Let $X$ be a smooth Fano manifold equipped with a “nice” $n$-blocks collection in the sense of [3] and $\mathcal{F}$ a coherent sheaf on $X$. Assume that $X$ is Fano and that all blocks are coherent sheaves. Here we prove that $\mathcal{F}$ has regularity $-\infty$ in the sense of [3] if $\text{Supp}(\mathcal{F})$ is finite, the converse being true under mild assumptions. The corresponding result is also true when $X$ has a geometric collection in the sense of [2].

1. Introduction.

Let $X$ be an $n$-dimensional smooth projective variety over $\mathbb{C}$. Let $\mathcal{D} := \mathcal{D}^b(\mathcal{O}_X - \text{mod})$ denote the bounded category of $\mathcal{O}_X$-sheaves. Let $\mathcal{F}$ be a coherent sheaf on $X$. Assume that $X$ has a geometric collection in the sense of [2] or an $n$-blocks collection in the sense of [3]. L. Costa and R. M. Miró-Roig defined the notion of regularity for $\mathcal{F}$ and asked a characterization of all $\mathcal{F}$ whose regularity is $-\infty$ ([2], Remark 3.3).

In section 2 we will recall the definitions contained in [2] and [3] and...
used in our statements below. After the statements we will discuss our motivations and give a very short list of interesting varieties to which these results may be applied.

We prove the following results.

**Theorem 1.** Assume that $X$ is Fano and that it has an $n$-blocks collection $\mathcal{B}$ whose members are coherent sheaves. Let $\mathcal{F}$ be a coherent sheaf on $X$. If $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{B}$, then $\text{Supp}(\mathcal{F})$ is finite. If all right mutations of all elements of $\mathcal{B}$ are locally free and $\text{Supp}(\mathcal{F})$ is finite, then $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{B}$.

**Corollary 1.** Assume that $X$ is Fano and that it has a geometric collection $\mathcal{G}$ whose members are coherent sheaves. Let $\mathcal{F}$ be a coherent sheaf on $X$. If $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{G}$, then $\text{Supp}(\mathcal{F})$ is finite. If all right mutations of all elements of $\mathcal{G}$ are locally free and $\text{Supp}(\mathcal{F})$ is finite, then $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{G}$.

We recall that any projective manifold with a geometric collection is Fano ([2], part (2) of Remark 2.16). Any $n$-dimensional smooth quadric $Q_n \subset \mathbb{P}^{n+1}$ has an $n$-block collection whose members are locally free ([3], Example 3.2 (2)). It has a geometric collection if and only if $n$ is odd. Any Grassmannian $G$ has an $n$-block collection (with $n := \dim(G)$) whose members are locally free sheaves ([3], Example 3.7 (4)). For the Fano 3-folds $V_5$ and $V_{22}$ D. Faenzi found a geometric collection whose members are locally free ([4], [5]).

Castelnuovo-Mumford regularity was introduced by Mumford in [8], Lecture 14, for a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$. He ascribed the idea to Castelnuovo for the following reason. Let $C \subset \mathbb{P}^n$ be a closed subvariety and $H \subset \mathbb{P}^n$ be a general hyperplane. Then we have an exact sequence

$$0 \rightarrow \mathcal{I}_C(t-1) \rightarrow \mathcal{I}_C(t) \rightarrow \mathcal{I}_{C \cap H}(t) \rightarrow 0$$

(1)

Castelnuovo used the corresponding classical (pre-sheaves) concepts of linear systems to get informations on $C$ from informations on $C \cap H$ plus other geometrical or numerical assumptions on $C$. The key properties of Castelnuovo-Mumford regularity is that if $\mathcal{F}$ is $m$-regular, then it is $(m+1)$-regular and $\mathcal{F}(m)$ (or $\mathcal{I}_C(m)$) is spanned. Since [8] several hundred papers studied this notion, which is now also a key property in computational algebra. Let $X$ be a projective scheme, $H$ an ample line
bundle on $X$ and $\mathcal{F}$ a coherent sheaf on $X$. The definition in [8], Lecture 14, apply verbatim, just writing $\mathcal{F} \otimes H^{\otimes t}$ instead of $\mathcal{F}(t)$. This is also called Castelnuovo-Mumford regularity with respect to the polarized pair $(X, H)$. $X$ may have several non-proportional polarizations. It is better to collect all informations for all polarizations in a single integer (the regularity) not in a string of integers, one for each proportional class of polarizations on $X$. This is the reason for the definitions given by Hoffman-Wang for products of projective varieties ([6]) and by Maclagan and Smith for toric varieties ([7]). Even when $X$ has only one polarization the search for generalizations of Beilinson’s spectral sequence from $\mathbb{P}^n$ to $X$ gave a strong motivation to introduce the notions of regularities for geometric collections ([2], Th. 2.21) and $n$-block collections ([3], Th. 3.10). The reader will notice that to prove Theorem 1 and Corollary 1 we will use neither the main definitions of [2] and [3] nor the machinery of derived categories. We will only use the formal properties (like “spannedness” or “$m$-regularity implies $(m+1)$-regularity”) proved in [2] and [3] (see eq. (2) in section 2 for an explanation of the word “spannedness”). We hope that our results will be extended and used if other notions of regularity will appear in the literature.

2. The main definitions and the proofs.

Let $X$ be an $n$-dimensional smooth projective variety over $\mathbb{C}$. Let $\mathcal{D} := \mathcal{D}^b(\mathcal{O}_X - \text{mod})$ denote the bounded category of $\mathcal{O}_X$-sheaves. For all objects $A, B \in \mathcal{D}$ set $\text{Hom}^* (A, B) := \oplus_{k \in \mathbb{Z}} \text{Ext}^k_D (A, B)$. An object $A \in \mathcal{D}$ is said to be exceptional if $\text{Hom}^* (A, A)$ is an 1-dimensional algebra generated by the identity. An ordered collection $(A_0, \ldots, A_m)$ of objects of $\mathcal{D}$ will be called an exceptional collection if each $A_i$ is exceptional and $\text{Ext}^*_D (A_k, A_j) = 0$ for all $0 \leq j < k \leq m$. A collection $(A_0, \ldots, A_m)$ is said to be strongly exceptional if it is exceptional and $\text{Ext}^*_D (A_j, A_k) = 0$ for all $(i, j, k)$ such that $i \neq 0$ and $j \leq k$. A collection $(A_0, \ldots, A_m)$ is said to be full if it generates $\mathcal{D}$. This implies $\mathcal{D} \cong \mathbb{Z}^{\oplus (m+1)}$. Now assume that $X$ admits a fully exceptional collection $\sigma = (A_0, \ldots, A_n)$. For any $A, B \in \mathcal{D}$ the right mutation $R_B A$ of $A$ and the left mutation $L_A B$ of $B$ are defined by the following distinguished triangles

$$R_B A[-1] \rightarrow A \rightarrow \text{Hom}^* (A, B) \otimes B \rightarrow R_B A$$
Let \( L_A B \to \text{Hom}^\bullet(A, B) \otimes A \to L_A B \) \([1]\)

([2], Definition 2.4). For every integer \( i \) such that \( 1 \leq i \leq n \), define the \( i \)-th right mutation \( R_i \sigma \) and the \( i \)-th left mutation \( L_i \sigma \) of \( \sigma \) by the formulas

\[
R_i \sigma := (A_0, \ldots, A_{i-2}, A_i, R_{A_{i-1}} A_{i-1}, A_{i+1}, \ldots, A_n)
\]

\[
L_i \sigma := (A_0, \ldots, A_{i-2}, L_{A_{i-1}} A_{i-1}, A_{i-1}, A_{i+1}, \ldots, A_n)
\]

(a switch of two elements of \( \sigma \) and the application to one of them of a right or left mutation) \([2]\), Definition 2.6). For any \( j \geq 2 \), set \( R^{(j)} A_i := R_{A_{i+j}} \cdots \circ R_{A_{i+1}} A_i \in \mathcal{D} \) and define in a similar way the iterated left mutations \( L^{(i)} \) \([2]\), Notation 2.7). Set \( A_{n+i} := R^{(n)} A_{i-1} \) for all \( 0 \leq i \leq n \) and \( A_{-j} := L^{(n)} A_{n-i+1} \) for all \( 1 \leq i \leq n \). Iterating the use of \( R^{(n)} \) and \( L^{(n)} \) we get the elix \( \{A_i\}_{i \in \mathbb{Z}} \) with \( A_j \in \mathcal{D} \) for all \( i \) \([2]\), Definition 2.12). For instance, if \( X = \mathbb{P}^n \), then \( (A_0, \ldots, A_n) := (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n)) \) is a geometric collection and \( \{\mathcal{O}_{\mathbb{P}^n}(i)\}_{i \in \mathbb{Z}} \) is the corresponding elix.

Let \( \mathcal{F} \) be a coherent sheaf on \( X \). \( \mathcal{F} \) is said to be \( m \)-regular with respect to the geometric collection \( \sigma = (A_0, \ldots, A_n) \) if \( \text{Ext}^q (R^{(-p)} A_{-m+p}, \mathcal{F}) = 0 \) for all integers \( q, p \) such that \( q > 0 \) and \( -n \leq p \leq 0 \). The regularity of \( \mathcal{F} \) is the minimal integer \( m \) such that \( \mathcal{F} \) is \( m \)-regular (or \(-\infty \) if it is \( m \)-regular for all \( m \in \mathbb{Z} \)). An exceptional collection \( (A_0, \ldots, A) \) is called a block if \( \text{Ext}^q_p (A_j, A_k) = 0 \) for all \( i, j, k \) such that \( k \neq j \). An \( m \)-block collection of elements of \( \mathcal{D} \) is an exceptional collection which may be partitioned into \( m + 1 \) consecutive blocks. Assume that \( X \) has an \( n \)-block collection whose elements generate \( \mathcal{D} \). Let \( \mathcal{F} \) be a coherent sheaf on \( X \). In \([3]\), Definition 4.5, there is a definition of regularity of \( \mathcal{F} \); it requires only technical modifications with respect to the simpler case of a geometric collection: they gave similar definitions of left and right mutations and elices. Then the definition of \( m \)-regularity is again given by certain Ext-vanishings. If a coherent sheaf \( \mathcal{F} \) is \( m \)-regular with respect to a geometric collection \( \sigma \) or an \( n \)-block collection \( \sigma \), then it gives a resolution

\[
0 \to \mathcal{L}_{-n} \to \cdots \to \mathcal{L}_{-1} \to \mathcal{L}_0 \to \mathcal{F} \to 0
\]

in which each \( \mathcal{L}_i \in \mathcal{D} \) is constructed from \( \mathcal{F} \) and the elements of \( \sigma \) taking tensor products \([1]\), between 3.1 and 3.2 for geometric collections, \([3]\), eq. (4.2), for \( n \)-blocks). If the elements of \( \sigma \) are coherent sheaves (resp. locally free coherent sheaves), then each \( \mathcal{L}_i \) is a coherent sheaf.
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(resp. a locally free coherent sheaf). In the case of Castelnuovo-Mumford regularity the corresponding result is true. It shows how the Castelnuovo-Mumford regularity bounds the degrees of the syzygies. This is the key reason for its use in computational algebra.

The following well-known result answers the corresponding problem for Castelnuovo-Mumford regularity.

**Lemma 1.** Let $X$ be a projective scheme, $L$ an ample line bundle on $X$ and $F$ a coherent sheaf on $X$. The following conditions are equivalent:

(a) $F$ is supported by finitely many points of $X$;
(b) $F \otimes L^{\otimes t}$ is spanned for all $t \ll 0$;
(c) $h^i(X, F \otimes L^{\otimes t}) = 0$ for all $i > 0$ and all $t \in \mathbb{Z}$.

**Proof.** Obviously, (a) implies (b) and (c). Now assume that (b) holds, but that $\dim(\text{Supp}(F)) > 0$. Take an integral projective curve $C \subseteq \text{Supp}(F)$. Since the restriction of a spanned sheaf is spanned, $F|C$ satisfies (c) with respect to the ample line bundle $R := L|C$. Let $f : D \to C$ be the normalization. Set $M := f^*(R)$. $M$ is an ample line bundle on $D$. Since $D$ is a smooth curve, the coherent sheaf $f^*(F)$ is either a torsion sheaf or the direct sum of a torsion sheaf $T$ and a vector bundle $E$ with positive rank. To prove (a) we must check that $f^*(F)$ is torsion. Assume $E \neq 0$. Since the pull-back of a spanned sheaf is spanned, $E \otimes M^{\otimes t}$ is spanned for all $t \in \mathbb{Z}$. Since $\deg(E \otimes R^{\otimes t}) = \deg(E) + t \cdot \text{rank}(E) \cdot \deg(M) < 0$ for $t \ll 0$, $E \otimes R^{\otimes t}$ is not spanned for $t \ll 0$, contradiction. Let $x \geq 1$ be an integer such that $L^{\otimes x}$ is very ample. If $F$ satisfies (c) for the line bundle $L$, then it satisfies the same condition for the line bundle $L' := L^{\otimes x}$. Hence to check that (c) implies (a) we may assume that $L$ is very ample. Fix an integer $t$. Since $h^i(X, F \otimes L^{\otimes t-i-1}) = 0$ for all $i > 0$, $F \otimes L^{\otimes t}$ is spanned ([8], p. 100). Thus (b) holds and hence (a) holds. 

**Proof of Theorem 1.** Fix a coherent sheaf $F$. Let $\mathcal{E}$ be the helix of blocks generated by $\mathcal{B}$ ([3], Definition 4.1). All elements of $\mathcal{E}$ are coherent sheaves, not just complexes ([3], Corollary 4.4) and their elements satisfies a periodicity modulo $n + 1$: $\mathcal{E}_i = \mathcal{E}_{i+n+1} \otimes \omega_X$ ([3], lines between 4.3 and 4.4). First assume that $F$ has regularity $-\infty$ with respect to $\mathcal{B}$, i.e. that it is $m$-regular with respect to $\mathcal{B}$ for all $m \ll 0$. Fix $m \in \mathbb{Z}$.
$m$-regularity of $\mathcal{F}$ implies that it is a quotient of a finite sum $\mathcal{L}_0$ of sheaves of the form $E^{-m}_j$ appearing in the blocks of $\mathcal{B}$ ([3], Definition 4.5). Since $\mathcal{F}$ is $t$-regular for all $t \ll 0$, the periodicity property of $\mathcal{E}$ shows that for all integers $t \leq 0$, $\mathcal{F}$ is a quotient of a finite direct sum of sheaves of the form $\mathcal{L}_0 \otimes \omega_X^{\oplus t}$. Since $X$ is Fano, $\omega_X^*$ is ample. Take $L := \omega_X^*$ and copy the proof that (b) implies (a) in Lemma 1. We get that $\text{Supp} (\mathcal{F})$ is finite.

Now assume that $\text{Supp} (\mathcal{F})$ is finite and that all right mutations of elements of $\mathcal{B}$ are locally free. Let $A$ be any of these mutations. Since $A$ is locally free, the local Ext-functors $\text{Ext}^i(A, \mathcal{F})$ vanish for all $i > 0$. Hence the local-to-global spectral sequence for the Ext-functors gives $\text{Ext}^i(A, \mathcal{F}) \cong H^i(X, \text{Hom}(A, \mathcal{F}))$ for all $i \geq 0$. Since $\text{Supp} (\mathcal{F})$ is finite, we get $\text{Ext}^q(A, \mathcal{F}) = 0$ for all $q > 0$. Hence for every integer $m$ the sheaf $\mathcal{F}$ satisfies the definition of $m$-regularity given in [3], Definition 4.5. Since $\mathcal{F}$ is $m$-regular with respect to $\mathcal{B}$ for all $m$, its regularity is $-\infty$. $\square$

Proof of Corollary 1. This result is a particular case of Theorem 1, because the definition of regularity for geometric collections given in [2] agrees with the definition of regularity for $n$-blocks collections given in [3] (see [3], Remark 4.7). It may be proved directly, just quoting [2], Remark 2.14, to get the periodicity property $\mathcal{E}_i = \mathcal{E}_{i+n+1} \otimes \omega_X$ and [2], Proposition 3.8, to get the surjection $\mathcal{L}_0 \to \mathcal{F}$. $\square$

Remark 1. In [1] J. V. Chipalkatti defined a notion of regularity for a coherent sheaf $\mathcal{F}$ on a Grassmannian. He remarked that $\mathcal{F}$ have regularity $-\infty$ (according to his definition) if and only if its support is finite ([1], part 4) of Remark 1.2).

Remark 2. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n \times \mathbb{P}^m$. Hoffman and Wang introduced a bigraded definition of regularity ([6]). The definition of ampleness and [6], Prop. 2.8, imply that if $\mathcal{F}$ is $(a, b)$-regular in the sense of Hoffman-Wang for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, then $\text{Supp} (\mathcal{F})$ is finite. The converse is obvious. As remarked in [3], Remark 5.2, Hoffman-Wang definition and its main properties may be extended verbatim to arbitrary multiprojective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$. 


REFERENCES


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