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ABOUT PERFECTION OF CIRCULAR MIXED HYPERGRAPHS

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A mixed hypergraph is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the vertex set and each of $\mathcal{C}$ and $\mathcal{D}$ is a family of subsets of X, the $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively. A proper k-coloring of $\mathcal{H}$ is a mapping $c : X \to \{1, \ldots, k\}$ such that each $\mathcal{C}$-edge has two vertices with a common color and each $\mathcal{D}$-edge has two vertices with different colors. Maximum number of colors in a coloring using all the colors is called upper chromatic number $\bar{\chi}(\mathcal{H})$. Maximum cardinality of subset of vertices which contains no $\mathcal{C}$-edge is $\mathcal{C}$-stability number $\alpha_{\mathcal{C}}(\mathcal{H})$. A mixed hypergraph is called $\mathcal{C}$-perfect if $\bar{\chi}(\mathcal{H}') = \alpha_{\mathcal{C}}(\mathcal{H}')$ for any induced subhypergraph $\mathcal{H}'$. A mixed hypergraph $\mathcal{H}$ is called *circular* if there exists a host cycle on the vertex set X such that every edge ($\mathcal{C}$- or $\mathcal{D}$-) induces a connected subgraph on the host cycle. We give a characterization of $\mathcal{C}$-perfect circular mixed hypergraphs.

1. Introduction and Definitions

A *mixed hypergraph* is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is the *vertex set* and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of X, the *$\mathcal{C}$-edges* and *$\mathcal{D}$-edges*, respectively. A proper k-coloring of a mixed hypergraph is a mapping from the vertex set to a set of k colors so that each $\mathcal{C}$-edge has two vertices with the same color and

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each $\mathcal{D}$-edge has two vertices with different colors. A mixed hypergraph is *k-colorable (uncolorable; uniquely colorable)* if it has a proper coloring with at most *k* colors (admits no proper colorings; admits precisely one proper coloring apart from permutations of colors). A *strict k*-coloring is a proper coloring using all *k* colors. The minimum number of colors in a strict coloring of $\mathcal{H}$ is the *lower chromatic number* $\chi(\mathcal{H})$; the maximum number is the *upper chromatic number* $\bar{\chi}(\mathcal{H})$. We use $c(x)$ for the color of the vertex x.

If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a mixed hypergraph, then the subhypergraph *induced* by $X' \subseteq X$ is the mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ defined by $\mathcal{C}' = \{C \in \mathcal{C} : C \subseteq X'\}$ and $\mathcal{D}' = \{D \in \mathcal{D} : D \subseteq X'\}$. A mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ is a *partial mixed hypergraph* of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ if $X' \subseteq X, \mathcal{C}' \subseteq \mathcal{C}, \mathcal{D}' \subseteq \mathcal{D}$. For the last, we use the notation $\mathcal{H}' \subseteq \mathcal{H}$. For short, sometimes we write $(X, \mathcal{D})$ or $\mathcal{H}_{\mathcal{D}}$ instead of $\mathcal{H} = (X, \emptyset, \mathcal{D})$, and write $(X, \mathcal{C})$ or $\mathcal{H}_{\mathcal{C}}$ instead of $\mathcal{H} = (X, \mathcal{C}, \emptyset)$, keeping in mind that the respective coloring restrictions are fulfilled.

A mixed hypergraph is *reduced* if it contains no included $\mathcal{C}$-edges and no included $\mathcal{D}$-edges, and moreover, the size of each $\mathcal{C}$-edge is at least 3, and the size of each $\mathcal{D}$-edge is at least 2. As it follows from the splitting-contraction algorithm [5], the coloring properties of arbitrary mixed hypergraph can be derived from the respective reduced mixed hypergraph. Therefore, without loss of generality, throughout the paper we consider (unless contrary is stated) the reduced mixed hypergraphs.

A *host graph* [3] for a hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D}) = (X, \mathcal{E})$ is a graph $G = (X, F)$ with vertex set X (same as the one of $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$) and some edge set F, such that every $E_i \in \mathcal{E}$ induces a connected subgraph in G. There can be many different host graphs for the same $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$. If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ admits a particular host graph G, then a specific term can be applied to $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$.

A mixed hypergraph $\mathcal{H}$ is called a *mixed interval hypergraph* [1] if there exists a linear ordering of its vertices such that every edge ($\mathcal{C}$- or $\mathcal{D}$-) induces an interval in that ordering. Mixed interval hypergraphs have been introduced and investigated in [1]. If we allow the interval to continue modulo n (for a graph on n vertices) we have a circular hypergraph. Voloshin and Voss have investigated these circular, mixed hypergraphs in [6, 7].

Definition 1.1. Let $S = (x_0, C_1, x_1, C_2, x_2, \ldots, x_{t-2}, C_{t-1}, x_{t-1}, C_t, x_t)$ be a sequence of different vertices $x_0, \ldots, x_t$ and different $\mathcal{C}$-edges $C_1, \ldots, C_t$ of size 3 such that x_{i-1} is the first vertex and x_i is the last vertex of C_i. The sequence S is called a *triple-chain* if $x_t \neq x_0$ and is called a *triple-circuit (triple-cycle)* if $x_t = x_0$.

We call a sequence of $\mathcal{C}$-edges without the restrictions a *chain*. Triple-chains (and chains) in mixed hypergraphs may pass (transport) the colors from the vertex x_0 on the vertex x_t. Namely, if every two consecutive vertices of the host path corresponding to S form a $\mathcal{D}$-edge, then necessarily $c(x_0) = c(x_1) = \ldots =$

$c(x_t)$, though intermediate vertices may receive arbitrary colors different from $c(x)$. Such a triple-chain is said to be strong.

Every graph G satisfies $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the size of the largest clique. The *perfect graphs* are the graphs such that $\chi(G') = \omega(G')$ for every induced subgraph G'.

Voloshin [5] introduced a natural analogue of perfection for the upper chromatic number. In a mixed hypergraph, a set of vertices is *$\mathcal{C}$-independent* or *$\mathcal{C}$-stable* if it contains no $\mathcal{C}$-edge. The *$\mathcal{C}$-stability number* $\alpha_{\mathcal{C}}(\mathcal{H})$ is the maximum cardinality of a $\mathcal{C}$-stable set of $\mathcal{H}$. Always $\bar{\chi}(\mathcal{H}) \leq \alpha_{\mathcal{C}}(\mathcal{H})$, because a set with more distinct colors than $\alpha_{\mathcal{C}}(\mathcal{H})$ would assign different colors to all the vertices of some $\mathcal{C}$-edge. A mixed hypergraph $\mathcal{H}$ is *$\mathcal{C}$-perfect* [5] if $\bar{\chi}(\mathcal{H}') = \alpha_{\mathcal{C}}(\mathcal{H}')$ for every induced subhypergraph $\mathcal{H}'$.

The study of $\mathcal{C}$-perfection has been developing including an article by Bujtás and Tuza [3] in which they looked at the $\mathcal{C}$-perfection of mixed hypertrees, simple circular hypergraphs, and simple cycloids. The authors further the information in the latter two results in the case of mixed hypergraphs continuing the work from [6] and [7]. Several classes of $\mathcal{C}$-perfect and minimal non-$\mathcal{C}$-perfect mixed hypergraphs have been found. A *cycloid* [5] is an r-uniform $\mathcal{C}$-hypergraph denoted $\mathcal{C}_n^r$ which has n $\mathcal{C}$-edges and admits a simple cycle on n vertices as a host graph. A *polystar* is a mixed hypergraph with at least two $\mathcal{C}$-edges in which the set Y of vertices common to all $\mathcal{C}$-edges (center) is nonempty, and every pair in Y forms a $\mathcal{D}$-edge. When the center consists of one vertex then the polystar is also called a monostar. Hence, each polystar in $\mathcal{C}$-hypergraph is a monostar. A *bistar* (called co-bistar in [5]) is a mixed hypergraph in which there exists a pair of distinct vertices common to all $\mathcal{C}$-edges and not forming a $\mathcal{D}$-edge.

Bistars are $\mathcal{C}$-perfect; polystars are not [5]. Also cycloids of the form $\mathcal{C}_{2r-1}^r$ are not $\mathcal{C}$-perfect [5]. When $n = 2r-1$, we have $\alpha_{\mathcal{C}}(\mathcal{C}_n^r) = 2r-3$ and $\overline{\chi}(\mathcal{C}_n^r) = 2r-4$. These cycloids are analogous to the known minimal imperfect graphs. Polystars and cycloids of the form $\mathcal{C}_{2r-1}^r$, $r \geq 3$, are minimal non-$\mathcal{C}$-perfect mixed hypergraphs in the sense that every proper induced subhypergraph of such a cycloid is $\mathcal{C}$-perfect, and every subhypergraph of a polystar that is not a polystar is $\mathcal{C}$-perfect.

Voloshin conjectured [5] that an r-uniform $\mathcal{C}$-hypergraph is $\mathcal{C}$-perfect if and only if it has no induced monostar or cycloid of the form $\mathcal{C}_{2r-1}^r, r \geq 3$. Král' [4] found a counter-example to Voloshin's hypergraph co-perfectness conjecture concerning r-uniform hypergraphs, the related topic in simple hypergraphs. The theorem stated: The r-uniform hypergraph H^r for any $r \geq 3$ contains neither a monostar nor the complete circular hypergraph $\mathcal{C}_{2r-1}^r$ on $2r-1$ vertices, but $\overline{\chi}(H^r) < \alpha(H^r)$.

These two natural non-$\mathcal{C}$-perfect families lead to an analogue of Berge's Strong Perfect Graph Conjecture, which states that a graph G is perfect if and only if no odd cycle of length at least 5 occurs as an induced subgraph of G or $\bar{G}$. However, the situation for hypergraphs is more complex than in the case of graphs.

2. Main Results

Let us strongly obey the agreement that all the vertices are ordered on the host cycle and in all expressions like '(u,v)-path' or '$\mathcal{D}$-distance between u and v' we mean the host-path or the number of vertices minus 1 in this path respectively, when starting at u and ending at v and going according to the given cyclic ordering. Moreover, for the vertex $u \in X$ the notation u^+, $u^{++}, \ldots$ or, equivalently, u^1, $u^2, \ldots$ will always mean the first successor, the second successor, $\ldots$ of u in the cyclic ordering.

Let C and C' be two $\mathcal{C}$-edges of a circular mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ meeting each other either in one vertex or in two vertices joined by a $\mathcal{D}$-edge of size 2. Let C and C' not cover X, i.e., $C \cup C' \neq X$. The subhypergraph $\mathcal{H}'$ induced by the vertex set $C \cup C'$ has only $\mathcal{C}$-edges $C^*, C \neq C^* \neq C'$, containing $C \cap C'$. Obviously, $\bar{\chi}(\mathcal{H}') = |C \cup C'| - 2 < |C \cup C'| - 1 = \alpha_{\mathcal{C}}(\mathcal{H}')$. Consequently, if $\mathcal{H}$ contains two $\mathcal{C}$-edges meeting in a K_1 or K_2 and not covering X, then $\mathcal{H}$ is not $\mathcal{C}$-perfect.

More generally, a mixed hypergraph $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is called a $\mathcal{C}$-monostar or covered $\mathcal{C}$-bistar if $\cap\{C | C \in \mathcal{C}\}$ consists of one vertex or two vertices joined by a $\mathcal{D}$-edge of size 2, respectively. A $\mathcal{C}$-monostar and a covered $\mathcal{C}$-bistar are not perfect. For interval mixed hypergraphs Bulgaru and Voloshin proved:

Theorem 2.1. *(E.Bulgaru and V.I.Voloshin [1]). An interval mixed hypergraph $\mathcal{H}$ is $\mathcal{C}$-perfect if and only if $\mathcal{H}$ does not contain induced $\mathcal{C}$-monostars and induced covered $\mathcal{C}$-bistars.*

A mixed hypergraph $\mathcal{H}$ is called *critically $\mathcal{C}$-imperfect* if $\bar{\chi}(\mathcal{H}) < \alpha_{\mathcal{C}}(\mathcal{H})$, and each proper induced subhypergraph of $\mathcal{H}$ is $\mathcal{C}$-perfect. Monostars and covered $\mathcal{C}$-bistars with two $\mathcal{C}$-edges are critically $\mathcal{C}$-imperfect.

Let $\mathbf{S}_n$ denote the class of all circular mixed hypergraphs of order $n \geq 7$ containing no induced monostar and no induced covered $\mathcal{C}$-bistar with the following property: there is an ordering of the $\mathcal{C}$-edges $C_0, C_1, C_2, \ldots, C_{s-1}$ so that $C_j \cup C_{j+1} = X$ and $C_j \cap C_{j+1}$ induces a K_1 or a K_2, $0 \leq j \leq s-1$ (indices modulo s). If $\mathcal{H}$ contains no $\mathcal{D}$-edges of size 2, then $\mathbf{S}_n$ is empty for even n, and $\mathbf{S}_n$ is the cycloid $\mathcal{C}^r_{2r-1}$ for odd $n = 2r-1$. An example for even $n = 2s$ is $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ with $X = \{0, 1, \ldots, 2s-1\}$, $(X, \mathcal{D})$ is the $2s$-cycle, and $\mathcal{C} = \{\{0, 1, \ldots, s\} + 2t \mid 0 \leq$

$t \leq s-1\}$, indices taken modulo n. A non-uniform example of a hypergraph in $\mathbf{S}_7$ would have $\mathcal{C} = \{\{0,1,2,3\},\{2,3,4,5,6\},\{6,0,1\},\{1,2,3,4,5\},\{4,5,6,0\}\}$.

Additionally, the lemma will help in understanding the construction of much of the following.

Lemma 2.2. *Let $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ contain two $\mathcal{C}$-edges $C_1 = \{x_1^1, x_2^1, \ldots, x_{k_1}^1\}$ and $C_2 = \{x_1^2, x_2^2, \ldots, x_{k_2}^2\}$ with $x_{k_1}^1 \in C_1 \cap C_2 \cong K_1$ or $\{x_{k_1-1}^1, x_{k_1}^1\} \in C_1 \cap C_2 \cong K_2$ and $C_1 \cup C_2 = X$. Let each $\mathcal{C}$-edge $C \notin \{C_1, C_2\}$ containing $x_{k_2}^2$ have a bad intersection with C_2, i.e., C also contains $x_{k_2-1}^2$, if $(x_{k_2-1}^2, x_{k_2}^2) \notin \mathcal{D}$, or $x_{k_2-2}^2$, if $(x_{k_2-1}^2, x_{k_2}^2) \in \mathcal{D}$. Then $\bar{\chi}(\mathcal{H}) = n-2$.*

Proof. We color the mixed hypergraph in the following way: $x_{k_1-1}^1$, $x_{k_1}^1$ or $x_{k_1-2}^1$, $x_{k_1}^1$ are colored by 1, if $(x_{k_1-1}^1, x_{k_1}^1) \notin \mathcal{D}$ or $(x_{k_1-1}^1, x_{k_1}^1) \in \mathcal{D}$, respectively; $x_{k_2-1}^2, x_{k_2}^2$ or $x_{k_2-2}^2, x_{k_2}^2$ are colored by 2, if $(x_{k_2-1}^2, x_{k_2}^2) \notin \mathcal{D}$ or $(x_{k_2-1}^2, x_{k_2}^2) \in \mathcal{D}$, respectively; all remaining vertices are now colored by pairwise different colors $3, 4, \ldots, n-2$. Each $\mathcal{C}$-edge $C, C \notin \{C_1, C_2\}$, contains $x_{k_1}^1$ or $x_{k_2}^2$ (because otherwise $C \subseteq C_1$ or $C \subseteq C_2$). If $x_{k_2}^2 \in C$ then by hypothesis $x_{k_2-1}^2 \in C$ for $(x_{k_2-1}^2, x_{k_2}^2) \notin \mathcal{D}$ and $x_{k_2-2}^2 \in C$ for $(x_{k_2-1}^2, x_{k_2}^2) \in \mathcal{D}$. Since $\mathcal{H}$ contains no two $\mathcal{C}$-edges with a *good* intersection not covering X, each $\mathcal{C}$-edge C with $x_{k_1}^1 \in C$ also contains $x_{k_1-1}^1$, if $(x_{k_1-1}^1, x_{k_1}^1) \notin \mathcal{D}$ and $x_{k_1-2}^1 \in C$, if $(x_{k_1-1}^1, x_{k_1}^1) \in \mathcal{D}$. Hence, $\mathcal{H}$ has a strict coloring by $n-2$ colors, and the proof of the lemma is complete. □

Proposition 2.3. *Every hypergraph $\mathcal{H}$ from $\mathbf{S}_n$ is critically $\mathcal{C}$-imperfect.*

Proof. Each proper induced subhypergraph of $\mathcal{H}$ is the union of interval mixed hypergraphs containing no induced monostar and no induced covered $\mathcal{C}$-bistar. Then by Bulgaru and Voloshin [1] each proper induced subhypergraph of $\mathcal{H}$ is $\mathcal{C}$-perfect. So we have only to show that $\bar{\chi}(\mathcal{H}) < \alpha_{\mathcal{C}}(\mathcal{H})$.

By using the vertices at each end of a $\mathcal{C}$-edge, $\mathcal{H}$ contains two vertices covering all $\mathcal{C}$-edges. For ease of explanation, let us call these vertices of $\mathcal{C}_0$, 0 and i. Then $\mathcal{C}_1 = \{(i-1 \text{ or } i), (i \text{ or } i+1), \ldots, n-1\}, \mathcal{C}_2 = \{(n-2 \text{ or } n-1), (n-1 \text{ or } 0), \ldots, (i-2 \text{ or } i-1)\}$, and so on, iterating back and forth containing 0, i, or both. So $\alpha_{\mathcal{C}}(\mathcal{H}) = n-2$.

Moreover, the vertices of the host cycle will be partitioned into M_0, M_1, $\ldots$, M_{s-1} being in this cyclic order on the host cycle by taking the intersection $C_j \cap C_{j+1}$ for each j. In doing so, each $M_k \cong K_1$ or K_2. This is due to how $\mathbf{S}_n$ is constructed. $C_0 \cap C_1 = M_1$ and C_1's last vertex will be $n-1$ since these edges must cover X. $C_1 \cap C_2 = M_2$ and C_2's last vertex will be the vertex prior to the vertex (or vertices) in M_1. As such C_3's first vertex is the said vertex or the vertex prior. This will continue in this fashion with each M_i decrementing by one or two vertices until the construction terminates. Now suppose $\mathcal{H}$ has a strict $(n-2)$-coloring.

We consider two cases:

Case 1. Three vertices x,y,z have the same color, say 1, and the remaining $n-3$ vertices have pairwise different colors $2,3,\dots,n-2$. The vertices $x \in M_{l_1}, y \in M_{l_2}$, and $z \in M_{l_3}, l_1 \neq l_2 \neq l_3 \neq l_1$. Since $n \geq 7$, there exists an $M_l, l \notin \{l_1,l_2,l_3\}$, such that no vertex of M_l is colored 1. Let $M_l = C_j \cap C_{j+1}$ for one j, $0 \leq j \leq s-1$. Then $C_j \setminus M_l \cap C_{j+1} \setminus M_l = \emptyset$, and C_j or C_{j+1} is not suitably colored, a contradiction.

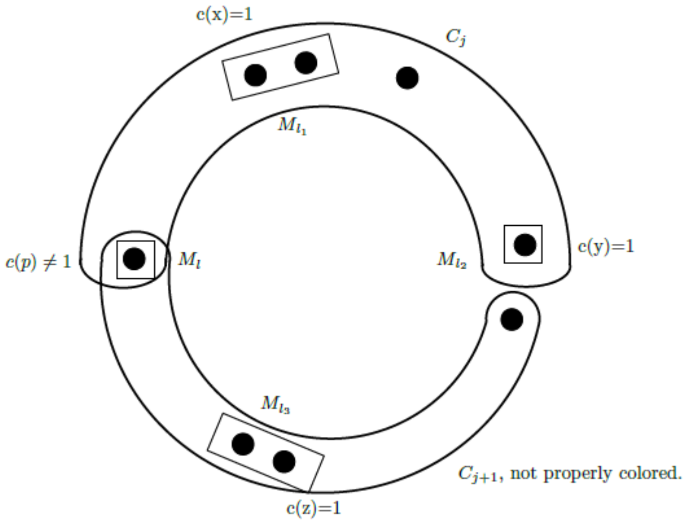

Figure 1: Case 1

Case 2. Two vertices x and y have the same color, say 1, and two other vertices p and q have the same color, say 2, and the remaining $n-4$ vertices have pairwise different colors $3,4,\dots,n-2$. The vertices $x \in M_{l_1}$ and $y \in M_{l_2}$, $l_1 \neq l_2$. If $l_1 - 1 \neq l_2 \neq l_1 + 1$ then there exists an $M_l = C_j \cap C_{j+1}$ so that $x \in C_j \setminus M_l$ and $y \in C_{j+1} \setminus M_l$. Since $(C_j \setminus M_l) \cap (C_{j+1} \setminus M_l) = \emptyset$ at least one of C_j or C_{j+1} has no two vertices of the same color and this $\mathcal{C}$-edge is not suitably colored.

Next suppose $l_2 = l_1 + 1$. Then there are a k and an l, $0 \leq k,l \leq s-1$, such that $M_l = C_k \cap C_{k+1}$ and $C_k \supseteq M_{l_1}$ and $C_{k+1} \supseteq M_{l_2}$. At least one of C_k and C_{k+1} has no two vertices of the same color and this $\mathcal{C}$-edge is not suitable colored. This contradiction proves the proposition. □

A set T of vertices is a $\mathcal{C}$-transversal (represents all $\mathcal{C}$-edges), if each $\mathcal{C}$-edge contains a vertex of T. If X' is a $\mathcal{C}$-independent set of vertices then $X \setminus X'$ is a $\mathcal{C}$-transversal of $\mathcal{H}$.

Theorem 2.4. *A circular mixed hypergraph $\mathcal{H}$ is $\mathcal{C}$-perfect if and only if $\mathcal{H} \notin \mathbf{S}_n$ and $\mathcal{H}$ does not contain induced $\mathcal{C}$-monostars and induced covered $\mathcal{C}$-bistars.*

Proof. $\Rightarrow$ If $\mathcal{H}$ is $\mathcal{C}$-perfect then $\mathcal{H} \notin \mathbf{S}_n$ and $\mathcal{H}$ does not contain induced $\mathcal{C}$-monostars and induced covered $\mathcal{C}$-bistars.

$\Leftarrow$ Suppose that $\mathcal{H} \notin \mathbf{S}_n$ and $\mathcal{H}$ does not contain induced $\mathcal{C}$-monostars and induced covered $\mathcal{C}$-bistars.
There are three cases:

Case 1. Let no two $\mathcal{C}$-edges cover all vertices of $\mathcal{H}$. Then they meet in the empty set or in an interval of the host cycle. Let $\{v_1, v_2, \ldots, v_t\}$ be a smallest $\mathcal{C}$-transversal of $\mathcal{H}$. Hence, $X \setminus \{v_1, v_2, \ldots, v_t\}$ is a maximum $\mathcal{C}$-independent set with $\alpha_{\mathcal{C}}(\mathcal{H}) = n - t$ vertices. Obviously, the upper chromatic number $\bar{\chi}(\mathcal{H}) \leq \alpha_{\mathcal{C}}(\mathcal{H}) = n - t$. We have to show that $\bar{\chi}(\mathcal{H}) = n - t$, i.e., $\mathcal{H}$ has a strict $(n-t)$-coloring.

Let S_i denote the set of all $\mathcal{C}$-edges containing v_i. Then $T_i := \cap\{C | C \in S_i\}$ is an interval of the host cycle containing v_i with end vertices y_i and z_i such that y_i precedes z_i. Since two different $\mathcal{C}$-edges which meet have a "bad" intersection (different from K_1 and K_2) $\{z_i^-, z_i\} \subseteq T_i$, if $\{z_i^- z_i\} \notin \mathcal{D}$, or $\{z_i^{--}, z_i^-, z_i\} \subseteq T_i$, if $\{z_i^- z_i\} \in \mathcal{D}$.

If $T_i \cap T_{i+1} \neq \emptyset$ with $w \in T_i \cap T_{i+1}$ then already $\{v_1, v_2, \ldots, v_{i-1}, w, v_{i+2}, \ldots, v_t\}$ is a $\mathcal{C}$-transversal with $t - 1$ vertices, a contradicton to the minimality of t. Hence, $T_i \cap T_{i+1} = \emptyset$ for all $1 \leq i \leq t$ (indices mod t). Color z_i, z_i^- or z_i, z_i^{--} with the color i, if $\{z_i z_i^-\} \notin \mathcal{D}$ or $\{z_i z_i^-\} \in \mathcal{D}$, respectively. All other vertices are colored with the colors $t+1, \ldots, n-t$ so that any two of the remaining vertices have different colors. So, $\mathcal{H}$ has a strict $(n-t)$-coloring, and $\alpha_{\mathcal{C}}(\mathcal{H}) = n - t = \bar{\chi}(\mathcal{H})$. Each induced subhypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ of $\mathcal{H}$ with $X' \neq X$ is the union of interval graphs; Theorem 2.1 implies $\bar{\chi}(\mathcal{H}') = \alpha_{\mathcal{C}}(\mathcal{H}')$. Consequently, $\mathcal{H}$ is $\mathcal{C}$-perfect.

Case 2. Let $\mathcal{H}$ contain two $\mathcal{C}$-edges A and B meeting in two intervals U and V, which do not form a common interval. Then the union $A \cup B = X$ covers all vertices of X. If $S := \cap\{C | C \in \mathcal{C}\} \neq \emptyset$ and $S \notin \{K_1, K_2\}$ then two vertices of S can be colored with the same color, say 1. Give all other vertices pairwise different colors $2, 3, \ldots, n-1$. The circular mixed hypergraph $\mathcal{H}$ has a strict $(n-1)$-coloring and $\bar{\chi}(\mathcal{H}) = n - 1 = \alpha_{\mathcal{C}}(\mathcal{H})$. Thus, $\mathcal{H}$ is $\mathcal{C}$-perfect.

If $S := \bigcap\{C | C \in \mathcal{C}\} \neq \emptyset$ and $S \cong K_1$ or K_2 then since the edges of $\mathcal{H}$ are intervals on the circular host graph it must be the case that $\mathcal{H}$ contains two $\mathcal{C}$-

edges C_1 and C_2 with $C_1 \cap C_2 = S \cup R$, where R is the other interval C_1 and C_2 must meet, since $\mathcal{H}$ does not contain a $\mathcal{C}$-monostar or a covered $\mathcal{C}$-bistar. Moreover, the union $C_1 \cup C_2 = X$. The conditions $S \subseteq A, S \subseteq B$, and $A \cup B = X$ imply that $C_1 \subseteq A$ or $C_1 \subseteq B$ or $C_2 \subseteq A$ or $C_2 \subseteq B$, a contradiction since hypergraphs exclude included edges.

In the rest of the proof, let $\cap\{C|C \in \mathcal{C}\} = \emptyset$, and $\alpha_{\mathcal{C}}(\mathcal{H}) \leq n-2$. Each $\mathcal{C}$-edge $C, C \notin \{A,B\}$, contains U and the vertices immediately preceding and succeeding U, or C contains V and the vertices immediately preceding and succeeding V. Hence, a vertex of U and a vertex of V form a transversal of $\mathcal{H}$. Thus, $\alpha_{\mathcal{C}}(\mathcal{H}) = n-2$.

In the following five subcases we shall show that $\bar{\chi}(\mathcal{H}) = n-2$. Consequently, $\mathcal{H}$ is $\mathcal{C}$-perfect.

Case 2.1. Let $U, V \ncong K_1$ or K_2. Then color the endvertices of the interval U by 1, the endvertices of the interval V by color 2, and give all other vertices the colors $3,4,\ldots,n-2$. The circular mixed hypergraph $\mathcal{H}$ has a strict $(n-2)$-coloring, and $\mathcal{H}$ is $\mathcal{C}$-perfect.

Case 2.2. Let $U \cong K_1$ (or $U \cong K_2$), $V \notin \{K_1, K_2\}$. Let u be the only vertex (or let u, u^+ be the vertices and $\{uu^+\}$ the edge) of U. Then color u^-, u^+ by color 1, the endvertices of the interval V by color 2, and give all other vertices pairwise different colors $3,4,\ldots,n-2$. The circular mixed hypergraph $\mathcal{H}$ has a strict $(n-2)$-coloring, and $\mathcal{H}$ is $\mathcal{C}$-perfect.

Case 2.3. Let $U \cong K_1 \cong V$ with $U = \{u\}$ and $V = \{v\}$. Each $\mathcal{C}$-edge C contains u, u^+, if $\{uu^+\} \notin \mathcal{D}$, and u, u^+, u^{++}, if $\{uu^+\} \in \mathcal{D}$, or C contains v, v^+, if $\{vv^+\} \notin \mathcal{D}$, and v, v^+, v^{++}, if $\{vv^+\} \in \mathcal{D}$. If $\{uu^+\} \notin \mathcal{D}$, then color u, u^+ by 1, and, if $\{uu^+\} \in \mathcal{D}$ and $u^{++} \neq v$, then color u, u^{++} by 1. If $\{vv^+\} \notin \mathcal{D}$, then color v, v^+ by 2, and, if $\{vv^+\} \in \mathcal{D}$ and $v^{++} \neq u$, then color v, v^{++} by 2. Give all other vertices pairwise different colors $3,4,\ldots,n-2$. The circular mixed hypergraph $\mathcal{H}$ has a strict $(n-2)$-coloring, and $\mathcal{H}$ is $\mathcal{C}$-perfect.

If $\{uu^+\} \in \mathcal{D}$, $u^{++} = v$ and either $\{vv^+\} \notin \mathcal{D}$ or $\{vv^+\} \in \mathcal{D}$, $v^{++} \neq u$ then color $u, u^{++} = v$, and v^+ or v^{++} by 1. Give all other vertices pairwise different colors $2,3,4,\ldots,n-2$. The circular mixed hypergraph $\mathcal{H}$ has a strict $(n-2)$-coloring, and $\mathcal{H}$ is $\mathcal{C}$-perfect.

If $\{uu^+\} \in \mathcal{D}$, $u^{++} = v$ and $\{vv^+\} \in \mathcal{D}$, $v^{++} = u$ then $\mathcal{H}$ has four vertices $v^{++} = u, u^+, u^{++} = v, v^+$, at least the edges $\{uu^+\}$ and $\{vv^+\}$, and $\mathcal{H}$ contains precisely the $\mathcal{C}$-edges $\{u, u^+, u^{++}\}$ and $\{v, v^+, v^{++}\}$. Then $\cap\{C|C \in \mathcal{C}\} = \{u, v\}$, $\bar{\chi}(\mathcal{H}) = n-1 = \alpha_{\mathcal{C}}(\mathcal{H})$, and $\mathcal{H}$ is $\mathcal{C}$-perfect.

Case 2.4. Let $U \cong K_1$ and $V \cong K_2$ with $U = \{u\}$ and $V = \{v^-, v, v^-v\}$. Let $S := \{u^-, u, u^+, \ldots, v^-\}$ and $T := \{v, v^+, \ldots, u^-, u, u^+\}$.

Case 2.4.1. Let $S \notin \mathcal{C}$. (By symmetry we could choose: $T \notin \mathcal{C}$.) Each $\mathcal{C}$-edge C contains u, u^+, if $\{uu^+\} \notin \mathcal{D}$, and u, u^+, u^{++}, if $\{uu^+\} \in \mathcal{D}$, or C contains v^-, v, v^+. If $\{uu^+\} \notin \mathcal{D}$, then color u, u^+ by 1, and, if $\{uu^+\} \in \mathcal{D},\ u^{++} \neq v$, then color u, u^{++} by 1; further, color v^-, v^+ by 2. Give all other vertices pairwise different colors $3, 4, \ldots, n-2$. If $\{uu^+\} \in \mathcal{D},\ u^{++} = v^-$, then color $u, u^{++} = v^-, v^+$ by 1. Give all other vertices pairwise different colors $3, 4, \ldots, n-2$. If $\{uu^+\} \in \mathcal{D},\ u^{++} = v^-$, then color $u, u^{++} = v^-, v^+$ by 1. In all other vertices pairwise different colors $2, 3, 4, \ldots, n-2$. In all cases the circular mixed hypergraph $\mathcal{H}$ has a strict $(n-2)$-coloring, and $\mathcal{H}$ is $\mathcal{C}$-perfect.

Case 2.4.2. Let $S, T \in \mathcal{C}$. Each $\mathcal{C}$-edge C contains u, u^+, if $\{uu^+\} \notin \mathcal{D}$, and u, u^+, u^{++}, if $\{uu^+\} \in \mathcal{D}$, or C contains v, v^+, if $\{vv^+\} \notin \mathcal{D}$, and v, v^+, v^{++}, if $\{vv^+\} \in \mathcal{D}$. (the latter assertion comes from the fact that each $\mathcal{C}$-edge C' meeting v, $C' \notin \{A, B, S, T\}$, has no "good" intersection with T as described in 2.2). Obviously, $u^{++} \neq v$. If $\{uu^+\} \notin \mathcal{D}$, the color u, u^+ by 1, and, if $\{uu^+\} \in \mathcal{D}$, then color u, u^{++} by 1. If $\{vv^+\} \notin \mathcal{D}$, then color v, v^+ by 2, and, if $\{vv^+\} \in \mathcal{D}, v^{++} \neq u$, then color v, v^{++} by 2. Give all other vertices pairwise different colors $3, 4, \ldots, n-2$. If $\{vv^+\} \in \mathcal{D},\ v^{++} = u$, then color $v, v^{++} = u, u^+$ or $v, v^{++} = u, u^{++}$ by 1. Give all other vertices pairwise different colors $2, 3, 4, \ldots, n-2$. In all cases the circular mixed hypergraph $\mathcal{H}$ has a strict $(n-2)$-coloring, and $\mathcal{H}$ is $\mathcal{C}$-perfect.

Case 2.5. Let $U \cong K_2 \cong V$ with $U = \{u^-, u, u^-u\}$ and $V = \{v^-, v, vv^-\}$. Each $\mathcal{C}$-edge contains u^-, u^+, v^-, v^+. Color u^-, u^+ by 1 and v^-, v^+ by 2. Give all other vertices pairwise different colors $3, 4, \ldots, n-2$. The circular mixed hypergraph $\mathcal{H}$ has a strict $(n-2)$-coloring, and $\mathcal{H}$ is $\mathcal{C}$-perfect.

Case 3. Let $\mathcal{H}$ not satisfy of the conditions of Cases 1 or 2. Hence, $\mathcal{H}$ contains two $\mathcal{C}$-edges A and B meeting in K_1 or K_2 and covering all vertices of X. Thus $(X, \{A, B\}, \mathcal{D})$ is a monostar or a covered $\mathcal{C}$-bistar, but it is not an induced subhypergraph. □

By 2.2, we shall show either $\bar{\chi}(\mathcal{H}) = n-2$ or the $\mathcal{C}$-edges can be ordered $C_0, C_1, \ldots, C_{s-1}$ so that $C_j \cup C_{j+1} = X$ and $M_j := C_j \cap C_{j+1}$ induces a K_1 or a K_2, $0 \leq j \leq s-1$ (indices modulo s), and $\mathcal{H} \in \mathbf{S}_n$. Let $C_0 = A,\ C_1 = B$ and $M_1 =$

$C_0 \cap C_1 \cong K_1$ or K_2. Let $x^1_{k_1}$ be the end vertex of C_1 not in M_1. If each $\mathcal{C}$-edge $C \neq C_1$ containing $x^1_{k_1}$ has a bad intersection with C_1 then the Lemma implies $\bar{\chi}(\mathcal{H}) = n-2$ and $\mathcal{H}$ is $\mathcal{C}$-perfect. Next, let there be a $\mathcal{C}$-edge C_2 with $M_2 := C_1 \cap C_2 \cong K_1$ or K_2. Since $\mathcal{H}$ has no induced monostar and no induced covered $\mathcal{C}$-bistar, the $\mathcal{C}$-edge C_2 is unique. Thus, a chain C_0, C_1, C_2 is uniquely constructed. By induction, the existence of the ordered sequence $C_0, C_1, C_2, \dots, C_{s-1}$ can be proved, or $\bar{\chi}(\mathcal{H}) = n-2$ and $\mathcal{H}$ is $\mathcal{C}$-perfect. In the first case, $\mathcal{H}$ cannot have additional $\mathcal{C}$-edges, and $\mathcal{H} \in \mathbf{S}_n$. □

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