We now describe how the node germs from Definition 2.3 together with the alignment conditions described in Definition 2.5 produce one of the circuits from Figure 3 inside the dual subdivision.

Figure 5 shows all node germs which lead to a parallelogram in the subdivision of the Newton polytope. If the node germ in a curve is dual to a parallelogram we have a picture as in Figure 5c. The right vertex of the floor of higher degree and the left vertex of the floor of lower degree form a bipyramid over the parallelogram as in Figure 3c. Figure 5a depicts the alignment of the horizontal end of the left string with a bounded horizontal edge of a curve of higher degree. In the floor plan, this translates to the dual vertical edges in the subdivisions forming a parallelogram. Since the string passes through the two vertices bounding the horizontal edge it aligns with, the dual polytope complex is a bipyramid over the parallelogram. The two top vertices of the pyramids are the vertices forming triangles with the vertical bounded edge in the dual subdivision to the floor of higher degree. Analogously, a right string aligning with a diagonal bounded edge (see Figure 5b) produces a bipyramid in the dual subdivision.

Figure 6a shows the alignment of a left string with a vertex not adjacent to a horizontal edge. The 5-valent vertex in this figure is dual to a type A circuit, as in Figure 3a. The analogous alignment of a right



(a) Intersection dual to a pentatope (b) Horizontal end of weight two (c) Diagonal end of weight two

Figure 6: Node germs leading to circuits of type A and type E.

string with a vertex not adjacent to a diagonal edge is very rare in our setting, since we consider surfaces of degree 3 and a smooth conic contains no such vertex. The occurring cases in our count are due to node germs in the conic and lead not to a pentatope as in Figure 3a, but to different complexes considered in Section 5.1.

Figures 6b and 6c show the node germs coming from an undivided edge of length two in the subdivision, as shown in Figure 3d. The node is contained in the dual 2-cell of the length two edge. Every intersection point of the weight two diagonal (resp. horizontal) end with the lower (resp. higher) degree curve of the floor plan can be selected to lift the node [9]. In the dual subdivision this corresponds to choosing three neighboring vertices which could form the polytope complex shown in Figure 3e. With our chosen point condition the neighboring vertex in the dual subdivision of the floor containing the undivided edge is always one of the three neighboring vertices. If the length two edge is diagonal (resp. vertical) the other two vertices have to form a vertical (resp. diagonal) length one edge in the boundary of the subdivision dual to the lower (resp. higher) degree curve of the floor plan.

The complex lifting multiplicity of the node germs in the floors can be determined combinatorially using [9].

Definition 2.7 (Specialized from Definition 5.4, [10])**.** Let F be a 2-nodal floor plan of degree 3. For each node germ $C^*_{i_j}$ in C_{i_j}, we define the following local complex multiplicity $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j})$:

1. If $C^*_{i_j}$ is dual to a parallelogram, then $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j}) = 2$.

2. If $C^*_{i_j}$ is a horizontal end of weight two, then $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j}) = 2(i_j + 1)$.

3. If $C^*_{i_j}$ is a diagonal end of weight two, then $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j}) = 2(i_j - 1)$.

4. If $C^*_{i_j}$ is a left string, then $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j}) = 2$.

5. If $C^*_{i_j}$ is a right string whose diagonal end aligns with a diagonal bounded edge, then $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j}) = 2$.

6. If $C^*_{i_j}$ is a right string whose diagonal end aligns with a vertex not adjacent to a diagonal edge, then $\mathrm{mult}_{\mathbb{C}}(C^*_{i_j}) = 1$.

The multiplicity of a 2-nodal floor plan is $\mathrm{mult}_{\mathbb{C}}(F) = \prod_{j=1}^{2} \mathrm{mult}_{\mathbb{C}}(C^*_{i_j})$.

To determine the real multiplicity, we have to fix the signs of the coordinates of the points in ω, as they determine the existence of real solutions of the initial equations in [9]. The dependence on the signs of the coordinates of the points is shown by including s in the notation $\mathrm{mult}_{\mathbb{R},s}$ for the real multiplicity. Here we only consider points where every coordinate is positive.

Definition 2.8 ([10], Definition 5.10)**.** For a node germ $C^*_{i_j}$ in C_{i_j}, we define the local real multiplicity $\mathrm{mult}_{\mathbb{R},s}(C^*_{i_j})$:

1. If $C^*_{i_j}$ is dual to a parallelogram, it depends on the position of the parallelogram in the Newton subdivision:

   - if the vertices are $(k,0)$, $(k,1)$, $(k-1,l)$ and $(k-1,l+1)$, then
   $$\mathrm{mult}_{\mathbb{R},s}(C^*_{i_j}) = \begin{cases} 2 \\ 0 \end{cases} \text{if } \left(\frac{3}{2}i_j + 1 + k + l\right)(i_j - 1) \equiv \begin{cases} 1 \\ 0 \end{cases} \text{modulo } 2.$$
   - if the vertices are $(k, 3-i_j-k)$, $(k, 3-i_j-k-1)$, $(k+1,l)$ and $(k+1,l+1)$, then
   $$\mathrm{mult}_{\mathbb{R},s}(C^*_{i_j}) = \begin{cases} 2 \\ 0 \end{cases} \text{if } \frac{1}{2}\cdot(i_j + 2 + 2l)(i_j - 1) \equiv \begin{cases} 1 \\ 0 \end{cases} \text{modulo } 2.$$

2. If $C^*_{i_j}$ is a diagonal edge of weight two, $\mathrm{mult}_{\mathbb{R},s}(C^*_{i_j}) = 2(i_j - 1)$.

3. If $C^*_{i_j}$ is a left string, then it depends on the position of the dual of the horizontal bounded edge of C_{i_j+1} with which it aligns. Assume it has the vertices (k,l) and $(k,l+1)$. Then
$$\mathrm{mult}_{\mathbb{R},s}(C^*_{i_j}) = \begin{cases} 2 \\ 0 \end{cases} \quad \text{if } i_j - k \equiv \begin{cases} 0 \\ 1 \end{cases} \quad \text{modulo } 2.$$

4. If $C^*_{i_j}$ is a right string whose diagonal end aligns with a a vertex not adjacent to a diagonal edge, then $\mathrm{mult}_{\mathbb{R},s}(C^*_{i_j}) = 1$.

A tropical 2-nodal surface S of degree 3 given by a 2-nodal floor plan F has at least $\mathrm{mult}_{\mathbb{R},s}(F) = \prod_{j=1}^{2} \mathrm{mult}_{\mathbb{R},s}(C^*_{i_j})$ real lifts with all positive coordinates satisfying the point conditions [10, Proposition 5.12]. Several cases are left out of the above definition because the number of real solutions is hard to control. We address this in Section 5.2. This is why we can only give a lower bound of real binodal cubic surfaces where the tropicalization contains separated nodes.

We now count surfaces from the floor plans defined in [10, Definition 5.2], which have node germs in the linear and cubic floors. Since we adhere exactly to Definition 2.5 the nodes will always be separated.

Proposition 2.9. *There are* 20 *cubic surfaces containing two nodes such that there is one node germ in the cubic floor and one in the linear floor. Of these binodal surfaces at least* 16 *are real.*

Proof. By Definition 2.5 a floor plan consists of a cubic curve C_3, a conic C_2, and a line C_1, where the tropical curves C_3 and C_1 contain node germs. Recall that the notation C^*_i stands for the node germ in C_i. By Definition 2.5 (6) the node germ of C_1 is a left string as in Figure 8a, which always aligns with the horizontal bounded edge in C_2, so $\mathrm{mult}_{\mathbb{C}}(C^*_1) = 2$. The node germs in C_3 possible by Definition 2.5 (5) are depicted in Figures 7b-7d and each one gives a different floor plan.

(7b) There is a right string in the cubic floor. In the smooth conic, there is no vertex which is not adjacent to a diagonal edge. So, the right string of the cubic must align with the diagonal bounded edge. This gives $\mathrm{mult}_{\mathbb{C}}(F) = \mathrm{mult}_{\mathbb{C}}(C^*_3) \cdot \mathrm{mult}_{\mathbb{C}}(C^*_1) = 2 \cdot 2 = 4$. In this case, $\mathrm{mult}_{\mathbb{R},s}(F)$ is undetermined, see Section 5.2.

(7c, 7d) The cubic has a weight two diagonal end. We have $2 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot \mathrm{mult}_{\mathbb{C}}(C^*_3) \cdot \mathrm{mult}_{\mathbb{C}}(C^*_1) = 2 \cdot (2(3-1) \cdot 2) = 16$. By Definition 2.8 (3) the real multiplicity of the left string depends on coordinates of the dual of the edge it aligns with: $(1,0)$ and $(1,1)$. This gives $2 \cdot \mathrm{mult}_{\mathbb{R},s}(F) = 2 \cdot \mathrm{mult}_{\mathbb{R},s}(C^*_3) \cdot \mathrm{mult}_{\mathbb{R},s}(C^*_1) = 2 \cdot (2(3-1) \cdot 2) = 16$.

□

Notice that having node germs separated by a floor only accounts for 20 of the 280 tropical cubic surfaces through our 17 points. As we will show, our extension of Definition 2.5 captures many more surfaces.

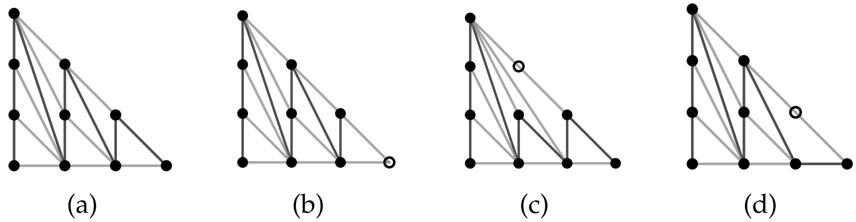
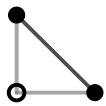
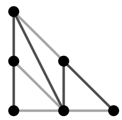


Figure 7: The triangulation dual to a smooth cubic floor and the three possible subdivisions dual to a tropical cubic curve with one node germ.



(a) Left string in C_1



(b) A triangulation dual to a smooth conic

Figure 8: The triangulations dual to linear and conic curves appearing as part of the floor plans to Proposition 2.9.

3. Nodes in adjacent floors

We now study cases where node germs are in adjacent floors of the floor plan, extending Definition 2.5, and check that the nodes are separated.

Lemma 3.1. *If a floor plan of a degree* d *surface in* $\mathbb{P}^3$ *contains a diagonal or horizontal end of weight two and a second node germ leading to a bipyramid in the subdivision, such that the bipyramid does not contain the weight two end, the nodes are separated.*

Proof. The bipyramid and the weight two end share at maximum one vertex. The neighboring points of the weight two end can be part of the bipyramid. This causes no obstructions to the conditions in [9] for the existence of a binodal surface tropicalizing to this. □

Lemma 3.2. *If a floor plan of a degree* d *surface in* $\mathbb{P}^3$ *has separated nodes,* C_2 *cannot have a right string.*

Proof. By Definition 2.5 (4) a right string in C_2 would have to align with a diagonal bounded edge of C_1 or with a vertex of C_1 not adjacent to a diagonal edge. Since C_1 is a tropical line, both cases can never occur. □

We now give the lemma used to eliminate cases with polytope complexes in the Newton subdivision that cannot accommodate two nodes.

We use obstructions arising from dimensional arguments, which are independent of the choice of generic points.

Lemma 3.3. *Let $\Gamma \subset \mathbb{Z}^3$ be finite, and let B_Γ be the variety of binodal hypersurfaces with defining polynomial having support Γ. If the dimension of B_Γ is less than $|\Gamma|-3$, then any tropical surface whose dual subdivision consists of unimodular tetrahedra away from $Conv(\Gamma)$ is not the tropicalization of a complex binodal cubic surface.*

Proof. If a binodal cubic surface had such a triangulation and satisfied our point conditions, then we could obtain from it a binodal surface with support Γ satisfying $|\Gamma|-3$ point conditions. However, if the dimension of B_Γ is less than $|\Gamma|-3$ we do not expect any such surfaces to satisfy $|\Gamma|-3$ generic point conditions. □

Typically the dimension of B_Γ is the expected dimension $|\Gamma|-3$. For some special point configurations Γ the dimension is less than this, and these are the cases we want to eliminate.

To apply the lemma, suppose $\mathrm{conv}(\Gamma)$ is a subcomplex of the subdivision of Δ. If apart from $\mathrm{conv}(\Gamma)$ the subdivision of Δ only contains unimodular simplices, cutting Δ down to $\mathrm{conv}(\Gamma)$ corresponds to removing the lattice points of $\Delta \setminus \mathrm{conv}(\Gamma)$, loosing one point condition each. Thus, if $\mathrm{conv}(\Gamma)$ cannot accommodate 2 nodes, neither can Δ.

Proposition 3.4. *There are* 24 *cubic surfaces containing two nodes such that the tropical cubic has two separated nodes and the corresponding node germs are contained in the conic and linear floors. Of these, at least* 4 *are real.*

Proof. Here a floor plan consists of a smooth cubic curve C_3 (see Figure 7a), a conic C_2 and a line C_1, both with a node germ. The node germ of C_1 is by Definition 2.5 (6) a left string, see Figure 8a. For C_2 all possibilities from Definition 2.3 are depicted in Figure 9. We examine all choices for the floor plan F and check whether the nodes are separated.

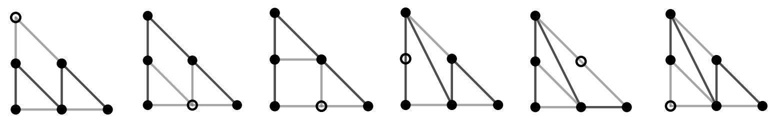
(9a)-(9c) By Definition 2.5 (3) the left string in C_1 must align with the horizontal bounded edge of C_2, which is dual to a face of the parallelogram in the subdivision. We obtain a prism polytope between the two floors, and by completion of the subdivision, we get two pyramids sitting over those two rectangle faces of the prism, that are not on the boundary of the Newton polytope. This complex may hold two nodes, see Section 5.1.

(9d) By Definition 2.5 (3), the left string of C_1 aligns with the horizontal bounded edge of C_2, giving a bipyramid in the subdivision, with top vertices the neighbors to the dual of the bounded diagonal edge in C_2. The length two edge dual to the horizontal end of weight two is surrounded by tetrahedra that only intersect the bipyramid in a unimodular face. So, the nodes are separated and we count their multiplicities: $\mathrm{mult}_{\mathbb{C}}(F) = \mathrm{mult}_{\mathbb{C}}(C_1^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2 \cdot 2(2+1) = 12$. In this case, $\mathrm{mult}_{\mathbb{R},s}(F)$ is undetermined, see Section 5.2.

(9e) The left string in C_1 must align with the vertex in C_2 not adjacent to a horizontal edge, but this vertex is dual to the area two triangle in the subdivision. The resulting volume two pentatope contains the neighbors of the length two edge. This configuration is eliminated using Lemma 3.3.

(9f) The left strings in C_1 and C_2 lead to two bipyramids in the subdivision. For each of the 3 alignment possibilities of the left string in C_2, the resulting bipyramids are disjoint and the nodes separate. We get $3 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 3 \cdot \mathrm{mult}_{\mathbb{C}}(C_1^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 3 \cdot (2 \cdot 2) = 12$. By Definition 2.8 (3) we need to consider the positions of the dual edges the left strings align with in order to compute the real multiplicities. The left string in C_1 aligns with the edge given by the vertices $(1,0),(1,1)$ in the conic floor, it has $\mathrm{mult}_{\mathbb{R},s}(C_1^*) = 2$. For the conic, two of the three choices have x-coordinate $k = 1$ in the cubic floor, so $\mathrm{mult}_{\mathbb{R},s}(C_2^*) = 0$. The last alignment is dual to x-coordinate $k = 2$, so we have $\mathrm{mult}_{\mathbb{R},s}(C_2^*) = 2$. We obtain $\mathrm{mult}_{\mathbb{R},s}(F) = 4$.

□



(a) (b) (c) (d) (e) (f)

Figure 9: The possible subdivisions dual to a tropical conic curve with one node germ appearing as part of a floor plan of a nodal cubic surface.

Proposition 3.5. *There are 90 cubic surfaces containing two nodes such that the tropical binodal cubic has separated nodes and the node germs are contained in the cubic and conic floors. Of these, at least 34 are real.*

Proof. A floor plan consists of a cubic C_3 with a node germ (Figures 7b-7d), a conic C_2 with a node germ (Figure 9), and a smooth line C_1. There are 18 combinations.

(7b, 9a-9b) The cubic contains a right string, which must align with a diagonal bounded edge by Definition 2.5 (4). The resulting subdivision contains a triangular prism with two pyramids. This complex may contain two nodes, see Section 5.1.

(7b, 9c) The right string in the cubic must align with the vertex of the conic dual to the square in the subdivision, giving rise to the polytope complex shown in Section 5.1.

(7b, 9d) The right string in the cubic must align with the vertex dual to the left triangle in the conic containing the weight two edge. The resulting complex may hold 2 nodes, see Section 5.1.

(7b, 9e) The resulting subdivision contains a bipyramid and a weight two configuration only overlapping in vertices, so the nodes are separated. We have $\text{mult}_{\mathbb{C}}(F) = \text{mult}_{\mathbb{C}}(C_3^*) \cdot \text{mult}_{\mathbb{C}}(C_2^*) = 2 \cdot 2(2-1) = 4$. In this case, $\text{mult}_{\mathbb{R},s}(F)$ is undetermined, see Section 5.2.

(7b, 9f) The left string in C_2 has to align with a horizontal bounded edge of C_3 by Definition 2.5 (4). There are 3 possibilities. If it aligns with the bounded edge adjacent to the right string in the cubic, we obtain a prism with two pyramids as in (7b, 9a). See Section 5.1. If it aligns with either of the other two horizontal bounded edges, we obtain two bipyramids in the dual subdivision. Because the diagonal bounded edge of C_2 is part of the left sting aligning with a horizontal bounded end not adjacent to the right string of C_3, we cannot align the right string with the diagonal edge, such that the end of the right string contains the whole horizontal bounded edge of C_2. Instead the end meets the bounded edge somewhere in the middle and passes only through one vertex. Therefore, in the subdivision the second pyramid over the alignment parallelogram must have its vertex in C_3 instead of in the C_2, see Figure 10. In total, we get two bipyramids that only share an edge, so the node germs are separated. We have $2 \cdot \text{mult}_{\mathbb{C}}(F) = 2 \cdot \text{mult}_{\mathbb{C}}(C_3^*) \cdot \text{mult}_{\mathbb{C}}(C_2^*) = 2 \cdot (2 \cdot 2) = 8$. In these two cases, the edge the string aligns with has x-coordinate $k = 1$ in the cubic floor and thus by Definition 2.8 they both give $\text{mult}_{\mathbb{R},s}(F) = 0$.

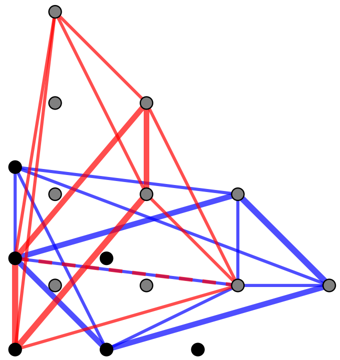


Figure 10: The two bipyramids for one alignment of (7b, 9f). The gray (resp. black) dots are the lattice points of the dual polytope to C_3 (resp. C_2). The shared edge of the bipyramids is marked blue and red.

(7c, 9a-9b) We obtain a bipyramid only overlapping with the configuration of the weight two end in vertices or edges. So the nodes are separated and $2 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2(2(3-1) \cdot 2) = 2 \cdot 8$. The parallelogram has vertices as in the first case of Definition 2.8 (1) with $k = 1, l = 1$ and $i_j = 2$, so $\mathrm{mult}_{\mathbb{R},s}(F) = 0$.

(7c, 9c) As in (7c, 9a) we have $\mathrm{mult}_{\mathbb{C}}(F) = 8$. For the real multiplicity we need the vertices of the parallelogram. They are as in the first case of Definition 2.8 (1) with $k = 1, l = 0$ and $i_j = 2$, so $\mathrm{mult}_{\mathbb{R},s}(C_2^*) = 2$. The weight 2 end in C_3 has $\mathrm{mult}_{\mathbb{R},s}(C_3^*) = 4$, so $\mathrm{mult}_{\mathbb{R},s}(F) = 8$.

(7c-7d, 9d) This subdivision contains a tetrahedron which is the convex hull of both weight two ends. We also need a choice of the neighboring points of the two weight two edges. By their special position to each other, it only remains to add the two vertices neighboring the edges in the respective subdivisions dual to their floors. Whether it can contain 2 nodes is so far undetermined, see Section 5.1.

(7c-7d, 9e) The nodes are separated, since the weight two ends with any choice of their neighboring points intersect in one vertex. So $2 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot \mathrm{mult}_{\mathbb{C}}(C_3^*) \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 2 \cdot (2(3-1) \cdot 2(2-1)) = 2 \cdot 8$ and $2 \cdot \mathrm{mult}_{\mathbb{R},s}(F) = 2 \cdot \mathrm{mult}_{\mathbb{R},s}(C_3^*) \cdot \mathrm{mult}_{\mathbb{R},s}(C_2^*) = 2 \cdot (2(3-1) \cdot 2(2-1)) = 2 \cdot 8$.

(7c, 9f) There are two possibilities to align the left string in C_2 with a horizontal bounded edge in C_3. If we select the left edge, we have a bipyramid, which does not contain the weight two end. By Lemma 3.1 the nodes are separate. However, we need to adjust the multiplicity formula from Definition 2.7 (3) to this case, because due to the alignment of the left string we obtain one intersection point less of the diagonal end of weight two with C_2. So instead of $3-1=2$ intersection points to chose from when lifting the node we have $3-2=1$. Thus, we obtain $\text{mult}_{\mathbb{C}}(F)=\text{mult}_{\mathbb{C}}(C_3^*)\cdot\text{mult}_{\mathbb{C}}(C_2^*)=2(3-2)\cdot 2=4$. Since the left edge has x-coordinate $k=1$, we obtain $\text{mult}_{\mathbb{R},s}(F)=0$. If we select the right edge, then the bipyramid contains the weight two end. See Section 5.1.

As the cubic floor contains a vertex of C_3 not adjacent to a horizontal edge, it is also possible to align the left string with this. In the dual subdivision this gives rise to a pentatope spanned by the triangle dual to the vertex in C_3 and the vertical edge in the conic floor dual to the horizontal end of the left string, see Figure 3a. The nodes dual to the length two edge and the pentatope are separated. By [9] we have $\text{mult}_{\mathbb{C}}(C_2^*)=\text{mult}_{\mathbb{R},s}(C_2^*)=1$. We count: $\text{mult}_{\mathbb{C}}(F)=\text{mult}_{\mathbb{C}}(C_3^*)\cdot\text{mult}_{\mathbb{C}}(C_2^*)=2(3-2)\cdot 1=2$ and $\text{mult}_{\mathbb{R},s}(F)=\text{mult}_{\mathbb{R},s}(C_3^*)\cdot\text{mult}_{\mathbb{R},s}(C_2^*)=2(3-2)\cdot 1=2$.

(7d, 9a-9b) We obtain a bipyramid overlapping with the weight two configuration in one or two vertices, so the nodes are separated and $2\cdot\text{mult}_{\mathbb{C}}(F)=2\cdot\text{mult}_{\mathbb{C}}(C_3^*)\cdot\text{mult}_{\mathbb{C}}(C_2^*)=2(2(3-1)\cdot 2(2-1))=2\cdot 8$. With the same parallelogram as in (7c, 9a): $\text{mult}_{\mathbb{R},s}(F)=0$.

(7d, 9c) This follows (7d, 9a), and we have $\text{mult}_{\mathbb{C}}(F)=8$. The real multiplicity follows (7c, 9c), and we have $\text{mult}_{\mathbb{R},s}(F)=8$.

(7d, 9f) For each of the two choices for the alignment of the left string of the conic with a horizontal bounded edge of the cubic, we obtain a bipyramid which may share two vertices with the neighbors of the edge of weight two. As in (7c, 9f) we need to adjust the multiplicity formula for the weight two end to $\text{mult}_{\mathbb{C}}(C_3^*)=2(3-2)=2$. We have $2\cdot\text{mult}_{\mathbb{C}}(F)=2\cdot 4$. For both alignments the dual edges have x-coordinate $k=1$ in the cubic floor, giving $\text{mult}_{\mathbb{R},s}(F)=0$. As C_3 also contains a vertex not adjacent to a horizontal edge, this opens a third alignment possibility. However, this vertex is adjacent to the weight two, so the nodes are not separated. The polytope complex can be seen in Figure 14.

□

4. Nodes in the same floor

We now examine cases where both node germs are in the same floor of the floor plan. By Lemma 3.2 we cannot have a right string in the conic part of the floor plan, if the nodes are separated. A few more cases, depicted in Figure 11, can be eliminated with the following Lemma 4.1.

Lemma 4.1. *The ways of omitting 2 points in the floor path in the conic floor shown in Figure 11 do not give separated nodes.*

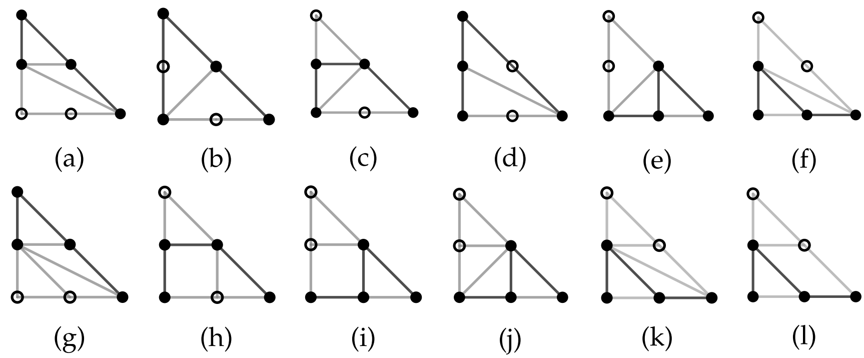


Figure 11: Conics through 3 points eliminated by Lemma 4.1.

Proof. If the conic in a floor plan has two node germs, it passes only through 3 points of the point configuration. In order to fix our cubic surface, every point we omit in the lattice path of the conic floor needs to compensate for the omitted point condition on our cubic surface.

A vertical weight two end does allow our conic to be fixed by fewer points. But our point configuration ensures the end has no interaction with the other floors and thus cannot give rise to a node-encoding circuit as in Figure 3. So, combined with a classical node germ this does not encode two separated nodes, dealing with 11a, 11b, 11c and 11d.

If the top vertex of the Newton polytope of C_2 is omitted in the floor path, we always obtain an upwards string. If the upwards string is to be pulled vertically upwards, it can never be aligned with any part of the other floors, thus not fixing the curve, eliminating 11h, 11j and 11l.

If the direction to pull the upwards string has some slope, as in 11e and 11f, or in the 2-dimensional strings in 11i and 11k, we still cannot align with any bounded edges of the other cubic, since we are above the line through the points due to our chosen point configuration. In 11g on the other hand we can align the vertical end of the string, but since we have two degrees of freedom this does not fix the curve, as we can still move the first vertical end. □

Remark 4.2. The last issue in the proof of Lemma 4.1 can be fixed, if we allow alignments with ends. These however do not give rise to separated nodes [9]. Therefore the cases 11a, 11e, 11f, 11g, 11i and 11k require further investigation, see Section 5.1. In this light the non-existence of right strings in the conic floor needs to be investigated.

Proposition 4.3. *There are 72 cubic surfaces containing two nodes, such that the tropical binodal conic has separated nodes and the corresponding node germs are both contained in the conic floor. Of these, at least 4 are real.*

Proof. See Figure 12.

(12a) Since the end of the left string, which aligns with a bounded horizontal edge of the conic, is of weight two, we obtain a bipyramid over a trapezoid. We get two different complexes depending upon the alignment, see Section 5.1.

(12b) We have a string with two degrees of freedom, because we can pull on both horizontal ends and vary their distance. Hence, we can align them both with the horizontal bounded edges of the cubic. There are three ways to do this. In the dual subdivisions this gives rise to two bipyramids. In all three cases they intersect maximally in two 2-dimensional unimodular faces, and thus are separated. Since the bipyramids arise not from classical node germs, we check their multiplicities via the underlying circuit. By [9, Lemma 4.8] we obtain multiplicity 2 for each, and thus $3 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 3 \cdot \mathrm{mult}_{\mathbb{C}}(C_2^*) = 3 \cdot 2 \cdot 2 = 12$. We get $\mathrm{mult}_{\mathbb{R},s}(F) = 0$, since one end has to align with a bounded edge in C_3 with dual edge of x-coordinate $k = 1$.

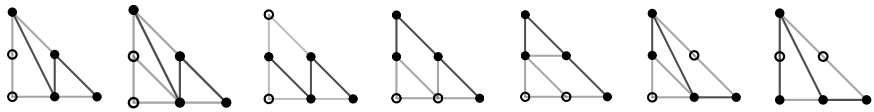
(12c-12d) The conic floor has a left string and a parallelogram. This gives two bipyramids in the subdivision which, depending on the choice of alignment for the left string, have a maximal intersection of an edge. We obtain $2 \cdot (3 \cdot \mathrm{mult}_{\mathbb{C}}(F)) = 2 \cdot 12$. The vertex positions of the parallelogram give $\mathrm{mult}_{\mathbb{R},s}(F) = 0$ as in Proposition 3.4 (9a).

(12e) As in (12c), we obtain $3 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 12$. The formulas for real multiplicities in Definition 2.8 do not match this case, see Section 5.2.

(12f) The bipyramids arising from the different alignment options only intersect with the neighboring points of the weight two end in one vertex, so $3 \cdot \mathrm{mult}_{\mathbb{C}}(F) = 12$. Only the alignment with the horizontal bounded edge of C_3 dual to the vertical edge of x-coordinate $k = 2$ has non-zero real multiplicity, giving $\mathrm{mult}_{\mathbb{R},s}(F) = 4$.

(12g) The two sets of neighboring points to the two weight two ends intersect in one vertex. So the nodes are separated and $\mathrm{mult}_{\mathbb{C}}(F) = 6 \cdot 2 = 12$, while $\mathrm{mult}_{\mathbb{R},s}(F)$ is undetermined, see Section 5.2.

□



(a) (b) (c) (d) (e) (f) (g)

Figure 12: Dual subdivisions of conics with two node germs.

Proposition 4.4. *There are 8 cubic surfaces containing two nodes, such that the tropical binodal cubic surface has separated nodes and the corresponding node germs are both contained in the cubic floor.*

So far the number of real surfaces is undetermined.

Proof. Only two types of node germs may occur in C_3, see Figure 13.

(13a) Since the weight two end is not contained in the bipyramid the two nodes are separated by Lemma 3.1, giving $\mathrm{mult}_{\mathbb{C}}(F) = 2 \cdot 4 = 8$. In this case, $\mathrm{mult}_{\mathbb{R},s}(F)$ is undetermined, see Section 5.2.

(13b) The classical alignment condition of the right string with diagonal end of weight two can not be satisfied, since the direction vector of the variable edge has a too high slope. Due to the point conditions the diagonal end of weight two and the diagonal bounded edge of the conic curve never meet.

(13c) Here we have a two-dimensional string. By the same argument as in (13b) we cannot align the middle diagonal end with the diagonal bounded edge of the conic. Aligning the right string with the diagonal bounded edge of the conic does not fixate our floor plan, since we can still move the middle diagonal end of the cubic.

(13d) We have three tetrahedra in the subdivision containing the weight three edge. This could contain two nodes, see Section 5.1.

In (13b), (13c) alignments with ends are an option, see Section 5.1. □

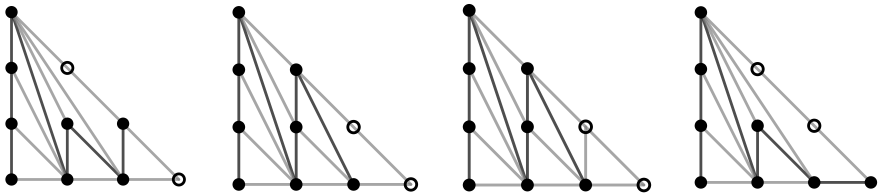


Figure 13: Cubics with two node germs.

5. Next steps

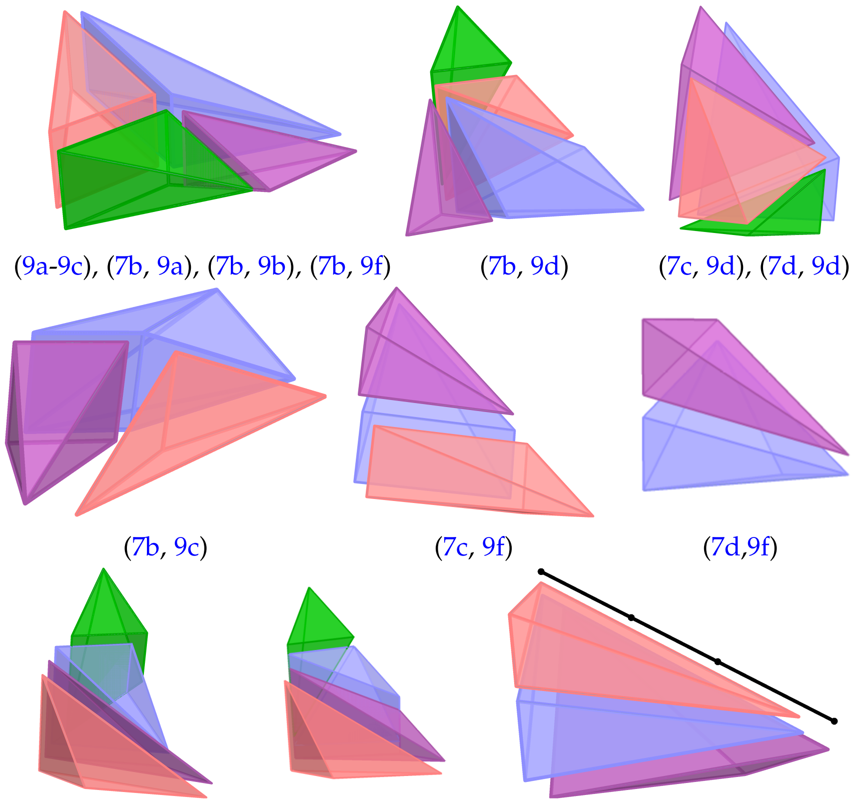
5.1. Dual complexes of unseparated nodes

In previous sections, we encountered cases where two distinct node germs did not give rise to separated nodes. The dual complexes arising from these cases are shown in Figure 14.

We also encountered the floors which do not give separated nodes in Figure 11 and in the proof of Proposition 4.4. By new alignment conditions, they might encode unseparated nodes, see Remark 4.2. Alignment with ends is not allowed for separated nodes, because circuit D (Figure 3b) is then contained in the boundary of the Newton polytope and cannot encode a single node [9]. However, with one point condition less than for one-nodal surfaces, we can obtain strings with one degree of freedom more and this makes not only the alignment of two ends possible, but additionally the alignment of the vertices the ends are adjacent to. This leads to a triangular prism shape in the subdivision, which has at least one parallelogram shaped face in the interior of the Newton polytope. At this time, we do not yet know whether any of these cases can contain two nodes or with what multiplicity they should be counted with, but in total they ought to give the 66 missing surfaces from our count.

5.2. Undetermined real multiplicities

In the previous sections, we encountered cases in which the real multiplicity was undefined. This happens when $C^*_{i_j}$ is a horizontal edge of weight two ((9d) and (12g)), and $C^*_{i_j}$ is a right string whose diagonal end aligns with a diagonal bounded edge ((7b), (7b, 9e), (12e), and (13a)). There might be real lifts satisfying the point conditions coming from floor plans containing these node germs, but the number of real



(9a-9c), (7b, 9a), (7b, 9b), (7b, 9f) (7b, 9d) (7c, 9d), (7d, 9d)

(7b, 9c) (7c, 9f) (7d,9f)

(12a) (12a) (13d)

Figure 14: Complexes whose duals could have to two nodes.

solutions is hard to control. An investigation of these cases is beyond the scope of this paper, so we leave Theorem 1.4 as a lower bound under these assumptions.

We may compute the real multiplicity of (12e), as well as of right strings aligning with diagonal bounded edges as follows. Shift the parallelogram to a special position used to prove [9, Lemma 4.8]. The equations of the proof of [9, Lemma 4.8] applied to our exact example then need to be checked for the existence of real solutions.

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