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# A NOTE ON TWIN PRACTICAL NUMBERS

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A positive integer $m$ is a practical number if every positive integer $n < m$ is a sum of distinct divisors of $m$. Let $P_2(x)$ be the counting function of the pairs $(m, m+2)$ of twin practical numbers. Margenstern conjectured that $P_2(x) \sim \lambda_2 x(\log x)^{-2}$. We prove that, for sufficiently large $x$ and for a suitable constant $k$, $P_2(x) > x \exp\{-k(\log x)^{1/2}\}$.

## 1. Introduction.

A positive integer $m$ is a practical number if every positive integer $n < m$ is a sum of distinct positive divisors of $m$.

A wide survey of results and conjectures on practical numbers is given by Margenstern [3]. Let $P(x)$ be the counting function of practical numbers. Erdös [2] proved that $P(x) = o(x)$. Saias [5], using suitable sieve methods introduced by Tenenbaum [8, 9], provided a good estimate in terms of a Chebishev-type theorem: for suitable constants $c_1$ and $c_2$,

$$c_1 \frac{x}{\log x} < P(x) < c_2 \frac{x}{\log x}.$$

Let $P_2(x)$ be the function counting practical numbers $m \le x$ such that $m + 2$ is also a practical number. Margenstern proved that $P_2(x) \to \infty$. By following his argument one easily gets $P_2(x) \gg \log\log x$. He also stated among other things the following prime-like conjectures.

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*Conjecture 1.1*. For a suitable $\lambda_1 > 0$:

$$P(x) \sim \lambda_1 \frac{x}{\log x}.$$

*Conjecture 1.2*. For a suitable $\lambda_2 > 0$:

$$P_2(x) \sim \lambda_2 \frac{x}{(\log x)^2}.$$

In his conjectures and computations, he proposed $\lambda_1 \simeq 1.341$ and $\lambda_2 \simeq 1.436$. In [4] the author proved a Goldbach-type result showing that every even positive integer is a sum of two practical numbers. The proof uses an auxiliary increasing sequence $m_n$ of practical numbers such that for every $n$, $m_n + 2$ is also a practical number and $m_{n+1}/m_n$ bounded by an absolute constant. This implies $P_2(x) \gg \log x$, but still very far from the conjecture. In this paper we prove the following result.

**Theorem 1.1.** *Let $k > 2 + \log(3/2)$. For sufficiently large $x$,*

$$P_2(x) > \frac{x}{\exp\{k(\log x)^{\frac{1}{2}}\}}.$$

This, in particular, proves for the first time that, for every $\alpha < 1$, for sufficiently large $x$, $P_2(x) > x^\alpha$.

## 2. Preliminary tools.

We briefly recall a structural theorem and a corollary which will be extensively used in the proof of Theorem 1.1.

**Theorem 2.1.** *An integer $m \geq 2$, $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\ell^{\alpha_\ell}$, with primes $p_1 < p_2 < \ldots < p_\ell$ and integers $\alpha_i \geq 1$, is practical if and only if $p_1 = 2$ and, for $i = 2, 3, \ldots, \ell$,*

$$p_i \leq \sigma\big(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{i-1}^{\alpha_{i-1}}\big) + 1,$$

*where $\sigma(n)$ denotes the sum of the positive divisors of $n$.*

**Corollary 2.1.** *Let $m$ be a practical number. If $n \leq \sigma(m) + 1$ then $mn$ is a practical number. In particular if $n \leq 2m$, $mn$ is practical.*

A proof of Theorem 2.1 can be found in Stewart paper [7]. Corollary 2.1 appears, for example, in [3].

Theorem 2.1 and Corollary 2.1 are the main tools to construct practical numbers. The following lemma that will be used in the proof of Theorem 1.1 is a kind of statement which follows by Corollary 2.1.

**Lemma 2.1.** *Let $z > 28$. Then it exists a practical number $m$ such that $m + 2$ is also practical and $m \leq z \leq \frac{3}{2}m$.*

*Proof.* For a proof, see [4], p. 207. □

In the following lemma, we will denote by $\pi(x)$ the number of primes not exceeding $x$.

**Lemma 2.2.** *Let $x$ sufficiently large. Let $N = [\log\log x/2\log 2]$ and $c = 2^{-2^{-N}}$. Let $m = m(x)$ be chosen in such a way that $m \leq x^{2^{-N-1}} \leq 2m$. Then*

$$\lim_{x\to\infty} \frac{\pi(m) - \pi(cm)}{(1-c)m\log m} = 1.$$

*Proof.* Note that $(1-c)$ tends to 0, and the above formula cannot be proved by using the prime number theorem in the usual form $\pi(x) \sim x/\log x$. However, a somewhat sharper form will suffice. We begin by estimating some of the quantities involved in the above expression. Let $\alpha = \{\log\log x/2\log 2\}$ and $\beta = 2^{\alpha}$. So $1 \leq \beta < 2$. We have

$$\begin{aligned}
(1-c) &= 1 - 2^{-2^{-N}} \\
&= 1 - 2^{-2^{-(\log\log x/2\log 2-\alpha)}} \\
&= 1 - e^{-(\log x)^{-\frac{1}{2}}\beta\log 2} \\
&= \beta\log 2(\log x)^{-\frac{1}{2}} + O((\log x)^{-1}).
\end{aligned}$$

Similarly, for suitable $\theta$ with $0 \leq \theta \leq \log 2$, we have

$$\begin{aligned}
\log m &= 2^{-N-1}\log x - \theta \\
&= 2^{-(\log\log x/2\log 2-\alpha)-1}\log x - \theta \\
&= \frac{\beta}{2}(\log x)^{\frac{1}{2}} - \theta.
\end{aligned}$$

In other terms $(1-c) \asymp 1/\log m$. By prime number theorem in his form $\pi(x) = \mathrm{li}\,(x) + O(x\exp\{-c\log^{-\frac{1}{2}} x\})$ (see for example [1]), we have

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + O\Big(\frac{x}{\log^3 x}\Big),$$

so

$$\pi(m)-\pi(cm)=\frac{m}{\log m}+\frac{m}{\log^2 m}-\frac{cm}{\log m}-\frac{cm}{\log^2 m}+O\Big(\frac{m}{\log^3 m}\Big)$$

$$=\frac{m}{\log m}\Big((1-c)\Big(1+\frac{1}{\log m}\Big)+O\Big(\frac{1}{\log^2 m}\Big)\Big),$$

therefore

$$\frac{\pi(m)-\pi(cm)}{(1-c)\frac{m}{\log m}}=1+\frac{1}{\log m}+O\Big(\frac{1}{(1-c)\log^2 m}\Big)=1+o(1)\,. \qquad \square$$

## 3. Main result.

*Proof of Theorem 1.1.* Let $x > e^{100}$ and $N = [\log\log x/2\log 2]$. By Lemma 2.1, there exists a pair $(m, m+2)$ of twin practical numbers with $m \le x^{2^{-N-1}} \le \frac{3}{2}m$. Let $c = 2^{-2^{-N}}$. Let $p_{1,1}, p_{2,1}, \dots, p_{k_1,1}$ be all primes between $cm$ and $m$. Let $p_{1,2}, p_{2,2}, \dots, p_{k_2,2}$ be all primes between $cm^2$ and $m^2$. For every $1 \le j \le N$ let $p_{i,j}$ be all $k_j$ primes between $cm^{2^{j-1}}$ and $m^{2^{j-1}}$. Note that for sufficiently large $x$, by Lemma 2.2, all $k_i$ are positive integers. For every $2N$-tuple $(a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N)$ of integers with $1 \le a_i, b_i \le k_i$, $a_i \ne b_i$ let define $m_1$ and $m_2$ as $m_1 = m\prod_{h=1}^N p_{a_h,h}$ and $m_2 = (m+2)\prod_{h=1}^N p_{b_h,h}$.

It is worth to spent some comments on the choice of $c$. Its value is sufficiently far from 1 to assure, by Lemma 2.2, that all $k_i$ are positive. On the other hand, it is sufficiently near to 1 to assure that the product of the primes involved in $m_1$ and $m_2$ is such that $m_1 \le 2m_2$ and $m_2 \le 2m_1$. In other words, $m_1$ and $m_2$ are made up of product of primes ‘picked up’ in certain tiny (in logarithmic scale) intervals of integers.

Note also that $m_1$ and $m_2$ are practical numbers by repeated application of Corollary 2.1 and that g.c.d.$(m_1, m_2) = 2$. So there exist positive integers $r = r(a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N)$ and $s = s(a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N)$ with $1 \le r < m_2/2$ and $1 \le s < m_1/2$ such that $m_2 s - m_1 r = 2$. Note that a pair $(r, s)$ with these properties is univocally determined as the smallest pair of positive integers $(r, s)$ such that $m_2 s - m_1 r = 2$. Since $m_1$ and $m_2$ are practical numbers and $r < m_1$, $s < m_2$, again by Corollary 2.1, $m_1 r$ and $m_2 s$ are practical numbers, and indeed $(m_1 r, m_2 s)$ is a pair of twin practical numbers. We get $\prod_{h=1}^N (k_h^2 - k_h)$ pairs of twin practical numbers, all bounded by $m^{2^{N+1}}$, some of which may be repeated.

We now estimate the number of times a pair may appear. Fix the $2N$-tuple $(a_1, a_2, \dots, a_N, b_1, b_2, \dots, b_N)$ and let $(a'_1, a'_2, \dots, a'_N, b'_1, b'_2, \dots, b'_N)$ be another $2N$-tuple of the same kind. Denote $m'_1 = m \prod_{h=1}^{N} p_{a'_h,h}$ and $m'_2 = (m+2) \prod_{h=1}^{N} p_{b'_h,h}$. Further let $r' = r(a'_1, a'_2, \dots, a'_N, b'_1, b'_2, \dots, b'_N)$ and $s' = s(a'_1, a'_2, \dots, a'_N, b'_1, b'_2, \dots, b'_N)$. How many choices for the $2N$-tuple $(a'_1, a'_2, \dots, a'_N, b'_1, b'_2, \dots, b'_N)$ do we have such that $m_1 r = m'_1 r'$ (and automatically $m_2 s = m'_2 s'$)?

Since $r$ (and $s$) is bounded by $m^{2^N}/2$, it cannot contain as divisors more than one prime between $cm^{2^{N-1}}$ and $m^{2^{N-1}}$. Since $m_1 r = m'_1 r'$ and $m_2 s = m'_2 s'$ we have at most two possible choices for $a'_N$, one of which is $a_N$, and at most two possible choices for $b'_N$, one of which is $b_N$.

Analogously, $r$ cannot contain as divisors more than three primes between $cm^{2^{N-2}}$ and $m^{2^{N-2}}$, nor more than seven primes between $cm^{2^{N-3}}$ and $m^{2^{N-3}}$. For every $h \le N$, $r$ cannot contain more than $2^h - 1$ primes between $cm^{2^{N-h}}$ and $m^{2^{N-h}}$. So we have no more than $2^h$ choices for $a'_h$, one of which is $a_h$ and no more than $2^h$ choices for $b'_h$, one of which is $b_h$. Hence $m_1 r = m'_1 r'$ (and $m_2 s = m'_2 s'$) in no more than $2^{N(N+1)}$ cases. This implies

$$P_2(x) \ge P_2(m^{2^{N+1}}) \ge \frac{\prod_{h=1}^{N}(k_h^2 - k_h)}{2^{N(N+1)}}.$$

Let $0 < \varepsilon_1 < 1 - 1/e(\log 2)^2$. For sufficiently large $x$ and for every $h \le N$, by Lemma 2.2

$$k_h^2 - k_h > (1-\varepsilon_1)(1-c)^2 \frac{m^{2^h}}{2^{2h-2}(\log m)^2}.$$

Hence, for sufficiently large $x$,

$$\begin{aligned}
P_2(x) &> \frac{1}{2^{N(N+1)}} \prod_{h=1}^{N} (1-\varepsilon_1)(1-c)^2 \frac{m^{2^h}}{2^{2h-2}(\log m)^2} \\
&= \frac{(1-\varepsilon_1)^N (1-2^{-2^{-N}})^{2N}}{2^{2N^2}} \cdot \frac{m^{2^{N+1}-2}}{(\log m)^{2N}} \\
&\ge \frac{(1-\varepsilon_1)^N (2^{-N-1}\log 2)^{2N}}{2^{2N^2}} \cdot \frac{(\frac{2}{3}x^{2^{-N-1}})^{2^{N+1}-2}}{(\log x^{2^{-N-1}})^{2N}} \\
&= \frac{(1-\varepsilon_1)^N (2^{-N-1}\log 2)^{2N}}{(3/2)^{2^{N+1}-2} 2^{2N^2} 2^{(-N-1)2N}} \cdot \frac{x^{1-2^{-N}}}{(\log x)^{2N}} \\
&> \frac{x}{(3/2)^{2^{N+1}} x^{2^{-N}} 2^{2N^2} (\log x)^{2N} e^N}\,.
\end{aligned}$$

Let $\varepsilon > 0$ and $\alpha = \{\log\log x/2\log 2\}$. For sufficiently large $x$, we have

$$\begin{aligned} P_2(x) &> \frac{x}{\exp\{2^{N+1}\log(3/2) + 2^{-N}\log x + 2N^2\log 2 + 2N\log\log x + N\}} \\ &> \frac{x}{\exp\{2^{1-\alpha}\log(3/2)(\log x)^{\frac{1}{2}} + 2^{\alpha}(\log x)^{-\frac{1}{2}}\log x + 2(\log\log x)^2\}} \\ &> \frac{x}{\exp\{(2 + \log(3/2) + \varepsilon)(\log x)^{\frac{1}{2}}\}}\,. \end{aligned}$$

The proof is complete $\square$

## 4. Some final remarks.

The proof of Theorem 1.1 is completely elementary. Its central point is the definition of suitable intervals where primes are chosen. These intervals may appear too thin at first sight, but any effort to improve their size causes technical problems in the count of all possible pairs of distinct twin practical numbers. So an improvement of the form $P_2(x) > x\exp\{-c(\log\log x)^2\}$ does not appear easily reachable.

One can hope to deal with this problem by a completely different approach, based on sieve methods, as Saias suggested me [6], but even taking in account the best preliminary results one can hope to get by sieve methods, the final result, applied to the pairs of twin practical numbers is not better that the previous one, proved in Theorem 1.1.

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## REFERENCES

[1] P.T. Bateman - H.G. Diamond, *A hundred years of prime numbers,* Amer. Math. Monthly, 103 (1996), pp. 729–741.

[2] P. Erdös, *On a Diophantine equation,* Mat. Lapok, 1 (1950), pp. 192–210, Hungarian. Russian and English summaries.

[3] M. Margenstern, *Les nombres pratiques: théorie, observations et conjectures,* J. Number Theory, 37 (1991), pp. 1–36.

[4] G. Melfi, *On two conjectures about practical numbers,* J. Number Theory, 56 (1996), pp. 205–210.

[5] E. Saias, *Entiers à diviseurs denses 1,* J. Number Theory, 62 (1997), pp. 163–191.

[6] E. Saias, *Personal communication,* (2000).

[7] B. M. Stewart, *Sums of distinct divisors,* Amer. J. Math., 76 (1954), pp. 779–785.

[8] G. Tenenbaum, *Sur un problème de crible et ses applications,* Ann. Sci. Éc. Norm. Sup. (4), 19 (1986), pp. 1–30.

[9] G. Tenenbaum, *Sur un problème de crible et ses applications, 2. Corrigendum et étude du graphe divisoriel,* Ann. Sci. Éc. Norm. Sup. (4), 28 (1995), pp. 115–127.

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