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96120: THE DEGREE OF THE LINEAR ORBIT OF A CUBIC SURFACE

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The projective linear group $\mathrm{PGL}(\mathbb{C},4)$ acts on cubic surfaces, considered as points of $\mathbb{P}^{19}_{\mathbb{C}}$. We compute the degree of the 15-dimensional projective variety defined by the Zariski closure of the orbit of a general cubic surface. The result, 96120, is obtained using methods from numerical algebraic geometry.

1. Introduction

Automorphism groups of varieties and group actions on varieties are of much interest to researchers of algebraic geometry, arithmetic geometry, and representation theory [1, 5, 14, 19]. Here, we study the

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action of the projective linear group $\mathrm{PGL}(\mathbb{C},4)$ on cubic surfaces parameterized by points in $\mathbb{P}^{19}_{\mathbb{C}}$. In particular, we compute the degree of the 15-dimensional projective variety in $\mathbb{P}^{19}_{\mathbb{C}}$ defined by the Zariski closure of the orbit of a general cubic surface under this action. This degree is also meaningful in enumerative geometry: It is the number of translates of a cubic surface that pass through 15 points in general position. This formulation provides an alternate method for obtaining the degree. Our work answers a question posed in [15].

Aluffi and Faber considered the analogous problem for plane curves of arbitrary degree, first the smooth case in [1] and second the general case in [2]. They obtained a closed formula for the degree of the orbit closure of a plane curve under the action of $\mathrm{PGL}(\mathbb{C},3)$. This was a significant undertaking, involving long and detailed calculations in intersection rings using advanced techniques from intersection theory.

Instead of adopting the techniques developed by Aluffi and Faber, we use tools from *numerical algebraic geometry* [9, 18]. The general idea is as follows. We fix a cubic surface f and 15 points in general position in $\mathbb{P}^3_{\mathbb{C}}$. The condition that a translate of f passes through these 15 points results in a polynomial system for which we compute all isolated numerical solutions by homotopy continuation and monodromy methods using the software `HomotopyContinuation.jl` [4]. The concept of an *approximate zero* [3] makes precise the definition of a numerical solution. We use Smale's α-theory and the software `alphaCertified` [10] to certify that the obtained numerical solutions indeed satisfy the system of polynomial equations. Finally, we use a *trace test* [12] to check that no solution is missing. With these techniques, we conclude that the number of numerical solutions we obtain, 96120, is the degree of the orbit closure. This result is a "numerical theorem" rather than a theorem in the classical sense.

Our presentation is organized as follows. In Section 2, we introduce the linear orbit problem in detail and derive the polynomial systems used in our computations. In Section 3, we discuss the techniques used from numerical algebraic geometry and in Section 4 we describe the computations performed to arrive at the result.

2. Linear Orbits and Polynomial Systems

Cubic surfaces in $\mathbb{P}^3_{\mathbb{C}}$ are defined by homogeneous cubic polynomials in 4 variables with complex coefficients. Their parameter space is $\mathbb{P}^{19}_{\mathbb{C}}$. We fix coordinates $(c_0 : \cdots : c_{19}) \in \mathbb{P}^{19}_{\mathbb{C}}$.

The projective linear group

$$\mathrm{PGL}(\mathbb{C},4) = \{\varphi \in \mathbb{P}^{15}_{\mathbb{C}} \mid \det\varphi \neq 0\} \subseteq \mathbb{P}^{15}_{\mathbb{C}}$$

acts on a cubic surface $f \in \mathbb{P}^{19}_{\mathbb{C}}$, with $\varphi \in \mathrm{PGL}(\mathbb{C},4)$, sending f to the cubic surface $\varphi \cdot f$ defined by the equation

$$f(\varphi(x,y,z,w)) = 0\,.$$

This corresponds to a linear change of the coordinates x,y,z,w. We say that $\varphi \cdot f$ is the *translate* of f by φ. Then $\mathrm{PGL}(\mathbb{C},4)\cdot f$ is the orbit of f in $\mathbb{P}^{19}_{\mathbb{C}}$ and its Zariski closure $\Omega_f := \overline{\mathrm{PGL}(\mathbb{C},4)\cdot f}$ is a 15-dimensional projective variety.

Example 2.1. To illustrate this idea, we consider the action of $\mathrm{PGL}(\mathbb{C},2)$ on pairs of points defined by homogeneous polynomials

$$f(x,y) = b_0x^2 + b_1xy + b_2y^2\,.$$

The parameter space for pairs of points is $\mathbb{P}^2_{\mathbb{C}}$, that is $f = (b_0 : b_1 : b_2) \in \mathbb{P}^2_{\mathbb{C}}$. Let

$$\varphi = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then

$$\begin{aligned} f(\varphi(x,y)) =& b_1(a_{11}x+a_{12}y)^2 + b_2(a_{11}x+a_{12}y)(a_{21}x+a_{22}y) + b_3(a_{21}x+a_{22}y)^2 \\ =& (b_1a_{11}^2 + b_2a_{11}a_{21} + b_3a_{21}^2)x^2 + \\ & (2b_1a_{11}a_{12} + b_2(a_{11}a_{22}+a_{12}a_{21}) + 2b_3a_{21}a_{22})xy + \\ & (b_1a_{12}^2 + b_2a_{12}a_{22} + b_3a_{22}^2)y^2 \end{aligned}$$

and thus

$$\begin{aligned} \varphi\cdot f = (&b_1a_{11}^2 + b_2a_{11}a_{21} + b_3a_{21}^2 : \\ &2b_1a_{11}a_{12} + b_2(a_{11}a_{22}+a_{12}a_{21}) + 2b_3a_{21}a_{22} : \\ &b_1a_{12}^2 + b_2a_{12}a_{22} + b_3a_{22}^2) \in \mathbb{P}^2_{\mathbb{C}}\,. \end{aligned}$$

To compute the degree of the orbit closure of a general cubic surface under the action of $\mathrm{PGL}(\mathbb{C},4)$, we construct as follows polynomial systems whose number of solutions correspond to the desired degree.

Fix a general cubic surface $f \in \mathbb{P}^{19}_{\mathbb{C}}$ and a general linear subspace $L \subseteq \mathbb{P}^{19}_{\mathbb{C}}$ of dimension 4, the codimension of Ω_f. Consider the map

$$\Theta_f : \mathrm{PGL}(\mathbb{C},4) \longrightarrow \mathbb{P}^{19}_{\mathbb{C}},\ \varphi \mapsto \varphi\cdot f\,,$$

whose image is the orbit of f by $\mathrm{PGL}(\mathbb{C},4)$. By [14, Theorem 5], a generic hypersurface of degree at least three in at least four variables has a trivial stabilizer. (In [5, Propostion 7.5] it is stated that the argument in [14] has an error but that it does not affect the correctness of the statement.) Hence, the map Θ_f is one-to-one.

Since L is general, we may assume that the intersection $\Omega_f \cap L$ is contained in the image of Θ_f. Therefore, the degrees of the zero-dimensional varieties $\Omega_f \cap L$ and $\Theta_f^{-1}(\Omega_f \cap L)$ coincide. It follows that the degree of Ω_f is the number of solutions of the polynomial system

$$\tilde{L}\,\varphi \cdot f = 0 \tag{1}$$

in the entries of $\varphi \in \mathrm{PGL}(\mathbb{C},4)$, where $\tilde{L} \in \mathbb{C}^{15\times 20}$ is a matrix representing the general linear subspace $L \subseteq \mathbb{P}^{19}_{\mathbb{C}}$ of dimension 4.

The degree of Ω_f is also of meaning in enumerative geometry. As we will see, it is the number of translates of f that pass through 15 points $p_1,\ldots,p_{15} \in \mathbb{P}^3_{\mathbb{C}}$ in general position. Here, the 15 points in general position play the same role as the general linear space L in formulation (1). The translated cubic surface $\varphi \cdot f$ passes through a point $p \in \mathbb{P}^3_{\mathbb{C}}$ if and only if $f(\varphi(p)) = 0$. This yields the polynomial system

$$f(\varphi(p_i)) = 0\,, \quad i = 1,\ldots,15 \tag{2}$$

in the entries of $\varphi \in \mathrm{PGL}(\mathbb{C},4)$. The degree of Ω_f is equal to the number of solutions of (2). To prove this claim, we apply Kleiman's transversality theorem to show that the fifteen hypersurfaces intersect transversally in a finite number of points.

The i-th equation in (2) is the pullback $Y_{p_i} \subseteq \mathrm{PGL}(\mathbb{C},4)$ by Θ_f of the hyperplane in $\mathbb{P}^{19}_{\mathbb{C}}$ of cubic surfaces passing through p_i. We show that every Y_{p_i} is smooth by proving that the tangent space at every point of Y_{p_i} is a hyperplane rather than the whole ambient space. Set $p = (1:0:0:0)$. We may assume $p_i = p$ since, for any $\phi \in \mathrm{PGL}(\mathbb{C},4)$ with $\phi(p_i) = p$, $Y_{p_i} \cong \phi(Y_{p_i}) = Y_p$. Now, the i-th equation in (2) is f evaluated at the first column of φ. That is, setting as coordinates $\varphi = (a_{ij})_{i,j} \in \mathrm{PGL}(\mathbb{C},4)$, we have $f(\varphi(p)) = f(a_{11},a_{21},a_{31},a_{41})$. Hence,

$$\partial_{a_{ij}} f(\varphi(p)) = \begin{cases} 0 & \text{if } j > 1 \\ (\partial_i f)(a_{11},a_{21},a_{31},a_{41}) & \text{otherwise,} \end{cases}$$

where $\partial_1 f = \partial_x f(x,y,z,w)$. Now, for $\varphi_0 = (a^0_{ij})_{i,j} \in Y_p$, the tangent space of Y_p at φ_0 is given by the equation

$$0 = \sum_{i,j} \partial_{a_{ij}} f(\varphi(p))|_{\varphi_0} \cdot (a_{ij} - a^0_{ij}) = \sum_{i=1}^{4} (\partial_i f)(a^0_{11},a^0_{21},a^0_{31},a^0_{41}) \cdot (a_{i1} - a^0_{i1}).$$

Since $\varphi_0 \in Y_p$ we have $f(a^0_{11}, a^0_{21}, a^0_{31}, a^0_{41}) = 0$. Now f is smooth, so its partial derivatives cannot all vanish at $(a^0_{11} : a^0_{21} : a^0_{31} : a^0_{41})$. Thus the tangent space of Y_p at ϕ_0 is indeed given by a hyperplane, so Y_p is smooth.

Now, we show that for general points $p_1, \ldots, p_{15}$ the smooth varieties Y_{p_i} intersect transversally. The group $\mathrm{PGL}(\mathbb{C}, 4)$ (as the group of automorphisms of $\mathbb{P}^3_{\mathbb{C}}$) acts transitively on itself (as the ambient space of Y_{p_i}) by precomposition. Hence, given an initial point p_1, for a general point p_2, the smooth varieties Y_{p_1} and Y_{p_2} intersect transversally by Kleiman's transversality theorem (see [11, Corollary 4] or [7, Theorem 1.7]). So $Y_{p_1} \cap Y_{p_2}$ is smooth of dimension $\dim(Y_{p_1}) - 1 = 13$ and, iterating the process for general points $p_3, \ldots, p_{15}$, we obtain that $Y_{p_1} \cap \cdots \cap Y_{p_{15}}$ is a smooth variety of dimension 0. That is, for general points $p_2, \ldots, p_{15}$, there is a finite number of $\varphi \in \mathrm{PGL}(\mathbb{C}, 4)$ satisfying (2) and this number is, by construction, the degree of Ω_f.

Formulations (1) and (2) both result in a system of 15 homogeneous cubic polynomials in the 16 unknowns $(a_{ij})_{1 \le i,j \le 4}$, but they have different computational advantages. To perform numerical homotopy continuation, it is beneficial to pass to an affine chart of projective space. This can be done in formulation (1) by fixing a coordinate, say adding the polynomial $a_{11} - 1 = 0$. But this introduces artificial solutions. For example, for every solution $\phi \in \mathbb{C}^{16}$, we have that $e^{i\frac{2}{3}\pi}\phi$ and $e^{i\frac{4}{3}\pi}\phi$ are also solutions. The formulation (2) does not produce these undesired artificial solutions. However, the formulation (1) is better suited for applying the trace test than (2). The reason is given in the following section.

3. Numerical Algebraic Geometry

Numerical algebraic geometry concerns numerical computations of objects describing algebraic sets defined over subfields of the complex numbers. The most basic of these objects are the *solution sets*, a data structure for representing solutions to polynomial systems. The term "numerical" refers to computations which are potentially inexact (e.g., floating-point arithmetic). However, this does not necessarily mean that the results obtained are unreliable. The certification of solutions plays an important role in the field. For a more in-depth definition and a brief history of numerical algebraic geometry see [9]. A comprehensive introduction to the subject is available in [18].

We now introduce the tools from numerical algebraic geometry needed to compute and certify the degree of the orbit closure. We fix a system of polynomials $F = (F_1, \dots, F_m)$ in n variables and assume that it has l isolated solutions $p_1, \dots, p_l \in \mathbb{C}^n$.

Homotopy Continuation. Numerical homotopy continuation [18, Section 8.4.1] is a fundamental method that underlies most of numerical algebraic geometry. The general idea is as follows. Suppose we want to compute the isolated solutions of F. We build a homotopy $H(x,t)$: $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^m$ which deforms a system of polynomials $G(x) = H(x,0)$ whose isolated solutions are known or easily computable into the system $F(x) = H(x,1)$. A well-defined homotopy requires that G has at least as many isolated solutions as F so that we are able to compute *all* isolated solutions of F. Given a solution x_0 of G, there is a solution path $x(t) : \mathbb{C} \longrightarrow \mathbb{C}^n$, which is a curve implicitly defined by the conditions $x(0) = x_0$ and $H(x(t),t) = 0$ for $t \in U \subseteq \mathbb{C}$ where U is the flat locus of the projection $\mathbb{C}^n \times \mathbb{C} \longrightarrow \mathbb{C}$ restricted to $H = 0$, which is dense in $\mathbb{C}$ by generic flatness. In particular, a well-defined homotopy requires $0 \in U$. The solution path is usually tracked using a predictor-corrector scheme. As t approaches 1 the solution path either diverges or converges to a solution of F.

A standard homotopy is the *total degree homotopy*. Bézout's theorem gives $N = \prod_{i=1}^m \deg(F_i)$ as an upper bound for the number of isolated solutions of F. A total degree homotopy uses a start system G with N isolated solutions and the homotopy $H(x,t) = (1-t)G(x) + tF(x)$. As the Bézout bound may be very high, for large computations the total degree homotopy is impractical and other methods are necessary.

Monodromy method. Monodromy (see [6, 13]) is an alternative method for finding isolated solutions to parameterized polynomial systems which is advantageous if the number of solutions is substantially lower than the Bézout bound. Embed our polynomial system F in a family of polynomial systems $\mathcal{F}_Q$, parameterized by a Zariski open subset Q of $\mathbb{C}^k$. Let l be the number of solutions of $F_q \in \mathcal{F}_Q$ for $q \in U$, where $U \subseteq Q$ is the flat locus of the family $\mathcal{F}_Q$.

Consider the incidence variety

$$Y := \left\{ (x,q) \in \mathbb{C}^n \times Q \mid F_q(x) = 0 \right\}.$$

Let π be the projection from $\mathbb{C}^n \times Q$ onto the second argument restricted to Y. For every $q \in U$, the fiber $Y_q = \pi^{-1}(q)$ has exactly l points. Given a loop O in U based at q, the preimage $\pi^{-1}(O)$ is a union of paths starting and ending at (possibly different) points of Y_q. So, giving a direction to the loop O, we may associate to each point y of Y_q the endpoint of the path starting at y. This defines an action, the *monodromy action*, of the fundamental group of U on the fiber Y_q, which in turn defines a map from the fundamental group of U to the symmetric group S_l. The *monodromy group* of our family at q is the image of such a map. This action is transitive if and only if Y is irreducible, which we assume.

Fix $q_0 \in U$ such that $F = F_{q_0} \in \mathcal{F}_Q$. Suppose a *start pair* (x_0, q_0) is given, that is, x_0 is a solution to the instance F_{q_0}. The start solution x_0 is numerically tracked along a directed loop in U, yielding a (possibly new) solution of F_{q_0} at the end. While a new solution is obtained, it is tracked along the *same* loop, yielding another possibly new solution.

Then, all solutions are tracked along a *new* loop, and the process is repeated until some stopping criterion is fulfilled.

We note that this method requires us to know one solution of our polynomial system to use as a start pair. Various strategies exist to find such a solution. We will describe one strategy in Section 4.

Certifying solutions. The above methods yield numerical approximations of solutions of our polynomial system F. How can we certify that the obtained approximations correspond to actual solutions of F and that they are all distinct? For systems F with an equal number n of polynomials and variables, Smale introduced the notion of an *approximate zero*, the α-number and the α-theorem, see [17]. In short, an approximate zero of F is any point $p \in \mathbb{C}^n$ such that Newton's method, when applied to p, converges quadratically to a zero of F. This means that the number of correct significant digits roughly doubles with each iteration of Newton's method.

Definition 3.1 (Approximate zero)**.** Let J_F be the $n \times n$ Jacobian matrix of F. A point $p \in \mathbb{C}^n$ is an *approximate zero* of F if there exists a zero $\zeta \in \mathbb{C}^n$ of F such that the sequence of Newton iterates

$$z_{k+1} = z_k - J_F(z_k)^{-1} F(z_k)$$

starting at $z_0 = p$ satisfies for all $k \geq 1$ that

$$\|z_k - \zeta\| \leq \left(\frac{1}{2}\right)^{2^k - 1} \|z_0 - \zeta\|^2.$$

If this holds, then we call ζ the *associated zero* of p. Here $\|x\|$ is the standard Euclidean norm in $\mathbb{C}^n$, and the zero ζ is assumed to be nonsingular (that is, $\det(J_F(\zeta)) \neq 0$).

To check whether a point $p \in \mathbb{C}^n$ is an approximate zero of F from Definition 3.1 requires infinitely many steps, one for each iteration of the Newton method. Nevertheless, when p is close enough to its associated zero, it is possible to certify that p is an approximate zero with only finitely many computations, as we now see. Smale's α-theorem (see [3, Theorem 4 in Chapter 8]) is an essential ingredient. The theorem uses the γ- and α-numbers

$$\begin{aligned} \gamma(F,x) &= \sup_{k\geq 2} \left\| \frac{1}{k!} J_F(x)^{-1} D^k F(x) \right\|^{\frac{1}{k-1}} \quad \text{and} \\ \alpha(F,x) &= \|J_F(x)^{-1} F(x)\| \cdot \gamma(F,x)\,, \end{aligned}$$

where D^kF is the tensor of order-k derivatives of F and the tensor $J_F^{-1} D^k F$ is understood as a multilinear map $A : (\mathbb{C}^n)^k \longrightarrow \mathbb{C}^n$ with norm $\|A\| := \max_{\|v\|=1} \|A(v,\ldots,v)\|$.

Theorem 3.2 (Smale's α-theorem)**.** If $\alpha(F,x) < 0.03$, then x is an approximate zero of F. Furthermore, if $y \in \mathbb{C}^n$ is any point with $\|y - x\|$ less than $(20\,\gamma(F,x))^{-1}$, then y is also an approximate zero of F with the same associated zero ζ as x.

Smale's α-theorem is more general than is stated above. The numbers 0.03 and 20 can be replaced by any pair of positive numbers satisfying certain constraints.

To avoid the computation of the γ-number Shub and Smale [16] derived an upper bound for $\gamma(F,x)$ which can be computed exactly and efficiently. Hence, one can decide algorithmically whether x is an approximate zero using only the data of the point x itself and F. Hauenstein and Sottile [10] implemented these ideas in an algorithm, called `alphaCertified`, which decides both whether a point $x \in \mathbb{C}^n$ is an approximate zero and whether two approximate zeros have distinct associated zeros.

Trace test The certification process explained above establishes a *lower* bound for the number of isolated solutions of F. The trace test can be used for polynomial systems satisfying certain conditions to show that *all* solutions have been found. See [12] for a more detailed explanation.

We first establish definitions of concepts used in the trace test. A *pencil of linear spaces* is a family M_t for $t \in \mathbb{C}$ of linear spaces that depends affinely on the parameter t. Each M_t is the span of a linear space L and a point t on a line disjoint from L. Suppose that $W \subset \mathbb{C}^n$ is an irreducible variety of dimension m of which we wish to verify the degree. Also suppose we are given a general pencil of linear spaces M_t for $t \in \mathbb{C}$ such that M_t is of codimension m for all t and W and M_0 intersect transversally. Fix a subset $W' \subseteq W \cap M_0$. We track W' along the pencil to obtain $W_t' \subseteq W \cap M_t$. Denote by $w(t)$ the sum of the points of W_t'. If $W_t' = W \cap M_t$ then $w(t)$ is the *trace* of $W \cap M_t$. The trace is an affine linear function of t [12, Proposition 3]. That is, there exist $a, b \in \mathbb{C}^n$ such that $w(t) = a + tb$. It can be shown that the sum of any proper subset of the points in $W \cap M_t$ is *not* an affine linear function of t.

This leads to the trace test: Let $t_1 \in \mathbb{C} \setminus \{0\}$, fix $W' \subseteq W \cap M_0$ and compute $\mathrm{tr}(t_1) := (w(t_1) - w(0)) - (w(0) - w(-t_1))$. Note that $\mathrm{tr}(t_1)$ is identically zero if and only if w is an affine linear function of t, which is true if and only if the cardinality of W' is the degree of W. Due to the generality assumption on M_t it is sufficient to compute $\mathrm{tr}(t_1)$ for only *one* $t_1 \in \mathbb{C} \setminus \{0\}$.

4. A Numerical Approach

In this section we explain our use of numerical algebraic geometry to obtain Theorem* 4.1 below. Reasonable mathematicians may differ as to whether it is appropriate to state this result as a theorem since we currently cannot certify the last step of our computation. We add the asterisk to acknowledge these differing opinions.

Theorem* 4.1. The degree of the orbit closure of a general cubic surface under the action of $\mathrm{PGL}(\mathbb{C}, 4)$ is 96120.

All computations performed to arrive at this result are available from the authors upon request.

To compute the degree of the orbit closure, we sample a general cubic surface $f \in \mathbb{P}^{19}_{\mathbb{C}}$ by drawing the real and imaginary parts of each of its coordinates independently from a univariate normal distribution. We then solve the polynomial system (2) encoding the enumerative geometry problem. A naive strategy is to sample 15 points $p_1, \ldots, p_{15} \in \mathbb{P}^3_{\mathbb{C}}$ in general position and use a total degree homotopy, but in this case the Bézout bound is $3^{15} = 14{,}348{,}907$. Here, the monodromy method is substantially more efficient.

To apply the monodromy method, we consider (2) as a polynomial system on the entries of φ parameterized by 15 points $p_1, \ldots, p_{15}$ in $\mathbb{P}^3_{\mathbb{C}}$. We consider the incidence variety

$$V = \{(\varphi, (p_1, \ldots, p_{15})) \in \mathrm{PGL}(\mathbb{C}, 4) \times (\mathbb{P}^3_{\mathbb{C}})^{15} \mid F(\varphi(p_i)) = 0, i = 1, \ldots, 15\}$$

and we denote by π the projection $\mathrm{PGL}(\mathbb{C},4)\times(\mathbb{P}^3_{\mathbb{C}})^{15}\longrightarrow(\mathbb{P}^3_{\mathbb{C}})^{15}$ restricted to V and by ρ the other projection restricted to V. The fiber of ρ over a point $\varphi\in\mathrm{PGL}(\mathbb{C},4)$ is the product of the cubic surface $\varphi\cdot f$ with itself in $(\mathbb{P}^3_{\mathbb{C}})^{15}$, hence V is irreducible.

Note that when we numerically solve the equations of V, we are considering an equivalent incidence variety W but in $\mathbb{P}^{15}_{\mathbb{C}}\times(\mathbb{P}^3_{\mathbb{C}})^{15}$, where $\mathbb{P}^{15}_{\mathbb{C}}$ is the projective space of all non-zero 4-by-4 matrices, a compactification of $\mathrm{PGL}(\mathbb{C},4)$. The closure $\overline{V}$ of V in $\mathbb{P}^{15}_{\mathbb{C}}\times(\mathbb{P}^3_{\mathbb{C}})^{15}$ is an irreducible component of W. If we start the monodromy method at some point of $\overline{V}$, then we will only find solutions in this irreducible component of W and thus we do not need to encode the condition $\varphi\in\mathrm{PGL}(4,\mathbb{C})$ explicitly.

Our strategy is to find a start pair $(\varphi_0;p_1,\dots,p_{15})\in V$ and then to use the monodromy action on the fiber $\pi^{-1}(p_1,\dots,p_{15})$ to find all solutions in this fiber. A start pair can be found by exchanging the role of variables and parameters. First, we sample a $\varphi_0\in\mathrm{PGL}(4,\mathbb{C})$ and the first three coordinates of 15 points $p_i\in\mathbb{P}^3_{\mathbb{C}}$ in general position. This yields a system of 15 polynomials each depending only on one variable: The ith polynomial depends only on the fourth coordinate of p_i. Such a system is easy to solve. Solving it yields a start pair $(\varphi_0;p_1,\dots,p_{15})\in V$, on which we run the monodromy method implemented in the software package `HomotopyContinuation.jl` [4]. In less than an hour on a single core, this method found 96120 approximate solutions corresponding to the start points $p_1,\dots,p_{15}\in\mathbb{P}^3_{\mathbb{C}}$.

Next we apply Smale's α-theory as implemented in the software `alphaCertified` [10] to certify two conditions of our numerical approximations: First, we show that each numerical approximation is indeed an approximate zero of our original polynomial system, and second that all 96120 approximate zeros have distinct associated zeros. Due to computational limits we were only able to obtain a certificate using (arbitrary precision) floating point arithmetic. Hauenstein and Sottile call this a "soft" certificate since it does not eliminate the possibility of floating point errors. It is preferable to use rational arithmetic for certification, but for a system of our size too much time is required to perform such a computation.

The certification process establishes a *lower* bound on the degree of the orbit closure. As a last step, we run a trace test to verify that we have indeed found *all* solutions. The trace test described in the previous section is only applicable to subvarieties of $\mathbb{P}^n_{\mathbb{C}}$. In [12] the authors derive a trace test to certify the completeness of a *collection of partial multihomogeneous witness sets*. Our formulation (2) is multihomogeneous but our computations provide only one partial multihomogeneous witness set, namely $\pi^{-1}(p_1,\dots,p_{15})$, and not the entire collection that would be necessary to run a trace test. To avoid these complications, we use formulation (1). We note that it is straightforward to construct a linear subspace L from the 15 points $p_1,\dots,p_{15}$ such that our solutions from the monodromy computation are also solutions to (1), so translating formulation (2) to (1) is not difficult.

In the language of numerical algebraic geometry our 96120 solutions together with the linear subspace L constitute a *pseudo witness set* [8]. We construct a general pencil M_t of linear spaces with $M_0 = L$. Working with approximate solutions refined to around 38 digits of accuracy we obtain for $\mathrm{tr}(1)$ a vector with norm of approximately 10^{-32}. Additionally, increasing the accuracy of the solutions decreases the norm of the trace test result. While this gives us very high certainty that we indeed obtained all solutions, we do not have a rigorous certificate that the trace test converges to zero when we increase the accuracy of the solutions. A certification of the trace test similar to Smale's α-theory for numerical solutions remains an important open problem.

From the described computations we conclude that degree of the orbit closure of a general cubic surface under the action $\mathrm{PGL}(\mathbb{C},4)$ is 96120.

We note that as a test of our methods, we confirmed known degrees of other varieties. In agreement with a theoretical result of Aluffi and Faber [1], we computed that the degree of the orbit closure of a general quartic curve in the plane is 14280. Additionally we computed that the degree of the orbit closure of the Cayley cubic, defined by the equation $yzw + xzw + xyw + xyz = 0$, is 305. Due to the symmetry of the variables in the Cayley cubic, there are 4! matrices corresponding to every polynomial in the orbit. As expected, we computed $7320 = 4! \cdot 305$ solutions. This coincides with a theoretical result of Vainsencher [19].

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