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THE SECOND DUAL VARIETY OF RATIONAL CONIC BUNDLES

ELENA CHIERICI

The second dual variety of a Segre-Hirzebruch surface linearly normally embedded in a projective space as a 2-regular rational conic bundle is studied by relating it to the classical dual variety of another linearly normal embedding of the underlying surface. These varieties are shown to be birational, and an upper bound on the dimensions of their singular loci is given.

Introduction.

Higher order dual varieties of projective manifolds have been studied in the last decades by many authors (Piene, Tai, Sacchiero, Shifrin), with special regard to some classes of manifolds, e.g. scrolls, whose osculatory behaviour is somehow pathological. More recently, thanks to the notions of k-jet spannedness and k-jet ampleness introduced by Beltrametti and Sommese [1], Lanteri and Mallavibarrena obtained some results on the physiology of higher order dual varieties of smooth projective manifolds and surfaces [6], [7]. However, a general theory is still far from being complete.

For this reason there is some interest in investigating particular classes of surfaces which satisfy all the hypothesis that make their osculatory behaviour

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as good as possible; in this paper we describe the second dual variety of rational conic bundles, more precisely of Segre-Hirzebruch surfaces $\mathbb{F}_e$ embedded by a 2-jet spanned line bundle L such that $L_f = \mathcal{O}_{\mathbb{P}^1}(2)$ for every fibre f.

The main tool in our study is the relationship between the second dual variety and the classical dual variety of another embedding of $\mathbb{F}_e$: the one given by $L - f$; actually, we show that the two varieties are birational (Prop. 2.1) providing a geometric interpretation of this fact. Moreover, focusing on the singular loci of these varieties, we obtain some upper bounds for their dimensions (Thm. 3.3); there a key role is played by the varieties parameterizing curves in a linear system having a given number of ordinary double points, sometimes called Severi varieties.

The paper is organized as follows: in Section 1 we recall some background material about higher order dual varieties, Segre-Hirzebruch surfaces and Severi varieties; in Section 2 we prove the birationality of the two dual varieties of $\mathbb{F}_e$, while in Section 3 we describe their singular loci.

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1. Background material.

We use standard notation from algebraic geometry. All varieties are defined over the complex number field. Tensor products of line bundles are written additively. Moreover, we do not distinguish between a vector bundle and the corresponding locally free sheaf.

Let $X \subset \mathbb{P}^N = \mathbb{P}(V)$ be a smooth variety of dimension $n \geq 2$ and set $L := \mathcal{O}_X(1)$; then V can be identified with the vector subspace of $H^0(X, L)$ giving rise to the embedding in $\mathbb{P}^N$. We denote by $|V|$ the linear system associated with V. Let $J_k L$ be the k-th jet bundle of L; for every integer $k \geq 1$ we consider the bundle homomorphism $j_k : V \otimes \mathcal{O}_X \to J_k L$, associating to each section $s \in V \subseteq H^0(X, L)$ its k-th jet $j_k(s)$. We set $\sigma_k + 1 = \max_{x \in X}\{\mathrm{rk}(j_{k,x})\}$, and consider the dense open subset $\mathcal{U} \subset X$ where $\mathrm{rk}(j_{k,x}) = \sigma_k + 1$. For every $x \in \mathcal{U}$, one can define the *k-th osculating space* to X at x as

$$\mathrm{Osc}^k_x(X) = \mathbb{P}(\mathrm{Im}\, j_{k,x}) \simeq \mathbb{P}^{\sigma_k};$$

we say that a hyperplane $H \subset \mathbb{P}^N$ is *k-osculating* to X at x if H contains $\mathrm{Osc}^k_x(X)$.

Thus H is a k-osculating hyperplane to X at a point $x \in \mathcal{U}$ if and only if it corresponds to a section $s \in |V|$ such that $j_{k,x}(s) = 0$, i.e. s and all its derivatives up to order k vanish at x; equivalently, H is k-osculating at x if and only if the hyperplane section $H \cap X$ has a singular point of order $\geq k+1$ at x.

Definition 1.1. The *k-th dual variety* $X^\vee_k$ of X is the closure in $\mathbb{P}^{N\vee} = \mathbb{P}(V^\vee) = |V|$ of the set of k-osculating hyperplanes to X at the points $x \in \mathcal{U}$, i.e.

$$X^\vee_k = \overline{\{H \in \mathbb{P}^N | H \supseteq \mathrm{Osc}^k_x(X),\ x \in \mathcal{U}\}}.$$

Note that $X^\vee_k$ is only defined when $\mathrm{Osc}^k_x(X) \neq \mathbb{P}$ for the general $x \in X$. This automatically rules out the Veronese manifolds $(X, L) = (\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))$, since for such manifolds $\mathrm{Osc}^k_x(X) = \mathbb{P}^N$ for all $x \in X$; on the other hand, there exist no more varieties with this property (see [4]).

Definition 1.2. Let X be a smooth n-fold and L a very ample line bundle on X. For any positive integer k we say that

(i) L is *k-jet spanned* w.r.t. V if the homomorphism

$$j_{k,x} : V \longrightarrow \Gamma(L \otimes (\mathcal{O}_X/\mathfrak{m}^{k+1}_x))$$

is surjective for every $x \in X$;

(ii) the pair (X, V) is *k-regular* if L is k-jet spanned w.r.t. V and the map $\varphi_{|V|}$ is an embedding. We say in particular that (X, L) is k-regular if $V = H^0(X, L)$.

If (X, V) is k-regular, then $\dim \mathrm{Im} j_{k,x} = \binom{k+n}{n}$ for every $x \in X$. Let $\mathcal{K}_k$ be the dual of the kernel of the vector bundle surjection $X \times V \longrightarrow J_k L$; then $X^\vee_k$ coincides with the image of $\mathbb{P}(\mathcal{K}_k)$ via the second projection $q : X \times |V| \to |V|$, as is shown in the following diagram:

$$\begin{array}{ccc} X \times |V| & \supset & \mathbb{P}(\mathcal{K}_k) \\ \Big\downarrow q & & \Big\downarrow \pi_k = q_{|\mathbb{P}(\mathcal{K}_k)} \\ |V| & \supset & X^\vee_k. \end{array}$$

Since $\dim \mathbb{P}(\mathcal{K}_k) = \dim X + \dim V - \dim \mathrm{Im} j_{k,x} - 1$, we have

$$\dim X^\vee_k \leq \dim \mathbb{P}(\mathcal{K}_k) = n + N - \binom{k+n}{n} =: \mathrm{expdim}\ X^\vee_k.$$

$X_k^\vee$ is said to be *degenerate* if $\dim X_k^\vee < \operatorname{expdim} X_k^\vee$, i.e. if π_k is not generically finite.

As to surfaces, i.e. $n = 2$, we know the following fact:

Theorem 1.3. ([7], Cor. 1.2) *Let (S, V) be a k-regular surface for some $k \geq 1$, and assume that $(S, V) \neq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$. Then the k'-th dual variety $S_{k'}^\vee$ of S is nondegenerate for every $k' \leq k$.*

Let $\mathbb{F}_e$ be the Segre-Hirzebruch surface of invariant e, let f be a fibre and let C_0 be a section of minimal self-intersection $C_0^2 = -e$ on $\mathbb{F}_e$. We briefly recall some useful results about this class of surfaces. For all results we need on linear systems on $\mathbb{F}_e$ we refer to [5], except for the following two lemmas, whose proof is not difficult (see [2], Section 1.4):

Lemma 1.4. *Let L be an ample line bundle on $\mathbb{F}_e$. Then $h^1(L) = h^2(L) = 0$.*

Lemma 1.5. *Let $L = [aC_0 + bf]$ be a line bundle on $\mathbb{F}_e$ such that $a \geq -1$, $b \geq ae$. Then $h^1(L) = 0$.*

Proposition 1.6. ([6], Rem. 2.2) *Let $L = [aC_0 + bf]$ be a line bundle on $\mathbb{F}_e$. Then the following conditions are equivalent:*

(i) *$(\mathbb{F}_e, L)$ is k-regular,*

(ii) *$a \geq k$, $b \geq k + ae$.*

We also recall the following fact:

Proposition 1.7. ([6], Prop. 2.6) *If $(\mathbb{F}_e, L)$ is k-regular, then the morphism π_k is birational.*

Let $\sigma : Y \longrightarrow \mathbb{F}_e$ be the blow-up of $\mathbb{F}_e$ at a point p; we denote with f_p the fiber of $\mathbb{F}_e$ through p, and with $\hat{f}_p$ its proper transform on Y. By well-known properties of blow-ups, if E is the exceptional curve introduced by σ, we have

$$0 = (\sigma^* f_p)^2 = (\hat{f}_p + E)^2 = \hat{f}_p^2 + 1,$$

so $\hat{f}_p$ too is an exceptional curve on Y. If $\tau : Y \longrightarrow Z$ is the map which contracts $\hat{f}_p$ to a point, we obtain a birational transformation

$$e_p = \tau \circ \sigma^{-1} : \mathbb{F}_e \dashrightarrow Z$$

which is called *elementary transformation* centered at p. Z is a rational ruled surface (see [9]), whose isomorphism class depends only on the position of the center of the transformation with respect to the fundamental section C_0; namely

$$Z = \begin{cases} \mathbb{F}_{e+1} & \text{if } p \in C_0 \\ \mathbb{F}_{e-1} & \text{if } p \notin C_0. \end{cases}$$

Definition 1.8. Let S be a projective algebraic surface, $|D|$ a basepoint free linear system on S. For each integer $\delta \geq 0$ we define the *Severi variety* $V_{|D|,\delta}$ to be the locally closed subscheme of $|D|$ which parameterizes the irreducible curves in $|D|$ having exactly δ ordinary double points as singularities.

We recall an important result of Chiantini and Sernesi [3 Thm. 1.1 and Rem. 1.2] about the dimension and smoothness of the Severi varieties of ruled surfaces, which generalizes the classical result of Severi for plane curves:

Theorem 1.9. *Let S be a ruled surface, and $C \subset S$ an irreducible curve with $K_S C < 0$ and such that the linear system $|C|$ is basepoint free. If $V_{|C|,\delta} \neq \emptyset$ for some $\delta \leq p_a(C)$, then the Severi variety $V_{|C|,\delta}$ is smooth of codimension δ in $|C|$.*

2. Relating the second dual variety to a classical dual variety.

Let $(S, L) = (\mathbb{F}_e, [2C_0 + bf])$ be a conic bundle over $\mathbb{P}^1$ which is 2-regular, i.e. $b \geq 2e + 2$ in view of 1.6. Consider the embedding

$$\varphi_{|L|} : S \longrightarrow \mathbb{P}^{h^0(L)-1} := \mathbb{P}^N,$$

where $N = 3(b-e)+2$. In $\mathbb{P}^{N\vee}$ we can consider the second dual variety $S_2^\vee$ of (S, L): it is the subvariety of $\mathbb{P}^{N\vee}$ parameterizing the 2-osculating hyperplanes to S, i.e. the elements H of the linear system $|L|$ such that the section $H \cap S$ has at least a singular point of multiplicity ≥ 3 on S. From now on we won't make any distinction between a hyperplane $H \subset \mathbb{P}^N$, the corresponding point in $\mathbb{P}^{N\vee}$ and the section cut out by H on S. We will say that H has a singular point at $p \in S$ if the curve $H \cap S$ is singular at p.

The following observation will be the starting point for our description of the second dual variety $S_2^\vee$: since $Hf = 2$, if $H \in |L|$ is a hyperplane with a singular point p of multiplicity ≥ 3 on S, then H must contain the fiber f_p through p as a component. If we set $R := H - f_p \in |L - f|$, then p must be at least a double point for R. The 2-regularity of L implies that the line bundle $L-f=[2C_0+(b-1)f]$ is very ample, so we can consider also the embedding

$$\varphi_{|L-f|}:\mathbb{F}_e\longrightarrow\mathbb{P}^{h^0(L-f)-1}=\mathbb{P}^{N-3};$$

from now on we will denote by Σ the image of $\mathbb{F}_e$ in $\mathbb{P}^{N-3}$.

In this embedding, R corresponds to a point in the dual projective space $(\mathbb{P}^{N-3})^\vee$, and since it has at least a singular point on Σ it belongs to the first dual variety $\Sigma_1^\vee\subset(\mathbb{P}^{N-3})^\vee$ of Σ. We can thus consider the rational map

$$\psi:\ S_2^\vee--\dashrightarrow\Sigma_1^\vee$$

associating to the general $H\in S_2^\vee$ the point $R=H-f_p\in\Sigma_1^\vee$, where p is the osculation point of H, whose uniqueness descends from the birationality of the map $\pi_2:\mathbb{P}(\mathcal{K}_2)\to S_2^\vee$. In order to describe this correspondence, first of all we need to compute the dimensions of $S_2^\vee$ and $\Sigma_1^\vee$:

Proposition 2.1. *The varieties* $\Sigma_1^\vee\subset(\mathbb{P}^{N-3})^\vee$ *and* $S_2^\vee\subset\mathbb{P}^{N\vee}$ *have the same dimension* $N-4$ *and they are birational; moreover,* $\deg\Sigma_1^\vee=8(b-e)-12$ *and* $\deg S_2^\vee=20(b-e-1)$.

Proof. Under the hypothesis that (S,L) is 2-regular, we know from Theorem 1.3 that both $\Sigma_1^\vee$ and $S_2^\vee$ have the expected dimension; so $\Sigma_1^\vee$ is a hypersurface in $(\mathbb{P}^{N-3})^\vee$ and $\dim S_2^\vee=N-4$.

As the map $\pi_2:\mathbb{P}(\mathcal{K}_2)\to S_2^\vee$ is birational, the generic hyperplane H has only one triple point on S: the map $\psi:S_2^\vee--\to\Sigma_1^\vee$ is thus defined on a dense open subset $\mathcal{U}\subseteq S_2^\vee$. In the same way, as the contact locus of the generic hyperplane $R\in\Sigma_1^\vee$ consists of a single point, ψ can be inverted on a dense open subset $\mathcal{V}\subseteq\Sigma_1^\vee$. The correspondence is not one-to-one between $\mathcal{U}$ and $\mathcal{V}$: while it is obvious that $\mathcal{V}\subseteq\psi(\mathcal{U})$, one can easily check that $\mathcal{V}\not\supseteq\psi(\mathcal{U})$, considering for example a hyperplane cutting on S a section, if it exists, having both a triple and a double point. Anyway, if we consider the open set $\mathcal{U}'=\psi^{-1}(\mathcal{V})\subseteq\mathcal{U}$, we conclude that ψ is one-to-one between $\mathcal{U}'$ and $\mathcal{V}$, so it is a birational map between the two dual varieties.

To compute the degrees of $S_2^\vee$ and $\Sigma_1^\vee$ we use [6], Thm. 1.4, and Prop. 1.7 above: we have $\deg S_2^\vee=c_2(J_2L)=20(b-e-1)$ and $\deg\Sigma_1^\vee=c_2(J_1(L-f))=8(b-e)-12$. $\square$

Let $\mathcal{K}_2$ and $\mathcal{K}_1'$ be the duals of the kernels of the vector bundles surjections in the exact sequences

$$0\to\mathcal{K}_2^\vee\to H^0(\mathbb{F}_e,L)\to J_2L\to0$$

$$0 \to \mathcal{K}_1^{'\vee} \to H^0(\mathbb{F}_e, L - f) \to J_1(L - f) \to 0$$

respectively. Due to the birationality of the maps ψ, π_2 and π_1', a birational map

$$\varphi : \mathbb{P}(\mathcal{K}_2) \dashrightarrow \mathbb{P}(\mathcal{K}_1')$$

is induced, making the following diagram commute:

$$\begin{tikzcd}
\mathbb{P}(\mathcal{K}_2) \arrow[r, dashed, "\varphi"] \arrow[d, "\pi_2"'] & \mathbb{P}(\mathcal{K}_1') \arrow[d, "\pi_1'"] \\
S_2^\vee \arrow[r, dashed, "\psi"'] & \Sigma_1^\vee .
\end{tikzcd}$$

Note that the birationality of $\mathbb{P}(\mathcal{K}_2)$ and $\mathbb{P}(\mathcal{K}_1')$ is granted a priori. In fact, since $\dim S_2^\vee = \dim \Sigma_1^\vee$, $\mathcal{K}_2^\vee$ and $\mathcal{K}_1^{'\vee}$ have the same rank, we know that the function fields of $\mathbb{P}(\mathcal{K}_2)$ and $\mathbb{P}(\mathcal{K}_1')$ are both isomorphic to the function field of $\mathbb{F}_e$.

The exceptional loci of ψ and ψ^{-1}, which we denote from now on with $\Delta_2 \subset S_2^\vee$ and $\Delta_1 \subset \Sigma_1^\vee$, parameterize the hyperplane sections of $|L|$ and $|L - f|$ respectively having more than one singular point (triple for Δ_2, double for Δ_1) lying on different fibers of $\mathbb{F}_e$. In particular, these sets are contained in the singular loci of $S_2^\vee$ and $\Sigma_1^\vee$ respectively, which are described in the following section.

3. The singular locus.

For the pair $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, 2))$, the special features of hyperplane sections having more than a triple point allow to give a detailed description of the singular loci of $S_2^\vee$ and $\Sigma_1^\vee$. For the details, see [8], Section 5. From now on we suppose that $(S, L) \neq (\mathbb{F}_0, [2C_0 + 2f])$.

A point $R \in (\mathbb{P}^{N-3})^\vee$ is singular for $\Sigma_1^\vee$ if and only if the corresponding hyperplane is tangent to Σ at more than one point, i.e. the section $R \cap \Sigma$ has more than a single ordinary quadratic singularity on Σ; this is the case also if $R \cap \Sigma$ has one point of multiplicity greater than two on Σ. In order to compute the codimension of the irreducible components of maximal dimension of the singular locus of $\Sigma_1^\vee$, which may a priori be reducible, we start with the computation of the codimension of the space of the singular elements of the linear system $|L - f|$; we can observe that, according to the dimension of this linear system, either all curves with more than one singularity are reducible (as for example in the case $(S, L) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2, 2)))$ or the generic one is irreducible; in this case it also happens that the generic element in Sing $\Sigma_1^\vee$ has exactly two ordinary double points as singularities.

This statement can be rephrased as follows: whenever the Severi variety $V_{|L-f|,2}$ is nonempty, the singular locus of $\Sigma_1^\vee$ contains an open subset which is isomorphic to it. Actually, the line bundle $L - f = [2C_0 + (b-1)f]$ on $\mathbb{F}_e$ satisfies the assumptions of Theorem 1.9: the linear system $|L - f|$ is very ample, thus basepoint free, and

$$(L - f)K_S = (2C_0 + (b-1)f)(-2C_0 - (e+2)f) = 2(e - b - 1) < 0.$$

Remark. The same technique can be used to describe also the singular locus of the second dual variety of the pair $(\mathbb{F}_e, L)$, even if we are dealing with points of multiplicity three instead of two. In fact, as we have already observed, a section $H \in |L|$ contains the fibers through its triple points as components, so that an element of Sing $S_2^\vee$ can be written in the form $H = f_p + f_q + D$, with p, $q \in S$ not necessarily distinct, and D an element of $|L - 2f|$ which is singular at p and q. Moreover, the general D satisfying this condition has only p and q as singularities, so that D determines H uniquely; thus we can state that if $V_{|L-2f|,2}$ is nonempty, then Sing $S_2^\vee$ contains an open subset isomorphic to it. We observe also that the linear system $|L - 2f|$ is 0-regular, i.e. basepoint free, and that

$$(L - 2f)K_S = (2C_0 + (b-2)f)(-2C_0 - (e+2)f) = 2(e - b) < 0,$$

so that if $V_{|L-2f|,2}$ is nonempty we can apply Theorem 1.9 again.

Our next step is to look for conditions on the line bundle L in order to guarantee that the Severi varieties $V_{|L-f|,2}$ and $V_{|L-2f|,2}$ are nonempty; for this purpose we will make use of elementary transformations, as in the proof of the following

Theorem 3.1. *Let $L = [2C_0 + hf]$ be a line bundle on $\mathbb{F}_e$ such that the linear system $|L|$ contains a smooth irreducible curve Γ. Then the Severi varieties*

(a) $V_{|2C_0+(h+2)f|,2}$ *on* $\mathbb{F}_e$

(b) $V_{|2C_0+(h+4)f|,2}$ *on* $\mathbb{F}_{e+2}$

(c) $V_{|2C_0+hf|,2}$ *on* $\mathbb{F}_{e-2}$

are nonempty (where C_0 and f denote respectively a curve of minimal self-intersection and a generic fiber on the corresponding surface).

Proof. (a) Let p be a point on $\mathbb{F}_e$ lying neither on Γ nor on the fundamental section of $\mathbb{F}_e$, which we will denote from now on with C_e; consider the elementary transformation e_p centered at p.

$$\begin{tikzcd} & Y \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & \\ \mathbb{F}_e \arrow[rr, dashed, "e_p"'] & & \mathbb{F}_{e-1} \end{tikzcd}$$

Let $\hat{\Gamma}$ be the proper transform of Γ in Y and set $\Gamma' := \tau(\hat{\Gamma})$; using the same notations as in Section 1.2, the following relations hold:

$\sigma^*\Gamma = \hat{\Gamma}$,

$\tau^*\Gamma' = \hat{\Gamma} + 2\hat{f}_p$,

$\sigma^* f_p = \hat{f}_p + E$,

$\sigma^* f = \tau^* f'$ for the generic fibers f of $\mathbb{F}_e$ and f' of $\mathbb{F}_{e-1}$.

Moreover, $\Gamma f_p = 2$ implies $\hat{\Gamma}\hat{f}_p = 2$. Since X is linearly equivalent to $2C_e + hf$ on $\mathbb{F}_e$, we can compute the equivalence class of Γ' in $\mathrm{Pic}(\mathbb{F}_{e-1})$, taking a generic fiber f' and the fundamental section C'_{e-1} as generators. Writing Γ' as $aC'_{e-1} + bf'$, we get

$$\begin{aligned} a &= \Gamma' f' = (\tau^*\Gamma')(\tau^* f') = (\hat{\Gamma} + 2\hat{f}_p)(\tau^* f') = (\sigma^*\Gamma + 2\hat{f}_p)(\sigma^* f) = \\ &= \Gamma f = 2, \end{aligned}$$

and

$$\begin{aligned} -a^2(e-1) + 2ab &= (\Gamma')^2 = (\hat{\Gamma} + 2\hat{f}_p)^2 = \\ &= (\sigma^*\Gamma)^2 + 4\hat{f}_p^2 + 4(\sigma^*\Gamma)(\sigma^* f_p - E) = \\ &= \Gamma^2 - 4 + 4\Gamma f_p = \Gamma^2 + 4 = -4e + 4h + 4. \end{aligned}$$

Then it must necessarily be $a = 2$ and $b = h$, so that Γ' belongs to the linear system $2C'_{e-1} + hf'$. It is also important to observe that, since the curve $\hat{\Gamma}$ on Y has exactly two intersections with the exceptional curve $\hat{f}_p$, the contraction of $\hat{f}_p$ gives rise to an ordinary double point on Γ'.

Now take p' to be a point on the section C'_{e-1} of $\mathbb{F}_{e-1}$, not lying on Γ', and consider the elementary transformation of $\mathbb{F}_{e-1}$ centered at p'.

$$\begin{tikzcd} & Y' \arrow[dl, "\sigma'"'] \arrow[dr, "\tau'"] & \\ \mathbb{F}_{e-1} \arrow[rr, dashed, "e_{p'}"'] & & \mathbb{F}_e \end{tikzcd}$$

We denote with f'' the generic fiber on $\mathbb{F}_e$, with $\hat{\Gamma}'$ the proper transform of Γ' on Y' and with $\Gamma'' \sim \alpha C''_e + \beta f''$ its image via τ'; arguing as before, we get

$$\alpha = \Gamma'' f'' = \Gamma' f' = 2,$$

$$-\alpha^2 e + 2\alpha\beta = (\Gamma'')^2 = (\Gamma')^2 + 4 = -4(e-1) + 4h + 4,$$

implying that $\alpha = 2$ and $\beta = h + 2$. These transformations allow us to obtain, starting from a smooth curve $\Gamma \in |2C_0 + hf|$, a curve Γ'' in the linear system $|2C''_e + (h+2)f''|$ on $\mathbb{F}_e$ which is irreducible and has exactly two ordinary double points as singularities.

The proof of (b) and (c) is essentially the same, considering elementary transformations centered at points lying both on the fundamental sections or both out of them respectively. □

We finally need to guarantee that, in the cases we are dealing with, the hypothesis of Theorem 3.1 are satisfied, i.e., that there exists a smooth irreducible curve Γ to which we can apply the transformations described in the proof of the theorem, and from which we can obtain an element of $|L - f|$ or $|L - 2f|$ on $\mathbb{F}_e$ having two ordinary double points.

Theorem 3.2. *Let $L = [2C_0 + bf]$ be a line bundle on $\mathbb{F}_e$. Then*

$$\text{(i) } V_{|L-f|,2} \neq \emptyset \quad for \ \ all \ \ b \geq \begin{cases} e+4 & \text{if } e \leq 2, \\ 2e+1 & \text{if } e \geq 3. \end{cases}$$

$$\text{(ii) } V_{|L-2f|,2} \neq \emptyset \quad for \ \ all \ \ b \geq \begin{cases} e+5 & \text{if } e \leq 2, \\ 2e+2 & \text{if } e \geq 3. \end{cases}$$

Proof. (i) First of all, we observe that we can obtain a curve in $|L - f| = |2C_0 + (b-1)f|$ on $\mathbb{F}_e$ having two ordinary double points starting from three different linear systems on three different surfaces:

$2C_0 + (b-3)f$ on $\mathbb{F}_e$,

$2C_0 + (b-5)f$ on $\mathbb{F}_{e-2}$,

$2C_0 + (b-1)f$ on $\mathbb{F}_{e+2}$.

From [5], Cor. 2.18, we know that the linear system $|aC_0 + bf|$ on $\mathbb{F}_e$ contains a smooth irreducible curve Γ, $\Gamma \neq C_0, f$ if and only if $a > 0$, $b > ae$ or $e > 0$, $a > 0$ and $b = ae$. Combining these two results, a straightforward calculation shows that we can obtain an element of $V_{|L-f|,2}$ on $\mathbb{F}_e$ if one of the following conditions holds:

$e \leq 2, b \geq e+4,$

$e \geq 3, b \geq 2e+1.$

(ii) is proved in the same way. □

Under these conditions on the line bundle L, we obtain from Theorem 1.9 that

$V_{|L-f|,2}$ is smooth of codimension 2 in $|L-f|$;

$V_{|L-2f|,2}$ is smooth of codimension 2 in $|L-2f|$.

From the first information we can directly state that $V_{|L-f|,2}$ has codimension 1 in $\Sigma_1^\vee$, while for the second dual variety $S_2^\vee$ we need to compute the dimension of the linear system $|L-2f| = |2C_0 + (b-2)f|$.

For this purpose write b as $b = 2e+2+r$, with $r \geq 0$; in our assumptions it cannot be $e = r = 0$. If $r \geq 1$, the line bundle $2C_0 + (2e+r)f$ is ample, and lemma 1.4 together with Riemann-Roch theorem imply that

$$\dim |2C_0 + (2e+r)f| = N - 6,$$

while if $r = 0$ and $e > 0$ we can apply lemma 1.5 and obtain that

$$\dim |2C_0 + 2ef| \leq N - 6.$$

Recalling that $\dim S_2^\vee = N - 4$, we can collect these informations in the following

Theorem 3.3. *Let $L = [2C_0 + bf]$ be a 2-jet spanned line bundle on $\mathbb{F}_e$, with $(b, e) \neq (2, 0)$. Then*

(a) *if $(\mathbb{F}_e, L) \neq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3))$ and $(\mathbb{F}_1, [2C_0 + 4f])$, Sing $\Sigma_1^\vee$ contains an irreducible open subset which is smooth of codimension one in $\Sigma_1^\vee$;*

(b) *if $(\mathbb{F}_e, L) \neq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3))$, $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,4))$, $(\mathbb{F}_1, [2C_0 + 4f])$ and $(\mathbb{F}_1, [2C_0 + 5f])$, Sing $S_2^\vee$ contains an irreducible open subset which is smooth of codimension ≥ 4 in $S_2^\vee$. In particular, equality holds if $b > 2e+2$.*

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Dipartimento di Matematica,
Università degli Studi di Trento,
via Sommarive 14 - 38050 Povo (TN)
e-mail: chierici@science.unitn.it