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TOWARDS THE DEGREE OF THE PGL(4)-ORBIT OF A CUBIC SURFACE

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We study the action of the group $\mathrm{PGL}(4)$ on the parameter space $\mathbb{P}^{19}$ of complex cubic surfaces. Specifically, we look at how the techniques used by Aluffi and Faber in [1] can be extended to compute the degree of the orbit closure $\overline{O}$ of a general cubic surface. We study the base locus of the induced rational map $\mathbb{P}^{15} \dashrightarrow \overline{O} \subset \mathbb{P}^{19}$, and the first steps in resolving this rational map by successively blowing up the reduced base locus.

1. Introduction

A complex cubic surface $\mathcal{S}$ in $\mathbb{P}^3$ is the vanishing locus of a homogenous degree-3 form of the type

$$F(\mathbf{x}) = a_0 x_0^3 + a_1 x_0^2 x_1 + \cdots + a_{19} x_3^3.$$

It is clear that cubic surfaces are parametrized by $\mathbb{P}\,\mathrm{Sym}^3\,(\mathbb{C}^4)^* \simeq \mathbb{P}^{19}$. However, two isomorphic surfaces correspond to different points in $\mathbb{P}^{19}$ and the simplest way this can happen is when changing coordinates. A natural question would then be: *Given a fixed $\mathcal{S}$ as above, which other cubic surfaces arise from $\mathcal{S}$ by coordinate change?* In other words, we are asking to describe the orbit O of $\mathcal{S}$ under the action of the group $\mathrm{PGL}(4)$ on the parameter space $\mathbb{P}^{19}$.

We would like to study the geometry of O: since this just forms a locally closed subset in $\mathbb{P}^{19}$, we will rather consider its closure $\overline{O}$. A first step in this

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direction is to compute its degree. The latter will depend on the choice of the surface $\mathcal{S}$ and in this paper we will primarily focus on the case where $\mathcal{S}$ is chosen to be general.

In the special case of the Cayley cubic, a surface with four distinct nodes, the degree of the orbit closure is already known. In [7], this number is computed to be 305, based on counting cubic surfaces with 4 distinct double points passing through 15 general points.

When $\mathcal{S}$ is general, the degree of the orbit closure will be significantly higher and different techniques will be needed. One could start by looking at the map $\phi\colon \mathrm{PGL}(4) \to \mathbb{P}^{19}$, sending the class of a matrix to its pre-composition with F. The image of this map is the orbit O and computing the degree of its closure would amount to count the number of points in the intersection of $\overline{O}$ with a general linear subspace of complementary dimension.

We can count the number of such points using intersection theory by finding a pair $(\widetilde{\mathcal{V}}, \widetilde{\phi})$ such that $\widetilde{\mathcal{V}}$ is a compactification of $\mathrm{PGL}(4)$ and $\widetilde{\phi}$ a dominant regular morphism from $\widetilde{\mathcal{V}}$ to $\overline{O}$ extending ϕ and the intersections of the pull-back of a hyperplane class by $\widetilde{\phi}$ is transversal. Then we can simply compute $\widetilde{\phi}^* c_1(\mathcal{O}_{\mathbb{P}^{19}}(1))^{15}$.

The first naïve compactification one could think of is $\mathbb{P}\mathrm{Hom}(\mathbb{C}^4, \mathbb{C}^4) \simeq \mathbb{P}^{15}$, which can be as well equipped with the pre-composition map which naturally extends ϕ. Unfortunately, this pair is not good enough since the given map is not regular. From a computational viewpoint, issues come from the fact that the pull-back classes we are considering will intersect in positive dimension.

The strategy that we would like to pursue here is to find an explicit resolution of ϕ where it is possible to keep track of how the intersections change. This approach was already considered by Aluffi and Faber who studied the case of plane curves of any degree. What we are going to do in the present paper is to adapt many of the ideas contained in there. In particular, we decide to regularize ϕ by a sequence of blow-ups at smooth centers. We will start by describing the support of the base locus $\mathrm{Bs}(\phi)$ from a set-theoretical point of view. We will then study the first steps towards the resolution of ϕ by successively blowing up the reduced components of the base locus.

Four of these steps will be analyzed, though currently it is not clear if they will be sufficient to give the desired resolution. This difficulty reflects an important difference from the case of plane curves: here the base locus of ϕ has many components, and this is a consequence of the fact that a general cubic surface contains 27 distinct lines. More specifically, we will see that problems can possibly arise from those morphisms in $\mathbb{P}^{15}$ whose image is spanned by a point contained in one of these lines.

The aim of this paper is to present a report of an on-going project, where the remaining work that needs to be done regards not only proving or disproving the existence of further components to blow up. Indeed, as mentioned above, there is also a computational aspect, namely showing how the different steps in the resolution contribute in finding the degree of $\overline{O}$. These computations will not be analyzed here, since the results would currently be very partial. They are hopefully going to appear in a future paper, as the natural conclusion of the work illustrated here. For this second part as well, we believe that a considerable inspiration could be taken by the techniques developed in [1].

An alternative approach to the same problem has been recently explored by Brustenga i Moncusí, Timme and Weinstein in [3]. There is however a significant difference between the methods. Indeed, in their paper the computation of the degree of the orbit closure is treated from a more numerical perspective. The idea is to count the number of solutions of a system of polynomial equations in an affine variety using homotopy continuation and monodromy methods. As a result, for a general $\mathcal{S}$, this number turns out to be 96120. On the other hand, applying intersection theory in the context of resolutions of singularities gives a more geometric flavor and we believe that this will help to shed some light on a complete understanding on the studied phenomena.

The problem was firstly introduced to us from the *27 Questions on Cubic Surfaces* (see [6]), in view of the First Meeting on Cubic Surfaces, that was held in Oslo on May 13, 2019. We would like to thank: Kristian Ranestad and Corey Harris for the valuable discussions and the patience with the many questions, Paolo Aluffi for very nice explanations about his paper [1], Maddie Weinstein for stimulating conversations, the anonymous referees for all the corrections and suggestions.

2. Setup

In this section we will first describe the action of $\mathrm{PGL}(4)$ on the parameter space of cubic surfaces. This will naturally produce rational maps

$$\mathbb{P}^{15} \simeq \mathbb{P}\mathrm{Hom}(W,W) \dashrightarrow \mathbb{P}\mathrm{Sym}^3(W^*) \simeq \mathbb{P}^{19},$$

one for every fixed cubic. If the latter is chosen to be general, it will be possible to illustrate how this map can be used to compute the degree of the orbit closure. Throughout the paper we will work over the field $\mathbb{C}$ of complex numbers.

2.1. The action of $\mathrm{PGL}(4)$

Let us denote with W the 4-dimensional vector space $\mathbb{C}^4$. A complex cubic surface $\mathcal{S} \subset \mathbb{P}W$ is the zero set of a homogeneous degree-3 polynomial in four variables, say

$$F(\mathbf{x}) = a_0x_0^3 + a_1x_0^2x_1 + \cdots + a_{18}x_2x_3^2 + a_{19}x_3^3,$$

which corresponds to a point $[F] := [a_0 : a_1 : \cdots : a_{19}]$ in the parameter space $\mathcal{F} := \mathbb{P}\mathrm{Sym}^3(W^*)$. The group PGL(4) acts on $\mathcal{F}$ by pre-composition (or, equivalently, by coordinate change):

$$\begin{aligned} \mathrm{PGL}(4) \times \mathcal{F} &\to \mathcal{F} \\ (\alpha, [F(\mathbf{x})]) &\mapsto [F(\alpha(\mathbf{x}))] \end{aligned}$$

For a fixed $\mathcal{S}$ (hence for a fixed F), this yields a map $\phi\colon \mathrm{PGL}(4) \to \mathcal{F}$, whose image is by definition the orbit O of F. Moreover, the fiber $\phi^{-1}(F)$ is the set of automorphisms of W leaving F unchanged, so it corresponds to group of linear automorphisms of $\mathcal{S}$. Our object of study is the degree of $\overline{O}$ in $\mathcal{F}$: to this purpose, we first need to understand the dimension of O and the degree of ϕ.

Let us denote with $\mathcal{V}$ the space $\mathbb{P}\mathrm{Hom}(W,W)$ of nonzero endomorphisms of W up to projective equivalence, which is also canonically isomorphic to the space of matrices $\mathbb{P}(W^* \otimes W)$.

Lemma 2.1. *Let $\mathcal{S}$ be a cubic surface with finite group of linear automorphisms. Then* $\dim \overline{O} = 15$.

Proof. By hypothesis ϕ is a finite map, so $\dim O = \dim \mathrm{PGL}(4)$. But $\dim O = \dim \overline{O}$ and PGL(4) embeds as an open subset $\mathcal{V}$, whose dimension is 15. □

From now on we will consider $\mathcal{S}$ to be *general*, meaning that its corresponding point in $\mathcal{F}$ lies in some proper Zariski open subset.

Lemma 2.2. *If $\mathcal{S}$ is a general cubic surface, the above map ϕ has degree* 1.

Proof. We will prove that each fiber of ϕ consists of a single point. Suppose that there exist two points α_1, α_2 of PGL(4) with the property that $F(\alpha_1(\mathbf{x})) = F(\alpha_2(\mathbf{x}))$: then the composite $\alpha_1^{-1} \circ \alpha_2$ would be a linear automorphism of $\mathcal{S}$. But a general cubic surface has no nontrivial linear automorphisms (see [5]), so $\alpha_1 = \alpha_2$. □

2.2. How to compute the degree of $\overline{O}$

As we noticed in the proof of Lemma 2.1, we can see PGL(4) as an open subset of $\mathcal{V}$; in particular the map ϕ can be understood as a rational map $\mathcal{V} \dashrightarrow \mathcal{F}$, which, by abuse of notation, we will keep calling ϕ. The strategy from [1] that we want to apply here is to resolve ϕ by a sequence of blow-ups in $\mathcal{V}$ and finally get a regular map $\widetilde{\phi}\colon \widetilde{\mathcal{V}} \to \mathcal{F}$, where $\widetilde{\mathcal{V}}$ is a smooth compactification of PGL(4) and $\operatorname{im}\widetilde{\phi} = \overline{O}$. The blow-ups will also produce a morphism $\pi\colon \widetilde{\mathcal{V}} \to \mathcal{V}$, such that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{PGL}(4) & \hookrightarrow & \widetilde{\mathcal{V}} & \xrightarrow{\widetilde{\phi}} & \mathcal{F} \\ \| & & \downarrow \pi & & \| \\ \mathrm{PGL}(4) & \hookrightarrow & \mathcal{V} & \overset{\phi}{\dashrightarrow} & \mathcal{F} \end{array}$$

Given this construction, we can compute the degree d of $\overline{O}$ as follows: let $\widetilde{\phi}_*\colon \mathrm{CH}(\widetilde{\mathcal{V}}) \to \mathrm{CH}(\mathcal{F})$ be the push-forward map between the corresponding Chow rings and let us recall that $\overline{O}$ is a 15-dimensional subvariety of $\mathcal{F}$. Then by definition $d = \int_{\mathcal{F}} [\overline{O}] \cdot H^{15}$, where $\int_{\mathcal{F}}(\cdot)$ denotes the degree of the 0-dimensional part, while H denotes the hyperplane class in $\mathrm{CH}(\mathcal{F}) \simeq \mathbb{Z}[H]/H^{16}$. On the other hand, by construction $\widetilde{\mathcal{V}}$ dominates $\overline{O}$, so $\deg \widetilde{\phi} \cdot [\overline{O}] = \widetilde{\phi}_*(1)$. Then, using the projection formula, we find:

$$\deg \widetilde{\phi} \cdot d = \int_{\mathcal{F}} \widetilde{\phi}_*(1 \cdot \widetilde{\phi}^* H^{15}) = \int_{\widetilde{\mathcal{V}}} \widetilde{\phi}^* H^{15}. \tag{1}$$

Definition 2.3. With notation as above, we define the *predegree* of $\overline{O}$ to be $\int_{\widetilde{\mathcal{V}}} (\widetilde{\phi}^*(H))^{15}$.

Note that, even when $\mathcal{S}$ has nontrivial linear automorphisms, it is possible to use equation (1) to find the degree of $\overline{O}$ by dividing the predegree by the order of the group of linear automorphisms. In the general case we have the following result:

Proposition 2.4. *For a general cubic surface $\mathcal{S}$, the degree of $\overline{O}$ equals its predegree.*

Proof. Indeed, thanks to Lemma 2.2, we know that if $\mathcal{S}$ is general, $\deg \widetilde{\phi} = 1$; then the expression (1) gives the desired equality. □

The first step towards the resolution of ϕ is to understand its base locus $\mathrm{Bs}(\phi)$. To this purpose we note that the linear system defining ϕ is spanned by a certain set of hypersurfaces having a nice geometric interpretation.

Definition 2.5. Let $\mathcal{S} = V(F)$ be a cubic surface in $\mathbb{P}W$. For every $p \in \mathbb{P}W$, the *point condition* P_p is defined as:

$$P_p = \{\alpha \in \mathcal{V} \mid F(\alpha(p)) = 0\}$$

i.e. the zero locus of $F(\alpha(p))$ as a polynomial in α.

Since the point conditions span the linear system defining ϕ, the base locus $\mathrm{Bs}(\phi)$ can be identified with the intersection $\bigcap_{p\in\mathbb{P}W} P_p$. After blowing up this locus in $\mathcal{V}$, we will get a new rational map, whose base locus will be described by the intersection of the proper transforms of the point conditions, and so on. Moreover, if we denote by $\widetilde{P}_p$ the proper transform of P_p in $\widetilde{\mathcal{V}}$, we see that $d = \int_{\widetilde{\mathcal{V}}} [\widetilde{P}_p]^{15}$.

Although the main focus of this paper is to illustrate the several steps needed to resolve ϕ, we would like to mention here a very important proposition, which can be (repeatedly) used to tell how the various blow-ups contribute in the computation of the degree of $\overline{O}$.

Proposition 2.6 ([1, Proposition 3.2]). *Let $i\colon B \to V$ be an inclusion of nonsingular projective varieties, and let $X \subset V$ be a codimension-1 subvariety, smooth along B. Let $\widetilde{V}$ be the blow-up of V along B, and let $\widetilde{X}$ be the proper transform of X. Then*

$$\int_{\widetilde{V}} [\widetilde{X}]^{\dim V} = \int_V [X]^{\dim V} - \int_B \frac{([B] + i^*[X])^{\dim V}}{c(N_{B/V})},$$

where $c(N_{B/V})$ denotes the total Chern class of the normal bundle of B in V.

In our situation, the role of V and X will be played by $\mathcal{V}$ and P_p, while B will represent each time a component of the reduced base locus that we are blowing up. Since the point-conditions are cubic hypersurfaces in $\mathcal{V}$, we have $\int_{\mathcal{V}} [P_p]^{15} = 3^{15}$. Then the degree of $\overline{O}$ will be $3^{15} - n_1 - \cdots - n_k$, where the n_i's are the contributions of the blown up loci that can be explicitly computed using Proposition 2.6.

At each step, the most difficult part will be to compute $c(N_{B/\mathcal{V}})$ in the Chow ring $\mathrm{CH}(B)$. This motivates us to look for a resolution, by picking a suitable sequence of blow-ups that allows to handle this computation easily.

The contributions coming from the sequence of blow-ups is left for a future paper, that is thought to be the natural continuation of the present one.

3. Towards the resolution of ϕ

In this section, we will describe the first steps necessary to regularize ϕ according to the strategy described in 2.2. It is not yet clear if these are enough or if more blow-ups are required. An important difference from the case of plane curves studied in [1] is that the base locus $\mathrm{Bs}(\phi)$ has not only one, but many components, reflecting the fact that a general cubic surface contains 27 distinct lines.

3.1. The base locus of ϕ

With the next proposition we are going to describe $\mathrm{Bs}(\phi)$ as a set. To this purpose, we look at $\mathcal{V}$ as the space of matrices $\mathbb{P}(W^* \otimes W)$, together with the Segre embedding

$$\mathbb{P}W^* \times \mathbb{P}W \hookrightarrow \mathbb{P}(W^* \otimes W)$$

given by

$$([k_0 : \cdots : k_3], [q_0 : \cdots : q_3]) \mapsto \begin{bmatrix} k_0q_0 & k_1q_0 & k_2q_0 & k_3q_0 \\ k_0q_1 & k_1q_1 & k_2q_1 & k_3q_1 \\ k_0q_2 & k_1q_2 & k_2q_2 & k_3q_2 \\ k_0q_3 & k_1q_3 & k_2q_3 & k_3q_3 \end{bmatrix},$$

where $k^\perp := \{\mathbf{x} \in \mathbb{P}W \mid k_0x_0 + \cdots + k_3x_3 = 0\}$ is the kernel of such a matrix and $q := [q_0 : \cdots : q_3]$ its image.

Proposition 3.1. *Let $\mathcal{S} = V(F)$ be a general smooth cubic surface in $\mathbb{P}W$. Let ϕ be the map defined above. Then $\mathrm{Bs}(\phi)$ is supported at the union of two closed components B and C, with:*

(i) $B \simeq \mathbb{P}W^* \times \mathcal{S}$;

(ii) $C \simeq \cup_{i=1}^{27} C_i$,

where the C_i's are the irreducible components of C and each C_i is isomorphic to $\mathbb{P}^7$.

Proof. The map ϕ is not defined over the set

$$\{\alpha \in \mathcal{V} \mid F(\alpha(\mathbf{x})) \equiv 0\} = \{\alpha \in \mathcal{V} \mid \mathrm{im}\,\alpha \subset V(F)\}.$$

Since $\mathcal{S}$ is taken to be general, its linear subspaces are points in $\mathcal{S}$ and the 27 lines, that we denote by $\ell_1, \ldots, \ell_{27}$. We can write the base locus as

$$B \cup C,$$

where

$$B := \{\alpha \in \mathcal{V} \mid \mathrm{rk}\,\alpha = 1, \mathrm{im}\,\alpha \in \mathcal{S}\},$$
$$C := \{\alpha \in \mathcal{V} \mid \mathrm{rk}\,\alpha \leq 2, \mathrm{im}\,\alpha \subseteq \ell_i \text{ for some } i\}.$$

(i) The matrices in B are parametrized by the choice of a point in $\mathcal{S}$ and the choice of a 4-tuple of coefficients in $\mathbb{P}W^*$ (indeed each column must be a multiple of the chosen point). Hence $B \simeq \mathbb{P}W^* \times \mathcal{S}$.

(ii) Regarding C, it consists of 27 components $\{C_i\}_{i=1}^{27}$, where C_i is the space of matrices whose image is spanned by ℓ_i. So for every i we can make the identification $C_i \simeq \mathbb{P}\mathrm{Hom}(W,U)$, where U is the 2-dimensional subspace of W for which $\mathbb{P}(U) = \ell_i$. This is a 7-dimensional projective linear space and in particular we get $C \simeq \cup_{i=1}^{27} \mathbb{P}^7$.

□

Remark 3.2. Alternatively, one can see the the elements of a fixed C_i as the sum of two rank-1 matrices parametrized by the choice of a point on the given line and the choice of a 4-tuple of coefficients in $\mathbb{P}W^*$. In other words, C_i is the union of the span of all pairs of points in $\mathbb{P}W^* \times \ell_i$ (including the degenerate case in which the two points coincide), so we are describing the secant variety $\mathrm{Sec}_2(\mathbb{P}W^* \times \ell_i) \simeq \mathrm{Sec}_2(\mathbb{P}^3 \times \mathbb{P}^1)$, which is a $\mathbb{P}^7$.

Remark 3.3. The subset, $\mathrm{PGL}(4) \subset \mathcal{V}$ does not intersect $\mathrm{Bs}(\phi)$, so as we resolve the rational map ϕ, we still get compactifications of $\mathrm{PGL}(4)$.

Remark 3.4. The above proof actually says more: the two components B and C intersect in

$$B \cap C = \{\alpha \in \mathcal{V} \mid \mathrm{rk}\,\alpha = 1, \mathrm{im}\,\alpha \text{ is a point on } \ell_i \text{ for some } i\}.$$

In particular, this implies the following Corollary.

Corollary 3.5. *Let $C_i, i = 1,\ldots,27$ be the components of C, each isomorphic to $\mathbb{P}^7$. Then*

$$B \cap C_i \simeq \mathbb{P}W^* \times \ell_i.$$

Moreover, for $i \neq j$ we have

$$C_i \cap C_j \simeq \begin{cases} \mathbb{P}W^* & \text{if } \ell_i \cap \ell_j \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

As we have mentioned at the end of Section 2, since $\mathrm{Bs}(\phi)$ has many components, there are many ways of resolving the map. The following order of blow-ups at smooth centers is suited for relating the base loci of the induced maps to properties of point conditions in $\mathcal{V}$.

We start by blowing up $\mathcal{V}$ along the component $B \simeq \mathbb{P}W^* \times \mathcal{S}$: this produces a morphism $\pi_1 : \mathcal{V}_1 \to \mathcal{V}$ and an exceptional divisor $E_1 \subset \mathcal{V}_1$. After blowing up B, the proper transforms of the point condition, denoted by $P_p^{(1)}$, will define a new rational map $\phi_1 : \mathcal{V}_1 \dashrightarrow \mathcal{F}$. Note that $B \cap \mathrm{PGL}(4) = \emptyset$ in $\mathcal{V}$, so $\mathcal{V}_1$ contains an open dense subset isomorphic to $\mathrm{PGL}(4)$ and with a little abuse of notation we will indicate it using the same symbol. Let us denote with $C_i^{(1)}$ the proper transform of C_i in $\mathcal{V}_1$ for every i.

Claim 3.6. The base locus $\mathrm{Bs}(\phi_1)$ is supported on the 27 components $C_i^{(1)}$'s, which are disjoint, plus a further component, denoted by B_1, contained in the exceptional divisor E_1, intersecting the $C_i^{(1)}$'s.

We will choose B_1 to be the center of the second blow-up. As before, this will produce a new morphism $\pi_2\colon \mathcal{V}_2 \to \mathcal{V}_1$, together with an exceptional divisor $E_2 \subset \mathcal{V}_2$. Again, the proper transforms of the point conditions, denoted by $P_p^{(2)}$, will define a rational map $\phi_2\colon \mathcal{V}_2 \dashrightarrow \mathcal{F}$.

Claim 3.7. The support of $\mathrm{Bs}(\phi_2)$ contains the 27 pairwise disjoint proper transforms $C_i^{(2)}$'s and a subvariety, denoted by B_2, which has a dominant 2:1 map to B.

Note that it is not clear whether the subvariety B_2 is irreducible or not. What we will prove is that it must consist of either 1 or 2 components. Moreover, we need to observe that Claim 3.7 refers to an inclusion, but not an equality, so there might be some other components in $\mathrm{Bs}(\phi_2)$, namely the ones dominating the intersections $B \cap C_i \simeq \mathbb{P}W^* \times \ell_i$.

If we assume that we have exactly the components listed in above, we can proceed by blowing up B_2. We get as usual a map $\pi_3\colon \mathcal{V}_3 \to \mathcal{V}_2$, an exceptional divisor $E_3 \subset \mathcal{V}_3$ and a rational map $\phi_3\colon \mathcal{V}_3 \dashrightarrow \mathcal{V}$ induced by the proper transforms of point conditions. We expect no component of the base locus of ϕ_3 to dominate B. In fact, one might hope that the only components of $\mathrm{Bs}(\phi_3)$ are the $C_i^{(3)}$, and that blowing up these components resolves the rational map.

We summarize the construction in Figure 1, which also fixes notation for the rest of the section.

3.2. The base locus after blowing up B

We now aim to prove Claim 3.6; in particular, we are interested in giving the set-theoretical description of $B_1 := E_1 \cap \mathrm{Bs}(\phi_1)$.

To this purpose, we recall that B is embedded in $\mathcal{V}$ via the Segre embedding. In particular, for every $\alpha \in \mathcal{V}$, we may identify the space $T_\alpha\mathcal{V}$ with the quotient $(W^* \otimes W)/\alpha\mathbb{C}$. Let $\alpha = (k,q)$ be a point in B and let us denote with $\sigma = T_q\mathcal{S}$ the tangent space of $\mathcal{S}$ at the point q.

Lemma 3.8. *With the identification $T_\alpha\mathcal{V} \simeq (W^* \otimes W)/\alpha\mathbb{C}$, we have:*

(i) $T_\alpha B = \{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \sigma,\ \tau(k^\perp) \subset q\}/\alpha\mathbb{C}$.

(ii) $T_\alpha(\mathbb{P}W^* \times \ell_i) = \{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \ell_i, \tau(k^\perp) \subset q\}/\alpha\mathbb{C}$.

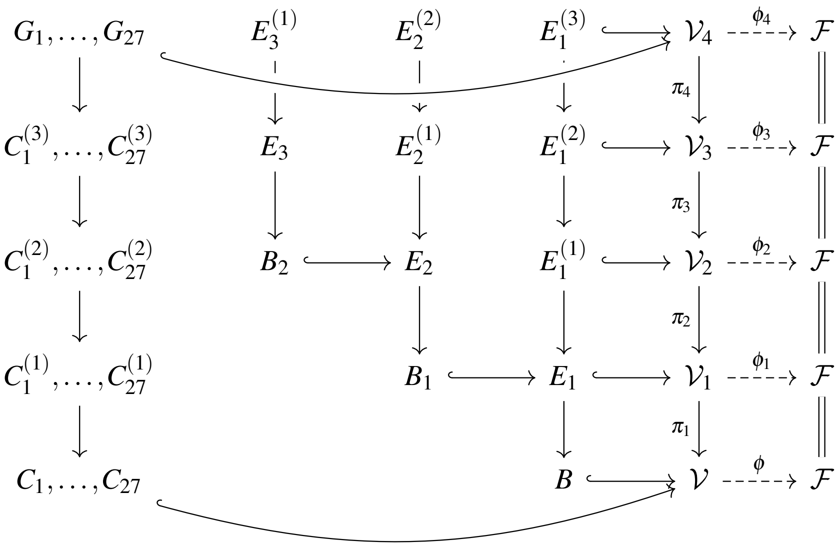

Figure 1: The sequence of blow-ups

(iii) The point condition P_p is non-singular at α and

$$T_\alpha P_p = \{\tau \in W^* \otimes W \mid \tau(p) \subset \sigma\}/\alpha\mathbb{C}.$$

Proof. The ideas in this proof are essentially the same of [1, Lemma 2.1].

(i) The (5-dimensional) tangent space of B at α is

$$\begin{aligned} T_\alpha B &= T_k(\mathbb{P}W^* \times \{q\}) \oplus T_q(\{k\} \times \mathcal{S}) \\ &= \frac{\{k' \otimes q \in W^* \otimes W \mid k' \in \mathbb{P}W^*\}}{k\mathbb{C}} \oplus \frac{\{k \otimes q' \in W^* \otimes W \mid q' \in \sigma\}}{q\mathbb{C}} \\ &= \frac{\{\tau \in W^* \otimes W \mid \operatorname{im}\tau = q\}}{k\mathbb{C}} \oplus \frac{\{\tau \in W^* \otimes W \mid \ker\tau = k^\perp,\ \operatorname{im}\tau \subset \sigma\}}{q\mathbb{C}}. \end{aligned}$$

The two spaces in the direct sum decomposition are both contained in the space

$$\frac{\{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \sigma,\ \tau(k^\perp) \subset q\}}{(k \otimes q)\mathbb{C}},$$

which is also of dimension 5, so they coincide.

(ii) Similarly we obtain the description for $T_\alpha(\mathbb{P}W^* \times \ell_i)$.

(iii) A line passing through α can be written as $\gamma_\alpha(s) = \alpha + \tau s$, for some $\tau \in \mathcal{V}$. Note that since $\operatorname{im}\alpha = q \in \mathcal{S}$, then $F(\gamma_\alpha(0)(p)) = F(\alpha(p)) = 0$. The intersection multiplicity $m_\alpha(P_p \cdot \gamma)$ is by definition the order of vanishing

$$\operatorname{ord}_{t=0}[F((\alpha+\tau s)(p))],$$

so the line γ is tangent to P_p if and only if that order is greater or equal than 2. By taking the Taylor expansion we get

$$F((\alpha+\tau s)(p)) = F(\alpha(p)) + \sum_{i=0}^{3} \left(\frac{\partial F}{\partial x_i}\right)_{\alpha(p)} \tau_i(p)s + \dots,$$

where $\tau_i(p)$ denotes the i-th coordinate of $\tau(p)$. Hence we need the constant and the linear term of this expression to vanish. We already know that $F(\alpha(p)) = 0$, while $\sum_i \left(\frac{\partial F}{\partial x_i}\right)_q \tau_i(p) = 0$ if and only if $\tau(p) \subset \sigma$, that is exactly the condition we claimed. The above computation says more: if $\tau(p) \not\subset \sigma$, then the line $\alpha + \tau s$ intersects P_p with multiplicity 1 at α, so P_p is non-singular at α.

□

We will also need a similar lemma describing various tangent spaces at points of C_i.

Lemma 3.9. *For every point $\alpha \in C_i$, we have:*

(i) $T_\alpha C_i = \{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \ell_i\}/\alpha\mathbb{C}$.

(ii) $T_\alpha P_p = \{\tau \in W^* \otimes W \mid \tau(p) \subset T_{\alpha(p)}\mathcal{S}\}/\alpha\mathbb{C}$.

Proof. (i) Since each $C_i \simeq \mathbb{P}^7$ is embedded in $\mathcal{V}$ as a linear space, if we identify the C_i with nonzero matrices with image in ℓ_i, then the tangent space to this linear space at any point is simply the linear space itself, i.e. the matrices with image in ℓ_i.

(ii) Exactly as the proof of Lemma 3.8(iii)

□

Lemma 3.10. *After blowing up B_1, the $C_i^{(1)}$ are all disjoint in $\mathcal{V}_1$.*

Proof. Recall that E_1 is defined as $\mathbb{P}(N_{B/\mathcal{V}})$, with $N_{B/\mathcal{V}} \simeq T\mathcal{V}/TB$. Then the intersection $C_i^{(1)} \cap E_1$ is the projectivization of the image of TC_i via the composition

$$TC_i \hookrightarrow T\mathcal{V} \to T\mathcal{V}/TB.$$

We need to prove that if C_i and C_j intersect in $\mathcal{V}$, then $C_i^{(1)}$ and $C_j^{(1)}$ are disjoint in the blow-up $\mathcal{V}_1$. We can check this fiberwise and show that for every $\alpha \in C_i \cap C_j$, the images of $T_\alpha C_i$ and $T_\alpha C_j$ in $T_\alpha \mathcal{V}/T_\alpha B$ do not intersect.

First observe that blowing up $\mathcal{V}$ along B affects C_i as if it was blown up along $\mathbb{P}W^* \times \ell_i$, producing an exceptional divisor $F_i := \mathbb{P}\left(\frac{TC_i}{T(\mathbb{P}W^* \times \ell_i)}\right)$, embedded in E_1. We may therefore instead prove that the image of $C_j^{(1)}$ in F_i is the empty set, i.e. that $T_\alpha C_i \cap \langle T_\alpha C_j, T_\alpha(\mathbb{P}W^* \times \ell_i)\rangle$ is contained in $T_\alpha(\mathbb{P}W^* \times \ell_i)$. Write q for the intersection $\ell_i \cap \ell_j$ and σ for $T_q\mathcal{S}$. Knowing the description of the tangent spaces in Lemma 3.8 and Lemma 3.9 and recalling that the two lines ℓ_i, ℓ_j span σ, we obtain

$$\langle T_\alpha C_j, T_\alpha(\mathbb{P}W^* \times \ell_i)\rangle = \{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \sigma,\ \tau(k^\perp) \subset \ell_j\}/\alpha\mathbb{C}.$$

Intersecting this span with $T_\alpha C_i$ we obtain exactly the tangent space $T_\alpha(\mathbb{P}W^* \times \ell_i)$, so $C_i^{(1)}$ and $C_j^{(1)}$ are disjoint in the blow-up. □

The tangent spaces appearing in Lemma 3.8 are also essential to describe the base locus of ϕ_1.

Proposition 3.11. *The base locus* $\mathrm{Bs}(\phi_1)$ *of the rational map* $\phi_1 \colon \mathcal{V}_1 \dashrightarrow \mathcal{F}$ *is supported on*

$$B_1 \cup \{C_1^{(1)}, \dots, C_{27}^{(1)}\},$$

where B_1 *is a* $\mathbb{P}^5$*-subbundle of* E_1*. Moreover,* $B_1 = (\bigcap_{p \in \mathbb{P}W} P_p^{(1)}) \cap E_1$ *both set and scheme-theoretically.*

Proof. This result refers to [1, Proposition 2.2]. As observed earlier, the base locus of ϕ_1 is set-theoretically $\bigcap_p P_p^{(1)}$. In particular, a point $\alpha_1 \in E_1$ lying in the fiber of $\alpha \in B$ is also in $\mathrm{Bs}(\phi_1)$ if it is determined by a vector in $\bigcap_p T_\alpha P_p$ which is normal to B. Thanks to Lemma 3.8(iii), we see that the intersection of all tangent spaces to the point conditions at α is given by the 11-dimensional space $\Sigma_\alpha := \{\tau \in W^* \otimes W \mid \operatorname{im}\tau \subset \sigma\}/\alpha\mathbb{C}$. This contains $T_\alpha B$ (see again Lemma 3.8) and the quotient $\Sigma_\alpha/T_\alpha B$ is a 6-dimensional subspace of the fiber of $N_{B/\mathcal{V}}$ over α. Moving α, we get a rank-6 subbundle of $N_{B/\mathcal{V}}$, so a $\mathbb{P}^5$-subbundle of $E_1 = \mathbb{P}(N_{B/\mathcal{V}})$, as we wanted. The $C_i^{(1)}$'s are also base loci, since the corresponding C_i's were so.

The second statement can be proved fiberwise: indeed, the fiber of B_1, a linear subspace, is cut out by fibers of the various $P_p^{(1)} \cap E_1$, which are linear spaces themselves. □

Corollary 3.12. *The component B_1 can be globally described as $\mathbb{P}\left(\frac{\bigcap_p T_\alpha P_p}{TB}\right)$ and its intersection with $C_i^{(1)}$ is the bundle over $\mathbb{P}W \times \ell_i$ given by:*

$$C_i^{(1)} \cap B_1 = \mathbb{P}\left(\frac{TC_i}{T(\mathbb{P}W^* \times \ell_i)}\right).$$

Proof. The global description of B_1 is straightforward from Proposition 3.11. Regarding the intersection $C_i^{(1)} \cap B_1$, this coincides with $C_i^{(1)} \cap E_1$. The description then holds by the same arguments used for Lemma 3.10. □

3.3. The base locus after blowing up B_1

We now address Claim 3.7: although a complete description of the components of $\mathrm{Bs}(\phi_2)$ will not be given, we will show which of those components are the ones dominating $B \simeq \mathbb{P}W^* \times \mathcal{S}$.

So, let us denote with B_2 the closed subvariety of $\mathrm{Bs}(\phi_2) \cap E_2$ dominating B. In order to understand B_2 we will need to look at the intersection of $\mathcal{S}$ with its tangent planes. We will focus on the points of $\mathcal{S}$ lying in the subset $\mathcal{S}_0 := \mathcal{S} \setminus \bigcap_{i=1}^{27} \ell_i$. Note that for every $q \in \mathcal{S}_0$, the plane cubic curve $T_q\mathcal{S} \cap \mathcal{S}$ is either a node or a cuspidal curve and if ℓ is a line in the tangent cone of such a cubic at its singular point q, then the intersection multiplicity is $m_q(\ell \cdot (T_q\mathcal{S} \cap \mathcal{S})) = 3$.

Definition 3.13. A line of matrices $\alpha + \tau s$ in $\mathcal{V}$, with $\alpha = (k,q) \in B$ is called a *special line* if $q \in \mathcal{S}_0$, $\tau(k^\perp) \not\subset q$ and the image of τ is contained in a line tangent to the cubic curve $T_q\mathcal{S} \cap \mathcal{S}$ at q.

We would like to translate properties of points in B_2 to properties of points in B and as we will soon see it will be useful to observe the following:

Lemma 3.14. *The base locus $\mathrm{Bs}(\phi_2)$ is disjoint from $E_1^{(1)}$.*

Proof. This is just a rephrase of the second part of Proposition 3.11, thanks to which we know that the point conditions $P_p^{(1)}$ intersect E_1 transversely in $\mathcal{V}_1$. □

Proposition 3.15. *Let α_2 be a point of E_2 and let us denote with $\alpha = (k,q)$ its image in B via the composite map $\pi_1 \circ \pi_2$. Suppose also that $q \in \mathcal{S}_0$. Then α_2 is in B_2 if and only if it can be written as the intersection of E_2 with the proper transform in $\mathcal{V}_2$ of a special line in $\mathcal{V}$. Moreover, the set of such α_2 is dense in B_2.*

Proof. By definition, $\alpha_2 \in \mathrm{Bs}(\phi_2)$ if and only if it is contained in the proper transform of a general point condition $P_p^{(2)}$. In particular, if α_2 is in B_2, it must represent a direction normal to B_1 and tangent to a general point condition $P_p^{(1)}$

at $\alpha_1 := \pi_2(\alpha_2)$. We can identify this direction with a smooth curve germ γ_{α_1} around α_1 in $\mathcal{V}_1$, satisfying normality to B_1 and the tangency condition:

$$m_{\alpha_1}(\gamma_{\alpha_1} \cdot P_p^{(1)}) \geq 2, \quad \text{for a general } p \in \mathbb{P}W.$$

Note that, using the above identification, we can write $\alpha_2 = E_2 \cap (\gamma_{\alpha_1})^{(1)}$.

Thanks to Lemma 3.14 we can rephrase everything in terms of curve germs in $\mathcal{V}$: indeed, γ_{α_1} turns out to be not only normal to B_1, but to the whole of E_1, so we can think of it as the proper transform of a line $\gamma_\alpha = \alpha + \tau s \subset V$, which is normal to B and intersects a general point condition P_p with multiplicity greater or equal than 3.

Denoting as usual with σ the tangent plane $T_q\mathcal{S}$, we can equivalently say that:

$$\alpha_2 \in B_2 \iff \alpha = E_2 \cap (\alpha + \tau t)^{(2)},$$

with $\operatorname{im}\tau \subset \sigma$ and $\tau(k^\perp) \not\subset q$ (see Lemma 3.8), such that for a general p we have $m_\alpha((\alpha + \tau s) \cdot P_p) \geq 3$.

This description reduces to study a special class of lines through α in $V = \mathcal{V}$: we divide in 3 cases, depending on the rank of τ, that can be either $1, 2$ or 3.

If $\operatorname{rk}\tau = 3$, then $\operatorname{im}\tau = \sigma$. In particular, for a general $p \in \mathbb{P}W$, we have $\tau(p) = q_p$, where q_p is a point varying on σ and different from q. Then the span

$$\langle \alpha(p) = q, \tau(p) = q_p \rangle$$

is a general line λ_p in σ passing through q and $(\alpha + \tau s)(p)$ is a parametrization of such a line. Then for a general p we have

$$\begin{aligned} m_\alpha((\alpha + \tau s) \cdot P_p) &= \operatorname{ord}_{t=0}(F((\alpha + \tau s)(p))) \\ &= m_q(\lambda_p \cdot \mathcal{S}) \\ &= m_q(\lambda_p \cdot (\mathcal{S} \cap \sigma)) = 2 < 3, \end{aligned}$$

so in this case α_2 is not in the base locus.

If $\operatorname{rk}\tau = 2$, then $\operatorname{im}\tau = \ell$, where ℓ is a line in the tangent plane σ. Again, for a general $p \in \mathbb{P}W$, the span of $\alpha(p)$ and $\tau(p)$ is a line through q and we are interested in computing $m_q(\lambda_p \cdot (\mathcal{S} \cap \sigma))$. There are two possibilities: if $q \notin \ell$, then for every two distinct points p_1 and p_2 in $\mathbb{P}W$ the lines λ_{p_1} and λ_{p_2} are distinct. In particular, for a general p, the above multiplicity will be 2, so in this case as well, α_2 is not in the base locus.

On the other hand, if $q \in \ell$, then for a general p we constantly have $\lambda_p = \ell$ and α_2 is in the base locus precisely when $m_q(\ell \cdot (\mathcal{S} \cap \sigma)) = 3$, i.e. when ℓ is one of the two tangent lines at the node q (or the double tangent line in the degenerate case). Note the multiplicity computation makes sense since we are assuming that $q \in \mathcal{S}_0$.

Finally, if $\operatorname{rk}\tau = 1$, then $\operatorname{im}\tau = q'$, a point in σ different from q (otherwise this would contradict $\tau(k^{\perp}) \not\subset q$). Then, arguing as above, for a general p, the span of $\alpha(p) = q$ and $\tau(p) = q'$ is a constant line ℓ and α_2 is in the base locus if and only if $m_q(\ell \cdot (\mathcal{S} \cap \sigma)) = 3$. Note that the rank-1 matrices τ satisfying this property come from taking the closure of the space of rank-2 matrices described at the previous step.

The density statement is a consequence of the fact that $\mathcal{S}_0$ is dense in $\mathcal{S}$, since a component dominates B if and only if it dominates $\mathbb{P}W^* \times \mathcal{S}_0 \subset B$. □

Our knowledge about the components of the base locus of ϕ_2 can be summarized in the following:

Proposition 3.16. *The components of the support of* $\mathrm{Bs}(\phi_2)$ *that dominate a component of the original base locus* $\mathrm{Bs}(\phi)$ *are* $C_1^{(2)}, \dots, C_{27}^{(2)}$ *and the irreducible components of* B_2*. Moreover, the map* $(\pi_1 \circ \pi_2)|_{B_2}$ *is a double cover of* B*, i.e.* B_2 *consists of at most* 2 *irreducible components.*

Proof. We just need to observe that B_2 is obtained by taking the closure of a subset of E_2 whose fibers over B correspond to two special lines of $\mathcal{V}$ (counted with multiplicity). □

Remark 3.17. While the $C_i^{(2)}$'s are clearly irreducible, we are still left lo understand if also B_2 is.

3.4. The base locus after blowing up the $C_i^{(3)}$

The last part of the paper is devoted to proving the following result:

Proposition 3.18. *After blowing up one of the components* $C_i^{(3)}$*, corresponding to matrices with image contained in a line, there will be no remaining base locus over the points in* C_i *corresponding to matrices of rank* 2*.*

Since, up to this point, the centers of all blow-ups have been away from matrices of rank 2, we will for simplicity consider the base locus after blowing up C_i in $\mathcal{V}$ instead of $C_i^{(3)}$ in $\mathcal{V}_2$.

We now wish to study the intersection of the tangent spaces of all point conditions. To this end, we will study the image of matrices contained in the intersection of all the tangent spaces. From Lemma 3.9 (ii) we see that for every $\alpha \in C_i$, the intersection of all the tangent spaces $T_\alpha P_p$ is:

$$\bigcap_{p \in W} \{\tau \in W^* \otimes W \mid \tau(p) \subset T_{\alpha(p)}\mathcal{S}\}/\alpha\mathbb{C}. \tag{2}$$

In fact, as we will prove now, this condition will imply that the image of the matrix τ must be contained in ℓ_i.

The proof relies on pencils of hyperplanes. The hyperplanes in W containing ℓ_i are parametrized by $\mathcal{H} \simeq \mathbb{P}^1$. A pencil of hyperplanes containing ℓ_i will be a morphism $\mathbb{P}^1 \to \mathcal{H}$, and the degree of the pencil is the degree of this morphism (if it is nonconstant).

In this and the following lemma, we will work with the affine space W instead of $\mathbb{P}W$.

Lemma 3.19. *Let $\alpha \in C_i$ be a point corresponding to a rank-2 matrix with image ℓ_i, and let $\tau \in W^* \otimes W$ be such that the image of τ in $T_\alpha \mathcal{V} \simeq W^* \otimes W/\alpha\mathbb{C}$ is in $\cap_p T_\alpha P_p$. Then for any two-dimensional subspace $U \subset W$ such that $\alpha(U) = \ell_i$, we have $\tau(U) \subseteq \ell_i$.*

Proof. From the two-dimensional subspace U we can construct a degree-two pencil $\mathcal{P}_1$ of hyperplanes in W containing ℓ_i by assigning to $u \in U$ the hyperplane defined by the equation

$$\sum_{i=0}^{3} \left(\frac{\partial F}{\partial x_i} \right)_{\alpha(u)} = 0,$$

where F is the general degree three polynomial defining the cubic surface $\mathcal{S}$. We think of $\mathcal{P}_1$ as assigning to $u \in U$ the tangent plane of $\mathcal{S}$ at $\alpha(s)$. This defines a map from $\mathbb{P}(U) \simeq \mathbb{P}^1$ to $\mathcal{H}$. This pencil will have degree two, as it is defined by degree two polynomials.

Assume for contradiction that $\tau(U) \not\subseteq \ell_i$. There are three cases: $\tau(U)$ is either a one-dimensional space not contained in ℓ_i, a two-dimensional space with one-dimensional intersection with ℓ_i, or a two-dimensional space with zero-dimensional intersection with ℓ_i. In all cases, we construct a second pencil $\mathcal{P}_2$ of hyperplanes containing ℓ_i, by assigning to $u \in U$ the hyperplane spanned by $\tau(u)$ and ℓ_i. This defines a map $\mathbb{P}(U) \dashrightarrow \mathcal{H}$ which is a priori at least rational, but extends to a morphism $\mathbb{P}(U) \to \mathcal{H}$ since the domain is a curve.

The condition (2) states that $\mathcal{P}_1$ and $\mathcal{P}_2$ are equal. Indeed, condition (2) requires that $\mathcal{P}_2(p) = \langle p, \ell_i \rangle$ is mapped to $T_{\alpha(p)}\mathcal{S}$ in $T_{\alpha(p)}\mathcal{V}$. But this can only happen if $\mathcal{P}_2(p) = \mathcal{P}_1(p)$. However, this cannot be true in any of the three cases, as we will see in the following:

- If $\tau(U)$ is a one-dimensional space not contained in ℓ_i, then $\mathcal{P}_2$ is constant, and therefore not equal to $\mathcal{P}_1$.

- In the case where $\tau(U)$ is two-dimensional and intersects ℓ_i in the one-dimensional space $q\mathbb{C}$, $\mathcal{P}_2(p)$ will be the hyperplane spanned by ℓ_i and $\tau(U)$ for any p, so the pencil is constant.

- If $\tau(U)$ is a two-dimensional space intersecting ℓ_i only in 0, then $\mathcal{P}_2$ is a pencil of degree 1. Therefore, again, it cannot be equal to $\mathcal{P}_1$.

□

From this lemma we can deduce that in fact the image of τ must be in ℓ_i.

Lemma 3.20. *With notation as above, let α be a matrix of rank 2 in C_i. If $\tau \in \bigcap_p T_\alpha P_p$, then τ is in the tangent space $T_\alpha C_i$.*

Proof. Let τ' be any element of $\in W^* \otimes W$ that is mapped to τ. For any vector $u \in W \setminus \ker \alpha$, it is possible to find a 2-dimensional subspace U containing u such that $\alpha(U) = \ell_i$. Then, thanks to Lemma 3.19, we have $\tau'(u) \in \ell_i$. But since u was arbitrarily chosen in $W \setminus \ker \alpha$ and this latter set spans W, we must have $\operatorname{im} \tau' \subset \ell_i$. □

Putting all this together we find that after blowing up a component of the base locus corresponding to matrices with image in a certain line, the remaining base locus is supported in the fibers over the rank-1 matrices.

Proposition 3.21. *Let*

$$\begin{array}{ccccc} \mathrm{PGL}(4) & \hookrightarrow & \mathcal{V}' & \overset{\phi'}{\dashrightarrow} & \mathcal{F} \\ \| & & \downarrow{\scriptstyle \pi} & & \| \\ \mathrm{PGL}(4) & \hookrightarrow & \mathcal{V} & \overset{\phi}{\dashrightarrow} & \mathcal{F} \end{array}$$

be the the diagram associated to the blow-up of $\mathcal{V}$, along one of the components $C_i \simeq \mathbb{P}^7$ and let $\phi' : \mathcal{V}' \dashrightarrow \mathcal{F}$ be the induced rational map. If we denote by G_i the exceptional divisor over C_i and by $\mathrm{Bs}(\phi')$ the base locus of ϕ', then $\pi(\mathrm{Bs}(\phi') \cap G_i)$ is contained in the $\mathbb{P}W^* \times \ell_i \subset \mathbb{C}_i$ consisting of rank-1 matrices.

Proof. We will prove the statement fiberwise. Let $\alpha \in C_i$ be a rank-2 matrix. Then we must show that $\mathrm{Bs}(\phi') \cap \pi^{-1}(\alpha)$ is empty. The fiber $\pi^{-1}(\alpha)$ is the projectivization of $(N_{C_i/\mathcal{V}})_\alpha$, the fiber of the normal bundle of C_i at α. If we denote with P_p the stict transform of a point condition, then $P_p \cap \pi^{-1}(\alpha)$ is the projectivization of the quotient $T_\alpha P_p / T_\alpha C_i$. Therefore $\mathrm{Bs}(\phi') \cap \pi^{-1}(\alpha)$, is obtained by projectivizing $\bigcap_p T_\alpha P_p / T_\alpha C_i$. But by Lemma 3.20 we know that $\bigcap_p T_\alpha P_p$ is actually contained in $T_\alpha C_i$, so the quotient described above must be trivial. After projectivizing, we see that $\mathrm{Bs}(\phi') \cap \pi^{-1}(\alpha)$ must be empty. □

Remark 3.22. In our resolution of $\phi : \mathcal{V} \dashrightarrow \mathcal{F}$, we actually want to blow up the proper transforms $C_i^{(3)}$ of the C_i in $\mathcal{V}_2$. However, over the matrices of rank 2, the blow-down $\mathcal{V}_2 \to \mathcal{V}$ is an isomorphism. We can therefore conclude from Propostion 3.21 that also in this case there is no further base locus over the rank-2 matrices.

Having Proposition 3.21 been proved, the natural question to ask is:

Question 3.23. After blowing up the $C_i^{(3)}$'s, is there any base locus over the subset of points that projects down to the locus of rank-1 matrices?

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