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STABILITY OF JENSEN'S EQUATIONS IN TWO NORMED SPACES

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Some stability questions of the Jensen's functional inequality in the setting of 2-normed spaces are derived in this article. Few more results are given on approximate isometries.

1. Introduction.

Several authors dealt about the stability of functional equations of various hues. To cite some important references we refer to the works of Hyers [4], Hyers and Rassias [5], Kominek [7], Parnami and Vasudeva [8], Rassias [9], Rassias and Semrl [10], Jung [6] and Ulam [11]. In fact some of these problems stemmed from the treatment given in reference [11]. It is our aim article to deal with Jensen's functional equation as was the case in [6], in the setting of two normed spaces extending the work of the same reference. In fact we investigate the Jensen's functional inequality of the following type:

$$(*) \qquad \Big\| \frac{1}{2} f\Big(\frac{x+y}{2}\Big) - f(x) - f(y) \Big\| \le \delta + \theta \big\{ \|x, z\|^p + \|y, z\|^p \big\},$$

where f is a mapping between Banach Spaces X into Y with X having 2-norm structure.

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Also z is a fixed element in X. In $(*) p \geq 0$ and $p \neq 1$. In fact we consider the stability of the inequality $(*)$. Moreover a little modification of example in [5] shows that $(*)$ is not stable for $p = 1$. for notational formulations and the properties of 2-normed spaced one can refer to [2]. We prove the following theorem

Theorem 1.1. *Let $p > 0$ and $p \neq 1$. Suppose f is a mapping from X into Y such that X is a 2-normed space, Y is a Banach Space. Let f satisfy the inequality $(*)$. Also suppose that for $p > 1, \delta = 0$ in the inequality $(*)$. Futher suppose that z is not in the linear span of x. Then the following inequalities hold for an additive mapping F from X into Y.*

$$\|f(x) - F(x)\| \leq \delta + \|f(0)\| + \theta/(2^{1-p} - 1)\{\|x, z\|^p\}(p < 1) \tag{1}$$

or

$$\|f(x) - F(x)\| \leq 2^{p-1}/(2^{1-p} - 1)\|x, z\|^p(p > 1) \tag{2}$$

Proof. If we put $y = 0$ in $(*)$ then we get the following inequality.

$$\|2f(x/2) - f(x)\| \leq \delta + \|f(0)\| + \theta\|x, z\|^p \tag{3}$$

for all x in X and fixed z in X.

We can prove by induction,

$$\|2^{-n} f(2^n x) - f(x)\| \leq (\delta + \|f(0)\|) \sum_{k-1}^{n} 2^{-k} + \theta\|x, z\|^p \sum_{k-1}^{n} 2^{-(1-p)k} \tag{4}$$

for the case when $0 < p < 1$. Substituting $2x$ for x and dividing both sides of (3) by 2 we see the validity of (4) for $n = 1$. Now assume that the inequality (4) holds for n in N. Now if we replace x in (3) by $2^{n+1}x$ and divide both side of (3) by 2 then it follows from (4) that

$$\|^{-(n+1)} f(2^{n+1}x) - f(x)\| \leq \tag{5}$$

$$2^{-n}\|2^{-1} f(2^{n+1}x) - f(2^n x)\| + \|2^{-n} f(2^n x) - f(x)\|$$

$$\leq (\delta + \|f(0)\|) \sum_{k=1}^{n+1} 2^{-k} + \theta\|x, z\|^p \sum_{k=1}^{n+1} 2^{-(1-p)}$$

This completes the proof of (4).

Now define

$$(5)a \qquad F(x) = \lim_{n\to\infty} 2^{-n} f(2^n x).$$

This is possible because Y is a Banach Space and we shall prove that the term defined in $F(x)$ through a sequence is a Cauchy sequence.

For $n > m$, using (4) we get

$$(6) \qquad \|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\| \leq$$
$$\leq 2^{-m}(\delta + \|f(0)\| + 2^{mp}/(2^{1-p} - 1)\theta\|x, z\|^p)$$

which tends to 0 as m tends to infinity. Let x, y in X be arbitrary. Then it follows from (5a) and $(*)$ that,

$$(7) \qquad \|F(x + y) - F(x) - F(y)\|$$
$$= \lim 2^{-(n+1)}\|2f(2^{n+1}(x + y)/2) - f(2^{n+1}x) - f(2^{n+1}y)\|$$
$$\leq \lim 2^{-(n+1)}(\delta + \theta 2^{(n+1)p}(\|x, z\|^p + \|y, z\|^p))$$

which tends to o as n tends to infinity. Hence F is an additive mapping. Now (4) and (5a) imply the validity of (6).

For uniqueness we simply see that for another additive G of similar nature we have the following inequality:

$$(8) \qquad \|F(x) - G(x)\| \leq 2^{-n}(2\delta + 2\|f(o)\| + 2\theta/(2^{1-p} - 1)2^{np}\|x, z\|^p)$$

which tends to o as n tends to infinity. Hence $F(x) = G(x)$. For the case when $p > 1$ and $\delta = o$, we can get the following equality.

$$(9) \qquad \|2^n f(2^{-n}x) - f(x)\| \leq \theta\|x, z\|^p \sum_{k=1}^{n-1} 2^{-(p-1)k}$$

instead of (4). There after the proof goes in the same fashion as in the previous case.

Examples of 2-normed spaces and isometries. [1] In R^2 an example of 2-norm would be given as follows. For x and y in R^2 we say

$$\|x, z\|^2 = \{((x_1)^2 + (x_2)^2)((z_1)^2 + (z_2)^2) - (x_1z_1 + x_2z_2)^2\}.$$

As such the above 2-norm satisfies:

(i) $\|x, z\| = o$ if x and z are linearly dependent, other wise it is $> o$.
(ii) $\|x, z\| = \|z, x\|$
(iii) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$

For $f(x_1, x_2) = x_1 + x_2$ the isometry condition $(x_1 + x_2)^2 = \|x, z\|^2$ is satisfied for $z = (1, 1)$.

2. Stability of the Jensen's inequality in a restricted domain.

In this section we give a version of Theorem 1.1 in a restricted domain and give an application of that result to derive some asymptotic property of some additive mappings.

Theorem 2.1. *Let $d \geq o$ and $\delta \geq o$ be given. Assume that a mapping f from X into Y satisfies the following Functional inequality.*

$$\|2f((x+y)/2) - f(x) - f(y)\| \leq \delta \tag{10}$$

for all x, y in X and fixed z in X such that the following 2-norms satisfy $\|x, z\| + \|y, z\| \geq d$. Also suppose that z does not belong to the linear span of x, y. Then there is an unique additive mapping $F : X \to Y$ which satisfies

$$\|f(x) - F(x)\| \leq 5\delta + \|f(o)\| \tag{11}$$

for all x in X.

The proof of the above theorem can be given on the same lines as Theorem 1.1.

We give a corollary of the above theorem which is interesting for the asymptotic property of additive mappings.

Corollary 2.2. *Suppose a mapping $f : X \to Y$ satisfies $f(0) = 0$ (X having 2-norm structure). Also f satisfies the following asymptotic condition.*

$$\|f(x+y) - f(x) - f(y)\| \to 0$$

as

$$\|x, z\| + \|y, z\| \to \infty$$

for a fixed z in X, with z not being in the linear span of x and y, then f is an additive mappings and the converse of this proposition holds.

Proof. If f is an additive mapping then the asymptotic condition is satisfied trivially. Next suppose the asymptotic condition of the theorem holds. Then there is a monotonically decreasing sequence δ_n such that the following inequality is true

$$\|2f((x+y)/2) - f(x) - f(y)\| \leq \delta_n, \text{ for } \|x, z\| + \|y, z\| \geq n. \tag{12}$$

Now from theorem 2.1 we can get a sequence of additive mapping $\{F_n\}$ such that

$$\|f(x) - F_n(x)\| \leq 5\delta_n \tag{13}$$

for all x in X. Let $m \geq \ell$. Obviously it follows from (13) that

$$\|f(x) - F_m(x)\| \leq 5\delta_m \leq \delta_\ell, \tag{14}$$

since δ_N is decreasing. Uniqueness of F_m implies $F_m = F_\ell$. Hence by letting $n \to \infty$ in (13) we get, f as additive.

In the next section we extend certain results of Dolinar [1] on stability of isometries in a generalized sense. Moreover these results are derived when the domain space has 2-norm structure.

3. Generalized Stability of isometries from 2-normed space to normed space.

Slight extension of the results from Lindenstrauss and Szankowski see [1] can eventually show the following.

Consider the function

$$\varphi_f(t) = \sup\{|\|f(x) - f(y)\| - \|x - y, z\|| : \|x - y, z\| \leq t \tag{15}$$

or

$$\|f(x) - f(y)\| \leq t\}$$

where z is a fixed element in X and $\|, \|$ stands for the symbol for 2-norm in X.

Suppose $f_1^\infty(\varphi_f)/t^2 dt < \infty$. Then there is an isometry $U : X \to Y$ such that

$$\|f(x) - U(x)\| = o(\|x, z\|), \text{ as } \|x, z\| \to \infty. \tag{15a}$$

Where $U(x) - \lim\limits_{n\to\infty} f((2^n x)/2^n)$. In the line of [1] we can define φ-isometry as follows:

$$|\|f(x) - f(y)\| - \|x - y, z\|| \leq \varphi(\|x - y, z\|) \tag{16}$$

The above inequality is satisfied for a given function φ and mapping $f : X \to Y$, where X does possess 2-norm structure. We shall prove the following proposition.

Theorem 3.1. *Let $f : X \to Y$ be a surjective φ- isometry and X has a 2-norm structure. Let $f(0, z) = 0$. Let $\varphi_s : [0, \infty) \to [0, \infty)$ be defined by $\varphi_s(t) = \sup_{u \leq t}\{\varphi(u)\}$.*

If

$$\int_1^\infty \frac{\varphi_s(t)}{t^2}\,dt < \infty,$$

then there is an isometry $U : X \to Y$ *defined by* $U(x) = \lim f((2^n x)/2^n)$ *which satisfies,*

$$\|f(x) - U(x)\| = o(\|x, z\|) \text{ as } \|x, z\| \to \infty.$$

Proof. Suppose

$$\int_1^\infty \frac{\varphi_s(t)}{t^2}\,dt < \infty. \tag{17}$$

Then there is a constant $M(\varphi)$ such that $t < 2(t - \varphi(t))$, for every $t > M(\varphi)$. Indeed if for every positive integer n we could find $t_n > n$ such that $\varphi_s(t_n) > t/2$, then we would have,

$$\int_{tn}^{2tn} \frac{\varphi_s(t)}{t^2}\,dt \geq \int_{tn}^{2tn} \frac{\varphi_s(t_n)}{t^2}\,dt = \varphi_s(t_n)(1/t_n) \geq \frac{1}{4},$$

which contradicts (17).

Let $\|f(x) - f(y)\| \leq t$. If $\|x - y, z\| > M(\varphi)$, then

$$\|x - y, z\| < 2(\|x - y, z\| - \varphi_s(\|x - y, z\|)) \leq 2\|f(x) - f(y)\| \leq 2t,$$

so

$$|\|f(x) - f(y)\| - \|x - y, z\|| \leq \varphi_s(2t).$$

If

$$\|x - y, z\| \leq M(\varphi),$$

then

$$|\|f(x) - f(y)\| - \|x - y, z\|| \leq \varphi_s(M(\varphi)).$$

Now let $\|x - y, z\| \leq t$. Then $|\|f(x) - f(y)\| - \|x - y, z\|| \leq \varphi(\|x - y, z\| \leq \varphi_s(t) \leq \varphi_s(2t)$. So if φ is given by (1)), we have,

$$\varphi_f(t) \leq \max\{\varphi_s(M(\varphi)), \varphi_s(2t)\} \text{ for } t \geq 0. \tag{18}$$

Then

$$\int_{M(\varphi)}^\infty \frac{\varphi_f(t)}{t^2}\,dt \leq \int_{M(\varphi)}^\infty \frac{\varphi_s(2t)}{t^2}\,dt \leq 2\int_{M(\varphi)}^\infty \frac{\varphi_s(t)}{t^2}\,dt < \infty.$$

Then by (17) we get the conclusion of the theorem.

4. Stability of approximate isometries when the range space is a Hilbert space.

In the line of [5] we can introduce approximate isometries as follows.

A mapping $f : X \to Y$ will be called (ε, p)-isometry where X is a 2-normed space and Y is a Banach space if it satisfies the following inequality for a fixed z in X.

(19) $$\big|\|f(x) - f(y)\| - \|x - y, z\|\big| \le \varepsilon\|x - y, z\|^p$$

A pair (X, Y) is said to be p-stable with respect to isomtries if there exists a function $\delta : [0, \infty \to [0, \infty)$ with $\lim \delta(\varepsilon) \to 0$ for every surjective isometry $f : X \to Y$ and there is a surjective isometry $U : X \to Y$ satisfying the estimate $\|f(x) - U(x)\| \le \varepsilon\|x, z\|^p$.

Theorem 3.1. *Let X be 2-normed space and Y be a real Hilbert space. Let ε and p be given such that $\varepsilon > 0$ and $p < 1$ also $f(0) = (0, z)$. Then there is a constant $K(\varepsilon, p)$ such that* $\lim K(\varepsilon, p) = o$ *and for (ε, p)-isometry $f : X \to Y$ there is a linear isometry $U : X \to Y$ such that*

$$\|f(x) - U(x)\| \le K(\varepsilon, p) \max\left\{\|x, z\|^p, \|x, z\|^{(1+p)/2}\right\}$$

The following lemma can be proved in the lines of Lemma 1 of [1].

Lemma 3.2. *Let X be a 2-normed space and Y is a Banach Space. Suppose $\varepsilon \ge 0$, $0 < p \le r < 1$, and $\delta \ge 0$. If $f : X \to Y$, $f(0) = (0, z)$, for a fixed z in X and f is an (ε, p) isometry satisfying*

$$\|f(x) - f(2x)/2\|\delta \max\left\{\|x, z\|^p, \|x, z\|^r\right\},$$

for all (x, z) with fixed z in X then there exists an isometry $U : X \to Y$ which satisfies the following

$$\|f(x) - U(x)\| \le \delta 2^{1-r}/(2^{1-r} - 1) \max\{\|x, z\|^p, \|x, z\|^r\}$$

where U is defined as $U(x) = \lim_{n\to\infty}(f(2^n x)/2^n)$.

Proof. of Theorem 3.1. Suppose $\varepsilon \ge 0$ and $0 < p < 1$. Let us estimate $\|f(x) - f(2x)/2\|$. Since f is an (ε, p) isometry,

$$\|f(x) - f(2x)\|^2 \le \left(\|x, z\| + \varepsilon\|x, z\|^p\right)^2$$

and thus

$$\|f(x)\|^2 + \|f(2x)\|^2 - 2 < f(x), f(2x) > \le \big(\|x, z\| + \varepsilon\|x, z\|^p\big)^2.$$

It now follows that,

$$2\|f(x) - f(2x)/2\|^2 = 2\|f(x)\|^2 + 2\|f(2x)/2\|^2 - 4 < f(x), f(2x)/2 >,$$

the right hand side of the previous inequality can be shown to be

$$\le 2\big(\|x, z\| + \varepsilon\|x, z\|^p\big)^2 - 2\|f(2x)/2\|^2.$$

There are two cases to be tackled. Suppose $\|x, z\| \ge 1/2\varepsilon^{1/1-p}$. In this case $\|2x\| - \varepsilon\|2x\|^p \le 0$. So, since f is an (ε, p)-isometry

$$\|f(2x)\|^2 \ge \big(\|2x\| - \varepsilon\|2x\|^p\big)^2$$

and therefore

$$\|f(x) - f(2x)/2\|^2 \le \big(\|x, z\| + \varepsilon\|x, z\|^p\big)^2 - \big(\|x, z\| - (1/2)^{1-p}\varepsilon\|x, z\|^p\big)^2,$$

after some simplification the right hand side of the previous inequality can be shown to be

$$\le 4\varepsilon\|x, z\|^{1+p} + \varepsilon^2\|x, z\|^{2p}. \tag{18}$$

If $\|x, z\| < 1$ then $\|x, z\| < \|x, z\|^p$ and therefore

$$\|f(x) - f(2x)/2\| \le \sqrt{\varepsilon(1+\varepsilon)}\|x, z\|^p .$$

On the other hand if $\|x, z\| \ge 1$ then $\|x, z\| \ge \|x, z\|^p$ and

$$\|f(x) - f(2x)/2\| \le \sqrt{\varepsilon(4+\varepsilon)}\|x, z\|^{(1+p)/2} .$$

In the second case when $\|x, z\| < 1/2\varepsilon^{1/(1-p)}$, that is

$$\|x, z\| > (1/2)^{1-p}\varepsilon\|x, z\|^p ,$$

it follows from (18) that

$$\|f(x) - f(2x)/2\| \le \|x, z\| + \varepsilon\|x, z\|^p \le$$

$$\leq (1/2)^{1-p}\varepsilon\|x, z\|^p + \varepsilon\|x, z\|^p \leq 2\varepsilon\|x, z\|^p.$$

So we have the final estimate as follows:

$$\|f(x) - f(2x)/2\| \leq 2\sqrt{\varepsilon(4+\varepsilon)}\max\left\{\|x, z\|^p, \|x, z\|^{(1+p)/2}\right\}.$$

Now applying Lemma 3.2 we get,

$$K(\varepsilon, p) = 2^{((3-p)/2)}/(2^{(1-p)/2} - 1)\sqrt{\varepsilon(4+\varepsilon)}.$$

Remark 3.1. Following can be observed . Suppose $C[0, 2]$ stand for continuous functions on $[0, 2]$. Let $E = \{\alpha e + f : f$ is in $C[0, 2]$ and e is a function from $[0, 2]$ into R such that $e(x) = 0, 0 < x < 1$, and $e(x) = 1$, for $1 \leq x \leq 2\}$. Define on E a 2-norm as follows

$$\|f, g\| = \{< f, f >< g, g > -| < f, g > |^2\}^2\},$$

where

$$< f, g >= \int_o^2 f\overline{g}\,dx\,.$$

In the above setting we can see that the subset A of E defined as $A = \{f : \|f, e\| \leq 1\}$ is sequentially closed in E but not complete with respect to e. Therefore the isometry U as in Theorem 3.1 does not exist when the range of U is a set like A, since A is not complete.

Remark 3.2. We observe that the isometry U in Theorem 3.1 is linear. This is because even if the domain space X becomes a real two normed space and if the range space Y happens to be strictly convex real Banach Space then the isometry U is always affine. In case $f(0) = 0$, then U becomes linear. Moreover in Theorem 3.1 the range space is taken as a real Hilbert space which is always strictly convex.

5. Some generalized form of stability of Jensen's inequality.

In the context of convex functions one of the generalization which is available in the literature is that of s-convex function where $0 < s < 1$. In that connection we see that a generalized form of Jensen's functional equation seems to be as follows. as in Section 2; we define

$$2^{1/\varepsilon} f((x+y)/2^{1/\varepsilon}) = f(x) + f(y), \tag{19}$$

where doman of function is a Banach space and the range space is a Banach space Y. Also $o < s < 1$.

We can prove the following stabilitry result in connection with the following generalized Jensen's inequality.

(20) $\|2^{1/s} f((x+y)/2^{1/s} - f(x) - f(y)\| \leq \delta + \|f(0)\| + \theta(\|x\|^p + \|y\|^p)$

Theorem 4.1. *Let $p > 0$ and $p > 1$. Let f be a mapping from X into Y where X and Y are Banach spaces. And f satisfies the inequality* (20). *Then there is a generalized additive mapping F in the sense of* (19) *such that either*

$$\|f(x)\| \leq \delta + f(0) + \theta/(2^{s(1-p)} - 1)\|x\|^p$$

or

$$\|f(x) - F(x)\| \leq 2^{ps-1}/(2^{ps-1} - 1)\|x\|^p$$

($0 < s < 1$ and $p > 1$ with $ps > 1$, $\delta = 0$, $f(0) = 0$).

Remark 4.2. Consider the case when a self mapping f from X into it self with X as a 2-normed space satisfies the following functional inequality (see Section 1):

(21) $$\|\{2f((x+y)/2 - f(x) - f(y)\}, w\| \leq \delta + \|f(0), w\| + \\ + \theta\{\|x, w\|^p + \|y, w\|^p\}(p < 1)$$

for all w in X and each x, y in X.

In such a case when we have to define the additive map F, the property of being Cauchy sequence is defined by the notion as follows. A sequence $\{x_n\}$ is a Cauchy sequence in a 2-normed space X with 2-norm $\|\cdot,\cdot\|$ if $\|x_n - x_m, w\| \to o$ as $n, m \to \infty$ for all w in X. Another property which is used to get an additive mapping F as in section 2 is the following.

$\|\{F(x+y) - F(x) - F(y)\}, w\| = 0$ for all w in X implies $F(x+y) - F(x) - F(y)$ and w are linearly dependent for all w in X. Which can happen only when $F(x+y) - F(x) - F(y) = 0$. Similarly case $p > 1$ can be tackled.

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REFERENCES

[1] G. Dolinar, *Generalized Stability of isometries,* J. Math. Anal. Appl., 242 (2000), pp. 39–56.

[2] Y.J. Cho - S.S. Kim - R.W. Freese - A. White, *Strict convexity and strict 2-convexity,* Math. Japonica, 38 (1993), pp. 27–33.

[3] G.L. Forti, *Hyers-Ulam stability of functional equations in several variables,* Aeq. Math., 50 (1995), pp. 143–190.

[4] D.H. Hyers, *On the stability of the linear functional equation,* Proc. National Acad. Sciences U.S.A., 27 (1941), pp. 222–224.

[5] D.H. Hyers - Th.M. Rassias, *Approximate Homomorphisms,* Aeq. Math., 44 (1992).

[6] S.M. Jung, *Hyer's-Ulam-Rassias Stability of Jensen's Equation and it's applications,* Proc. Amer. Math. Soc., 126 (1998), pp. 3137–3143.

[7] Z. Kominek, *On a local stability of a Jensen functional equation,* Demonstratio Math., 22 (1989), pp. 499–507.

[8] J.C. Pamami - H.L. Vasudeva, *On Jensen's functional equation,* Aeq. Math., 43 (1992), pp. 211–218.

[9] Th.M. Rassias, *On the stability of the linear mapping in BanachSpaces,* Proc. Amer. Math. Soc., 72 (1978), pp. 197–300.

[10] Th.M. Rassias - P. Semrl, *On the behaviour mapping which do not satisfy Hyer's Ulam Stability,* Proc. Amer. Math. Soc., 114 (1992).

[11] S.M. Ulam, *Problems in Modern Mathematics,* Chap. VI, Science Edition Wiley, New York, 1964.

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