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CUBIC SURFACES ON THE SINGULAR LOCUS OF THE ECKARDT HYPERSURFACE

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The Eckardt hypersurface in $\mathbb{P}^{19}$ is the closure of the locus of smooth cubic surfaces with an Eckardt point, which is a point common to three of the 27 lines on a smooth cubic surface. We describe the cubic surfaces lying on the singular locus of the model of this hypersurface in $\mathbb{P}^4$, obtained via restriction to the space of cubic surfaces possessing a so-called Sylvester form. We prove that, inside the moduli of cubics, the singular locus corresponds to a reducible surface with two rational irreducible components intersecting along two rational curves. The two curves intersect at two points representing the Clebsch and the Fermat cubic surfaces. We observe that the cubic surfaces parameterized by the two components or the two rational curves are distinguished by the number of Eckardt points and automorphism groups.

Introduction

The moduli of cubic surfaces $\mathcal{M}_{cub}$ is defined as the geometric invariant quotient of the space of quaternary cubics $\mathbb{P}^{19} \cong \mathbb{P}(\mathrm{H}^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)))$ by the induced action of the special linear group $\mathrm{SL}(4)$. The classical description of this space is due to G. Salmon [11] and A. Clebsch [1]. They proved $\mathcal{M}_{cub}$ is isomorphic to the weighted projective space $\mathbb{P}(1,2,3,4,5)$ by showing that the corresponding

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graded ring of invariants is generated by six homogeneous invariant polynomials $\bar{I}_n$ of degrees $n = 8, 16, 24, 32, 40, 100$. The first five polynomials are algebraically independent, however $\bar{I}_{100}^2$ can be expressed as a polynomial in terms of the other invariants. In fact, they described a birational model of $\mathcal{M}_{cub}$ as the quotient space of $\mathbb{P}^4$, the parameter space of cubic surfaces with a so-called Sylvester form, by the action of the Symmetric group $\mathcal{S}_5$. Under this birational equivalence, each SL(4)-invariant polynomial $\bar{I}_n$ can be regarded as continuation of an $\mathcal{S}_5$-invariant polynomial I_n of the same degree in the coordinate ring of $\mathbb{P}^4$.

One can use these invariants to describe further interesting subspaces of the space of cubic surfaces. By classical results ([12], App. III), the vanishing of $\bar{I}_{32}$ is a necessary and sufficient condition for a cubic surface to be singular. In this line, the famous discriminant hypersurface $V(\bar{I}_{32})$ parameterizes singular cubic surfaces generically having a node. Turning to smooth cubic surfaces equipped with 27 lines, a point common to three lines is called an Eckardt point. The Salmon invariant $\bar{I}_{100}$ vanishes on the closure of the locus of smooth cubic surfaces with an Eckardt point. We call $V(\bar{I}_{100}) \subset \mathbb{P}^{19}$ the Eckardt hypersurface.

Let $E = V(I_{100})$ be the model of the Eckardt hypersurface in $\mathbb{P}^4$, parameterizing the cubic surfaces in Sylvester form having an Eckardt point. By abuse of language, we may refer to E as Eckardt hypersurface as well. Motivated by question 13 in [9], the main contribution of this paper is to study the cubic surfaces determined by the singular locus of this hypersurface. We prove (Theorem 2.1), up to linear change of coordinates in $\mathbb{P}^3$, the singular locus determines two 2-dimensional rational families of cubic surfaces intersecting along two rational curves. The two curves intersect at two points which correspond to the Clebsch and the Fermat cubic surfaces possessing respectively 10 and 18 Eckardt points. The generic elements of the two families are smooth cubic surfaces with respectively 2 and 3 Eckardt points. Moreover, the two rational curves parameterize the cubic surfaces with respectively 4 and 6 Eckardt points. The difference in number of the Eckardt points implies further difference in the automorphism group of the cubic surfaces parameterized in different families.

The paper is structured as follows. In the first section, we recall the construction of the moduli of cubic surfaces as a weighted projective space. Section 2 deals with the general description of the singular locus of the Eckardt hypersurface E corresponding to a reducible surface with two components inside the moduli of cubics. In Section 3, we investigate the geometric features and the differences of the cubic surfaces lying on the two components and the two rational curves. Our results rely on the computations done by the computer algebra system *Macaulay2* [6], and uses the supporting functions in [7].

1. Preliminaries

In this section we briefly review the construction of the moduli spaces of cubic surfaces using the Sylvester forms and the Salmon invariants.

Let $\mathbb{P}^{19} = \mathbb{P}(V)$ be the parameter space of cubic forms $V = \mathbb{C}[x_0, \dots, x_3]_{(3)}$ in four variables. One considers the induced action of $G := \mathrm{SL}(4)$ on $\mathbb{P}^{19}$, from the standard action on $\mathbb{P}^3$. The geometric invariant quotient

$$\mathcal{M}_{cub} := \mathbb{P}^{19} /\!\!/ G$$

is called the moduli space of cubic surfaces. By the following result, $\mathcal{M}_{cub}$ is a projective variety.

Proposition 1.1. *There is an isomorphism $\mathcal{M}_{cub} \cong \mathrm{Proj}(R^G)$, where R is the coordinate ring of $\mathbb{P}^{19}$ and R^G is the ring of invariants.*

Proof. See [3], Proposition 8.1. □

The computation from classical invariant theory due to Salmon [11] and Clebsch [1] shows that the graded ring of invariants is generated by homogeneous polynomials $\bar{I}_n$ of degrees

$$n = 8, 16, 24, 32, 40, 100$$

such that the first five invariants are algebraically independent. Since 100 is not divisible by 8 and there is a relation expressing $\bar{I}_{100}^2$ as a polynomial in terms of the remaining invariants, the graded subalgebra generated by elements of degree divisible by 8 is freely generated by the first five invariants, and $\mathcal{M}_{cub}$ has structure of the weighted projective space

$$\mathcal{M}_{cub} \cong \mathrm{Proj}(R^G) \cong \mathrm{Proj}(\bigoplus_{k \in \mathbb{Z}} R^G_{8k}) \cong \mathbb{P}(1,2,3,4,5).$$

See [10] for a modern proof of this isomorphism. One can restrict the invariants to an open subset of cubic surfaces with somewhat easier form, that is the set of the cubic surfaces with Sylvester forms. This then would allow to express the invariants in terms of symmetric functions of the coefficients of the Sylvester representation.

Theorem 1.2. *A general cubic surface is projectively isomorphic to a surface in $\mathbb{P}^4$ given by equations*

$$a_0 z_0^3 + a_1 z_1^3 + a_2 z_2^3 + a_3 z_3^3 + a_4 z_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0. \quad (*)$$

The coefficients $a_0, \dots, a_4$ are determined uniquely up to permutation and a common scaling.

Proof. See [4], Corollary 9.4.2. □

Definition 1.3. The equations $(*)$ in above theorem is called a Sylvester form of a cubic surface. The Sylvester form is non-degenerate if $a_i \neq 0$ for all $i = 1,\dots,4$. Otherwise, it is called degenerate.

Let σ_i be the elementary symmetric polynomial of degree i in $a_0,\dots,a_4$. Then, Salmon's computations ([11], p.197) provide the following easy formulation of the invariants for cubic surfaces possessing a Sylvester representation:

$$I_8 = \sigma_4^2 - 4\sigma_3\sigma_5,\ I_{16} = \sigma_1\sigma_5^3,\ I_{24} = \sigma_4\sigma_5^4,\ I_{32} = \sigma_2\sigma_5^6,\ I_{40} = \sigma_5^8, \quad (\star)$$

and

$$I_{100} = \sigma_5^{18}.\det \begin{pmatrix} 1 & a_0 & a_0^2 & a_0^3 & a_0^4 \\ \vdots & & \ddots & & \vdots \\ 1 & a_4 & a_4^2 & a_4^3 & a_4^4 \end{pmatrix}.$$

One can use the Algorithm 3.1 in [8] to compute the coefficients a_i's and to evaluate these invariants for a general quaternary cubic. Moreover, one can consider the divisors of the zero sets of the invariants inside the moduli of cubics [2]. The invariant $\bar{I}_{32}$ defines the boundary divisor in $\mathcal{M}_{cub}$ as the locus of singular cubic surfaces. The invariant $\bar{I}_{40}$ restricts to $(a_0a_1a_2a_3a_4)^8$, and vanishes on the closure of the locus of smooth cubic surfaces with a degenerate Sylvester form. The invariant $\bar{I}_{100}$ vanishes on the closure of the locus of smooth cubic surfaces with an Eckardt point. This is a point common to three of the 27 lines on a smooth cubic surface. In particular, a general cubic surface with such a point has a Sylvester form in which at least two of the a_i coincide.

We observe that in the above formulation $(\star)$, the G-invariant polynomials can be viewed as invariants under the action of the Symmetric group $\mathcal{S}_5$. Furthermore, let $\mathbb{P}^4 /\!\!/ \mathcal{S}_5$ be the quotient of the parameter space $\mathbb{P}^4$ of Sylvester forms by the action of the symmetric group $\mathcal{S}_5$. This quotient space is isomorphic to the weighted projective space $\mathbb{P}(1,2,3,4,5)_\sigma$ equipped with natural coordinates $\sigma_1,\dots,\sigma_5$. Following [2], the above formulas defines a birational map

$$\begin{aligned} \mathbb{P}^4 /\!\!/ \mathcal{S}_5 \cong \mathbb{P}(1,2,3,4,5)_\sigma & \overset{\Phi}{\dashrightarrow} \mathcal{M}_{cub} \cong \mathbb{P}(1,2,3,4,5)_I \\ (\sigma_1 : \sigma_2 : \sigma_3 : \sigma_4 : \sigma_5) & \longmapsto (I_8 : I_{16} : I_{24} : I_{32} : I_{40}). \end{aligned}$$

with base locus $V(\sigma_4,\sigma_5)$. Note that considering the symmetric polynomials as polynomials in coordinates a_i's, the base locus $V(\sigma_4,\sigma_5) \subset \mathbb{P}^4$ is the union of the loci $V(a_i,a_j)$ for all $i \neq j$. On the other hand , we have

$$\sigma_1 = \frac{I_{16}}{\sigma_5^3}, \quad \sigma_2 = \frac{I_{32}}{\sigma_5^6}, \quad \sigma_3 = \frac{I_{24}^2 - I_8I_{40}}{4\sigma_5^9}, \quad \sigma_4 = \frac{I_{24}I_{40}}{\sigma_5^{12}}, \quad \sigma_5 = \frac{I_{40}^2}{\sigma_5^{15}} \quad (**)$$

which defines the birational inverse map

$$(I_8 : I_{16} : I_{24} : I_{32} : I_{40}) \longmapsto \left(I_{16} : I_{32} : I_{24}^2 - I_8 I_{40} : I_{24} I_{40} : I_{40}^2\right).$$

Considering the coordinates I_{8d}'s as polynomials in terms of a_i's and formulas in $(**)$ shows that this map is not defined at the points for which $\sigma_5 = 0$. This is the point $Q = (1 : 0 : 0 : 0 : 0)$. It is shown in [2], Theorem 6.6, that inside the moduli of cubics, Q stands for the Fermat cubic surface given by

$$S_f : \quad (z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0) \subset \mathbb{P}^3.$$

2. The singular locus of the Eckardt hypersurface

A point where three lines on a smooth cubic surface intersect is called an Eckardt point. In this case, the three lines are cut out by the intersection of the cubic surface with the tangent plane at this point. The Eckardt hypersurface $V(\bar{I}_{100}) \subset \mathbb{P}^{19}$ is the closure of the locus of smooth cubic surfaces with an Eckardt point. Let $E := V(I_{100}) \subset \mathbb{P}^4$ be the model of the Eckardt hypersurface inside the space of cubic surfaces possessing a Sylvester form. The following theorem describes a general cubic surface (up to linear change of coordinates in $\mathbb{P}^3$) lying on the singular locus $\Gamma \subset E$ of this hypersurface. More precisely, let

$$\begin{aligned} \Psi : \mathbb{P}^4 &\longrightarrow \mathbb{P}(1,2,3,4,5)_\sigma \\ (a_0 : \cdots : a_4) &\mapsto (\sigma_1 : \sigma_2 : \sigma_3 : \sigma_4 : \sigma_5) \end{aligned}$$

be the quotient map from $\mathbb{P}^4$, and set $\Delta := \overline{\Phi(\Psi(\Gamma) \setminus [V(\sigma_4, \sigma_5) \cap \Psi(\Gamma)])}$, then we have:

Theorem 2.1. *With the above notation, Δ is the union of two rational irreducible surfaces $S_{[2,2,1]}$ and $S_{[3,1,1]}$, as the two irreducible components of Δ. A general point of each component is a smooth cubic surface with the Sylvester form as follows, respectively:*

$$S_{[2,2,1]} : \quad az_0^3 + bz_1^3 + bz_2^3 + cz_3^3 + cz_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0, \quad a, b, c \in \mathbb{C}$$

$$S_{[3,1,1]} : \quad az_0^3 + bz_1^3 + bz_2^3 + bz_3^3 + cz_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0, \quad a, b, c \in \mathbb{C}.$$

The two components intersect along two rational curves, and the two curves in turn intersect at two points which represent the Clebsch and the Fermat cubic surfaces.

Remark 2.2. We remark that, as it is suggested by naming of the two surfaces, and in shadow of Theorem 1.2, one can consider other possible permutations of the three coefficients a, b, c in Sylvester presentation of a general element on the two components.

Proof. An explicit computation (see `VerifyAssertion1`, [7]) of the singular locus Γ and its decomposition into irreducible components shows that Γ has 30 irreducible linear components as follows:

- 5 irreducible components corresponding to the coordinate hyperplanes

$$V(a_i) \subset \mathbb{P}^4, \quad i = 0, \ldots, 4.$$

Therefore, under the map Ψ, the five components parameterizing the cubic surfaces with a degenerate Sylvester form are mapped to the hypersurface $V(\sigma_5)$ which is contacted to the point Q via the map Φ. Therefore, the five components correspond to the Fermat cubic surface.

- 15 irreducible components as copies of $\mathbb{P}^2$ given by

$$V_{ijkl} = V(a_i - a_j, a_k - a_l) \subset \mathbb{P}^4, \quad i \neq j \neq k \neq l$$

for (i, j, k, l) among

$$\begin{array}{ccccc} (2,3,1,4), & (1,4,0,2), & (1,4,0,3), & (2,4,1,3), & (1,3,0,2) \\ (1,3,0,4), & (3,4,1,2), & (3,4,0,1), & (3,4,0,2), & (2,3,0,1) \\ (2,3,0,4), & (1,2,0,3), & (1,2,0,4), & (2,4,0,1), & (2,4,0,3) \end{array}$$

Since for a pair of 4-tuples $(i, j, k, l), (i', j', k', l')$, one can map the corresponding components $V_{ijkl}, V_{i'j'k'l'}$, one to another, by a permutation of the coordinates in $\mathbb{P}^4$, under the map Ψ, the two components, and therefore all the components V_{ijkl}'s are mapped to a rational surface $S_1 \subset \mathbb{P}(1,2,3,4,5)_\sigma$, whose general point is a cubic surface with Sylvester form:

$$az_0^3 + bz_1^3 + bz_2^3 + cz_3^3 + cz_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0, \quad a, b, c \in \mathbb{C}.$$

In particular, under the birational map Φ, the surface S_1 has a birational model $S_{[2,2,1]} \subset \mathcal{M}_{cub}$ whose general point is a smooth cubic surface with Sylvester form as above. In fact,

$$S_{[2,2,1]} = \overline{\Phi(S_1 \setminus [V(\sigma_4, \sigma_5) \cap S_1])}.$$

More precisely, choosing the coordinate $(a : b : c)$ for $\mathbb{P}^2$, as the representative of the components V_{ijkl}, we have

$$\mathbb{P}^2 \setminus V(abc) \longrightarrow S_1 \setminus [V(\sigma_5) \cap S_1] \longrightarrow S_{[2,2,1]} \setminus \{Q\}.$$

- 10 irreducible components as copies of $\mathbb{P}^2$ given by

$$V_{ijk} = V(a_i - a_j,\ a_k - a_j) \subset \mathbb{P}^4, \quad i \neq j \neq k$$

for (i,j,k) among the triples

$$\begin{gathered}(2,4,1),\ (3,4,2),\ (3,4,1),\ (3,4,0),\ (1,4,0)\\ (2,3,0),\ (2,3,1),\ (1,2,0),\ (1,3,0),\ (2,4,0)\end{gathered}$$

With the same argument as above, for two choices of (i,j,k), the points on the corresponding components differ only by a coordinate permutation of $\mathbb{P}^4$, and hence they correspond to an another rational surface $S_{[3,1,1]} \subset \mathcal{M}_{cub}$ whose general point is a cubic surface of type:

$$az_0^3 + bz_1^3 + bz_2^3 + bz_3^3 + cz_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0, \quad a,b,c \in \mathbb{C}.$$

The two surfaces $S_{[2,2,1]}$ and $S_{[3,1,1]}$ inside $\mathcal{M}_{cub}$ intersect along two rational curves $C_{[3,2]}$ and $C_{[4,1]}$, arised by the images of the two lines $\ell_1 : \mathbb{P}^1 \cong V(a-c) \subset \mathbb{P}^2$ and $\ell_2 : \mathbb{P}^1 \cong V(b-c) \subset \mathbb{P}^2$, respectively. Therefore, they parameterize the cubic surfaces with the Sylvester forms as follows:

$$C_{[3,2]}: \quad az_0^3 + bz_1^3 + bz_2^3 + bz_3^3 + az_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0, \quad a,b \in \mathbb{C}.$$

$$C_{[4,1]}: \quad az_0^3 + bz_1^3 + bz_2^3 + bz_3^3 + bz_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0, \quad a,b \in \mathbb{C}$$

In particular, the two curves intersect at two points, Q which comes by the choice of one of the coordinates a,b,c to be zero, and the point arised by $a = b = c$ which represents the Clebsch cubic surface given by the following equations:

$$S_c: \quad z_0^3 + z_1^3 + z_2^3 + z_3^3 + z_4^3 = 0, \quad z_0 + z_1 + z_2 + z_3 + z_4 = 0.$$

□

3. How different are the two components?

In this section, we study the geometric nature and the differences of the cubic surfaces lying on the two components or the two curves. Look at the table 1 for a quick overview of the data of a general point on $S_{[2,2,1]}, S_{[3,1,1]}, C_{[3,2]}$ and $C_{[4,1]}$, as well as that of the Clebsch and the Fermat cubic surfaces. In the rows "no. Epts" and "Aut", we have marked respectively the number of the Eckardt points and the automorphism group.

As the first step of our study, an explicit examination demonstrates that the two components stand for two different types of singularities of E. In fact, identifying each of the components with its possible birational model as an irreducible component of Γ we have:

Theorem 3.1. *A general point of $S_{[2,2,1]}$ (resp. $S_{[3,1,1]}$) corresponds to an ordinary double (resp. triple) point on E. Moreover, a general point of each of the two rational curves corresponds to an ordinary triple point on E.*

Proof. See (`VerifyAssertion2`, [7]) for the explicit computation. □

The nicely prescribed Sylvester forms of the cubic surfaces on the two components and the curves reveal the number of Eckardt points and their arrangements. More precisely, let $\pi \subset \mathbb{P}^3$ be the pentahedron with faces

$$\pi_i := (z_i = 0), \quad i = 0, \dots, 3$$
$$\pi_4 := -(z_0 + z_1 + z_2 + z_3)$$

where $z_0, \dots, z_3$ are considered as the coordinates of $\mathbb{P}^3$. The two faces π_i, π_j intersect along the edge $e_{ij} := (\pi_i = \pi_j = 0)$, and the vertex A_{ij} is the intersection point of the three faces with indices different from i, j. This is the so-called Sylvester pentahedron associated to a cubic surface with Sylvester representation as in $(*)$. We remark that, as followed from the Theorem 1.2, for a general cubic surface given by a homogeneous cubic form F, there are five linear forms ℓ_i's in four variables of $\mathbb{P}^3$ such that any four of them are linearly independent and such that

$$\sum_{i=0}^{4} \ell_i = 0, \quad F = \sum_{i=0}^{4} a_i \ell_i^3.$$

Therefore, and more generally, one can consider the Sylvester pentahedron of F as the pentahedron associated to these five linear forms, similarly. Note that the five linear forms are uniquely determined by F (up to permutations and a common non-zero scaling). See ([12], page 125-137) for more details on this.

Let π be the pentahedron defined above, and for a cubic surface with Sylvester representation as $(*)$, consider its equations in $\mathbb{P}^3$ obtained by substituting z_4 with $-(z_0 + z_1 + z_2 + z_3)$ in $(*)$. We choose the remaining coordinates $z_0, \dots, z_3$ as the coordinates of $\mathbb{P}^3$. By classical results ([12], page 148), one can see that, with this notation, a general cubic surface S lying on

- $S_{[2,2,1]}$ has 2 Eckardt points A_{12}, A_{34} such that the joining line $(z_1 + z_2 = z_0 = 0)$ is contained in S.
- $S_{[3,1,1]}$ has 3 Eckardt points given by the vertices A_{12}, A_{23}, A_{13}, which are collinear and the common line $(z_0 = z_1 + z_2 + z_3 = 0)$ is not contained in the surface S.
- $C_{[3,2]}$ has 4 Eckardt points A_{12}, A_{23}, A_{13} which are collinear and the point A_{04}, which is joined to the former points by the three lines (through it) cut out by the tangent hyperplane at this point.

- $C_{[4,1]}$ has 6 Eckardt points. The 6 points are the 6 vertices of the quadrilateral intersected on π_0 by four other faces of the pentahedron, that is $A_{12}, A_{23}, A_{13}, A_{14}, A_{24}, A_{34}$. Naturally, in this case the hyperplane π_0 cuts out a cubic curve on the surface passing through the 6 points.

In particular, the Clebsch cubic surface has 10 Eckardt points as the 10 vertices of the pentahedron π. The Fermat cubic surface possesses 18 Eckardt points.

The diversity in number of Eckardt points causes further difference of cubic surfaces in terms of the automorphism group. An automorphism of the projective space $\mathbb{P}^3$ fixing a hyperplane and a point is called a homology. The single point is called the center of homology. In terminology of classical projective geometry, a homology of order 2 is usually referred to as an involution. There is a one-to-one correspondence between the set of Eckardt points of a smooth cubic surface and the set of involutions of $\mathbb{P}^3$ keeping the surface invariant ([5], Theorem 9.2).

To avoid iteration, let S denote the cubic surface corresponding to a general point of one of the two surfaces or the two curves. By classification of the possible groups of automorphisms of a smooth cubic surface [5], a general cubic lying on $S_{[2,2,1]}$ has automophism group generated by the two involutions associated to the two Eckardt points and $\mathrm{Aut}(S) \cong (\mathbb{Z}_2)^2$. On the other hand, the automorphism group of a general cubic surface lying on $S_{[3,1,1]}$ is generated by involutions permuting the three Eckardt points and keeping the common line invariant, that is $\mathrm{Aut}(S) = \mathcal{S}_3$. For a general cubic surface on the curves $C_{[3,2]}$ and $C_{[4,1]}$, one has $\mathrm{Aut}(S) \cong \mathcal{S}_3 \times \mathcal{S}_2$ and $\mathrm{Aut}(S) \cong \mathcal{S}_4$, respectively. The Clebsch cubic surface has the automorphism group $\mathrm{Aut}(S_c) \cong \mathcal{S}_5$ acting by permutations of the coordinates in $\mathbb{P}^4$. Up to isomorphism, the Clebsch surface is the only cubic surface with this automorphism group. The automorphism group of the Fermat cubic surface is isomorphic to $H \rtimes \mathcal{S}_4$ where the subgroup H acts by multiplying the coordinates by a primitive third root of unity and $\mathcal{S}_4$ acts by permuting the coordinates in $\mathbb{P}^3$. For more on automorphism group of cubic surfaces classified in any characteristic, we refer the reader to the recent paper [5].

	$S_{[2,2,1]}$	$S_{[3,1,1]}$	$C_{[3,2]}$	$C_{[4,1]}$	S_c	S_f
no. Epts	2	3	4	6	10	18
Aut	$(\mathbb{Z}_2)^2$	$\mathcal{S}_3$	$\mathcal{S}_3 \times \mathcal{S}_2$	$\mathcal{S}_4$	$\mathcal{S}_5$	$H \rtimes \mathcal{S}_4$

Table 1: The data of a general cubic surface on different components

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