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FAMILIES OF LINES IN FANO VARIETIES COMPLETE INTERSECTION IN A GRASSMANNIAN

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In this paper we determine all Fano varieties which are complete intersections of hypersurfaces in a Grassmannian. Then, in the case Fano's conjecture is satisfied, we give a formula in order to compute the dimension of the Hilbert scheme that parametrises their lines.

1. Introduction.

In this paper we give a method to determine all Fano varieties that are complete intersections of one or more hypersurfaces in a Grassmannian and we study the dimension of the Hilbert scheme of the families of their lines. First, we give a numerical charecterization of these Fano varieties and we classify them up to dimension 5 (see section 2). Then we extend the so-called Tennison method, that is a modern treatment of a classic work by Predonzan ([9],[13]) concerning a complete study of the family of lines that are contained in the quartic hypersurface of $\mathbb{P}^4$. This method can be extended studying the projections of the incidence variety

$$\mathbb{I} = \{(\ell, X) \in G(2, r+n+1) \times \mathbb{Q}/\ell \subseteq X\} \subseteq G(2, r+n+1) \times \mathbb{Q},$$

where $\mathbb{Q}$ is the algebraic variety parametrising all the n-dimensional complete intersections of r hypersurfaces of degrees n_i respectively in a projective space

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(over $\mathbb{C}$) and $G(2, r + n + 1)$ is the Grassmannian of the lines in $\mathbb{P}^{n+r}$. In this form we determine exactly the dimension of the Hilbert scheme $\mathcal{F}(X)$ that parametrises the lines of a complete intersection X of a finite number of generic hypersurfaces in a projective space and, when this is a Fano threefold, we obtain that $1 \leq \dim \mathcal{F}(X) \leq 4$. Our aim is to extend the Tennison-Predonzan's result for the determination of the dimension of the Hilbert scheme $\mathcal{F}(X)$ that parametrises the lines of any Fano complete intersection in a Grassmannian. This is possible provided that there exists at least one line. (In general, we know that when the dimension of a smooth Fano variety plus one is twice minor to its index, then it contains at least one line ([6], page 248), but the complete intersection of hypersurfaces in a Grassmannian does not verify this property). Besides using the incidence varieties, this extension also requires the techniques of Schubert's calculus in the cohomology ring of the Grassmannian and the use of the universal quotient bundle.

Precisely, we prove that for any n-fold X which is a Fano variety complete intersection in a Grassmannian containing at least one line, the dimension of $\mathcal{F}(X)$ is greater than or equal to $n-3+\text{index}(X)$. As an immediate consequence we obtain that the family of lines of a smooth Fano threefold which is a complete intersection of generic hypersurfaces in a projective space or in a Grassmannian, has dimension equal to the index of the variety. In particular, we find again that the dimension of the Hilbert scheme that parametrises the lines for the intersection $G(2, 6) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5$ of the Plücker image of $G(2, 6)$ with five generic hyperplanes of $\mathbb{P}^{14}$ and the intersection $G(2, 5) \cap H_1 \cap H_2 \cap Q$ of the Plücker image of G(2,5) with two generic hyperplanes and a generic quadric in $\mathbb{P}^9$, denoted by $\mathbb{V}_3^{14}$ and $\mathbb{V}_3^{10}$ respectively, is exactly one. In fact, Shŏkurov proves Fano's conjecture for Fano threefolds of the type $\mathbb{V}_{2g-2}$ in $\mathbb{P}^{g+1}$ and of the first kind ([12], [5]). There is a general result obtained by Iskovskih (1989) ([8]), which says that for the varieties that have index 1 and genus ≥ 3, the Hilbert scheme which parametrises the lines has dimension exactly one. Since the intersection X of the Grassmannian of the lines on $\mathbb{P}^4$ with three generic hyperplanes contains a Del Pezzo surface and therefore it contains some lines, then we find again that the family of lines of X is parametrised by a 2-dimensional scheme.

2. Fano Varieties, complete intersection in Grassmannians.

Let X be an algebraic variety and let $-K_X$ be its anticanonical divisor, then:

Definition 1. *A smooth projective algebraic variety X is Fano if $-K_X$ is ample.*

The largest integer $r \geq 1$ such that $-K_X \sim rH$ for some divisor H of X is called the index of the variety. A Fano variety is called of the principal series if its anticanonical divisor is very ample, and of the first kind if its Picard Group is isomorphic to $\mathbb{Z}$.

Remark 1. A smooth complete intersection $X = V(n_1, \ldots, n_r) \subseteq \mathbb{P}^{r+n}$ of r generic hypersurfaces of degrees $n_1, \ldots, n_r$ respectively has canonical divisor $K_X \sim (\sum_{i=1}^{r} n_i - r - n - 1)H$, with H the hyperplane divisor of X.
Then X is a Fano variety if and only if $\sum_{i=1}^{r} n_i \leq r + n$. When $\dim X \geq 3$, by Noether-Lefschetz theorem, the Picard group is isomorphic to $\mathbb{Z}$ and it is generated by the hyperplane section, then the index of X is $r+n+1-\sum_{i=1}^{r} n_i$. [2], page 179, Cor. 3.2 . When $\dim X = 2$ there are only three Fano complete intersections, which are Del Pezzo surfaces, namely the quadric $V(2)$ of $\mathbb{P}^3$, the cubic $V(3)$ of $\mathbb{P}^3$ and the complete intersection of two quadrics $V(2,2)$ of $\mathbb{P}^4$. These surfaces have index respectively 2, 1 and 1 [11], page 233, so also in dimension two the above formula for the index of a Fano complete intersection holds.
We remark that the three two-dimensional Fano complete intersections are the only Fano complete intersections with Picard group not isomorphic to $\mathbb{Z}$, i.e. not of the first kind; on the other hand every Fano complete intersection is of the principal series.

Let $pl : G(k,m) \hookrightarrow \mathbb{P}^{\binom{m}{k}-1}$ be the Plücker embedding (often we omit to explicity mention the embedding pl), and let us denote by

$$X = G(k,m) \cap V(n_1) \cap \ldots \cap V(n_r)$$

where $V(n_i)$ $(i = 1, \ldots, r)$ are r generic hypersurfaces of degrees n_i respectively in $\mathbb{P}^{\binom{m}{k}-1}$, with $r = \dim G(k,m) - n$, where n is the dimension of X. For this type of n-dimensional varieties, we now prove two important properties.

Proposition 1.
a) *X is Fano* $\Leftrightarrow \sum_{i=1}^{r} n_i - m < 0$.
b) *X is Fano and* $\dim X \geq 2 \Rightarrow \operatorname{index}(X) = m - \sum_{i=1}^{r} n_i$.

Proof.
a) Recall first that the canonical divisor of the Grassmannian $G(k,m)$ of the $(k-1)$-planes in $\mathbb{P}^{m-1}$ is $K_{G(k,m)} \sim -mE$ [1], page 79, where E is the hyperplane divisor.

Let us consider

$$X_1 = G(k,m) \cap V(n_1)$$

$$
\begin{array}{lllll}
X_2 & = & X_1 \cap V(n_2) & = & G(k,m) \cap V(n_1) \cap V(n_2) \\
 & \vdots & & & \\
X_{r-1} & = & X_{r-2} \cap V(n_{r-1}) & = & G(k,m) \cap V(n_1) \cap \ldots \cap V(n_{r-1}) \\
X_r & = & X_{r-1} \cap V(n_r) & = & G(k,m) \cap V(n_1) \cap \ldots \cap V(n_r) = X
\end{array}
$$

Denote by E_i the hyperplane divisor of X_i, for $i = 1, \ldots, r-1$, and by H the hyperplane divisor of X. Using the adjunction formula, if we consider the smooth one-codimensional subvariety X_1 of $G(k,m)$, we have $K_{X_1} \sim (K_{G(k,m)} + X_1) \cdot X_1 \sim (-mE + n_1 E) \cdot X_1 \sim (n_1 - m)E_1$. Iterating until $X_r = X$, for the canonical divisor of X we obtain that $K_X \sim \left(\sum_{i=1}^r n_i - m\right) H$. Hence: X is Fano $\Leftrightarrow$ $\sum_{i=1}^r n_i - m < 0$.

b) If dim ≥ 3, since $Pic(G(k,m)) \cong \mathbb{Z}$, then, again by Noether-Lefschetz Theorem, $Pic(X) \cong \mathbb{Z}$ ([2], page 180, Cor. 3.3) and then index $(X) = m - \sum_{i=1}^r n_i$. If $\dim X = 2$, then there are only three possibility (see the following classification), so by a direct computation we find that the above formula for the index is verified also in this case. □

Corollary 1. *Let X be Fano then $V(n_1) \cap \ldots \cap V(n_r)$ is Fano.*

Proof. In fact, $m - 1 \leq \binom{m}{k} - 1$ because $m > k$. □

Remark 2. Note that every Fano variety section of a Grassmannian is of the principal series and of the first kind except the three Fano surfaces (see the following classification) which have a very ample anticanonical divisor, but their Picard group is not isomorphic to $\mathbb{Z}$.

Now we want to classify the Fano varieties X as above, with $\dim X \leq 5$. As a matter of fact one finds a curve, three surfaces, six threefolds, nine fourfolds, thirteen fivefolds. The method can be easily extended. In fact, since $n_i \geq 1 \quad \forall i$, then $r - m < 0$. Being $r = k(m-k) - n$, we have $m(k-1) - k^2 - n < 0$; and since $G(k,m) \cong G(m-k,m)$, we impose that $m \geq 2k$. Therefore using the formulas:

- $k \geq 2$,
- $k^2 - 2k - n < 0$,
- $2k \leq m < \frac{k^2+n}{k-1}$,
- $\sum_{i=1}^r n_i < m$;

we give a classification including for each of them the degree, the index, the number $d = (-K_X)^n$, where d is the degree of the variety $\phi_{|-K_X|}(X)$ under the anticanonical embedding (in $\mathbb{P}^{\dim|-K_X|}$) and furthermore, when X is a threefold, the integer invariant $g = \frac{1}{2}(-K_X^3) + 1$ corresponding to the genus of the canonical curve contained into the $K3$ surface embedded in X when $-K_X$ is very ample, ([4], page 488, [1], page 65). Let us denote by H, H_i the hyperplanes; by $V(2)$, $V(2)'$, $V(3)$, $V(4)$ the hypersurfaces (quadrics, cubics, quartics respectively) and by $V(n_1, \dots, n_r)$ the complete intersections in a projective space, here is a list with the varietes that we found:

Curve:
$G(2,4) \cap H_1 \cap H_2 \cap H_3$, the Grassmannian of the lines of $\mathbb{P}^3$, i.e. the Klein Quadric, cut out by three generic hyperplanes and is a rational curve, as known.

Surfaces:

	Fano surface	degree	$-K$	$d = (-K)^2$
1	$G(2,4) \cap H_1 \cap H_2 = V(2) \subseteq \mathbb{P}^3$	2	$2H$	8
2	$G(2,4) \cap H \cap V(2) = V(2,2) \subseteq \mathbb{P}^4$	4	H	4
3	$G(2,5) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \subseteq \mathbb{P}^5$	5	H	5

Threefolds:

	Fano threefold	degree	$-K$	$d = (-K)^3$	g
1	$G(2,4) \cap H = V(2) \subseteq \mathbb{P}^4$	2	$3H$	54	28
2	$G(2,4) \cap V(2) = V(2,2) \subseteq \mathbb{P}^5$	4	$2H$	32	17
3	$G(2,4) \cap V(3) = V(2,3) \subseteq \mathbb{P}^5$	6	H	6	4
4	$G(2,5) \cap H_1 \cap H_2 \cap H_3 \subseteq \mathbb{P}^6$	5	$2H$	40	21
5	$G(2,5) \cap H_1 \cap H_2 \cap V(2) \subseteq \mathbb{P}^7$	10	H	10	6
6	$G(2,6) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \subseteq \mathbb{P}^9$	14	H	14	8

Fourfolds:

	Fano fourfold	degree	$-K$	$d = (K)^4$
1	$G(2,4) = V(2) \subseteq \mathbb{P}^5$	2	$4H$	512
2	$G(2,5) \cap H_1 \cap H_2 \subseteq \mathbb{P}^7$	5	$3H$	405
3	$G(2,5) \cap H \cap V(2) \subseteq \mathbb{P}^8$	10	$2H$	160
4	$G(2,5) \cap H \cap V(3) \subseteq \mathbb{P}^8$	15	H	15
5	$G(2,5) \cap V(2) \cap V(2)' \subseteq \mathbb{P}^9$	20	H	20
6	$G(2,6) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \subseteq \mathbb{P}^{10}$	14	$2H$	224
7	$G(2,6) \cap H_1 \cap H_2 \cap H_3 \cap V(2) \subseteq \mathbb{P}^{11}$	28	H	28
8	$G(2,7) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \cap H_6 \subseteq \mathbb{P}^{14}$	42	H	42
9	$G(3,6) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \subseteq \mathbb{P}^{14}$	42	H	42

Fivefolds:

	Fano fivefold	degree	$-K$	$d = (-K)^5$
1	$G(2,5) \cap H \subseteq \mathbb{P}^8$	5	$4H$	5120
2	$G(2,5) \cap V(2) \subseteq \mathbb{P}^9$	10	$3H$	2430
3	$G(2,5) \cap V(3) \subseteq \mathbb{P}^9$	15	$2H$	480
4	$G(2,5) \cap V(4) \subseteq \mathbb{P}^9$	20	H	20
5	$G(2,6) \cap H_1 \cap H_2 \cap H_3 \subseteq \mathbb{P}^{11}$	14	$3H$	3402
6	$G(2,6) \cap H_1 \cap H_2 \cap V(2) \subseteq \mathbb{P}^{12}$	28	$2H$	896
7	$G(2,6) \cap H_1 \cap H_2 \cap V(3) \subseteq \mathbb{P}^{12}$	42	H	42
8	$G(2,6) \cap H \cap V(2) \cap V(2)' \subseteq \mathbb{P}^{13}$	56	H	56
9	$G(2,7) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \subseteq \mathbb{P}^{15}$	42	$2H$	1344
10	$G(2,7) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap V(2) \subseteq \mathbb{P}^{16}$	84	H	84
11	$G(2,8) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5 \cap H_6 \cap H_7 \subseteq \mathbb{P}^{20}$	122	H	122
12	$G(3,6) \cap H_1 \cap H_2 \cap H_3 \cap H_4 \subseteq \mathbb{P}^{15}$	42	$2H$	1344
13	$G(3,6) \cap H_1 \cap H_2 \cap H_3 \cap V(2) \subseteq \mathbb{P}^{16}$	84	H	84

3. The family of lines in a *n*-dimensional Fano complete intersection.

In this section we study the dimension of the family of lines of a Fano complete intersection of hypersurfaces in a projective space. First we show some classical important examples of threefolds. Then, in Theorem 1, we prove an extension of Tennison's method [13] which he realized for the study of the family of the lines of the quartic hypersurface of $\mathbb{P}^4$.

Example 1. For the study of the family of the lines of a complete intersection $V(2,3)$ in $\mathbb{P}^5$, we consider the Hilbert scheme $\mathbb{Q}$ that parametrises the smooth complete intersections of type $V(2,3) \subseteq \mathbb{P}^5$. We have the natural morphism

$$\pi : \mathbb{Q} \longrightarrow \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))).$$

If Q is a generic smooth quadric in $\mathbb{P}^5(\mathrm{rk}Q = 6)$, then we determine exactly the dimension of the generic fiber $\pi^{-1}(Q)$. Using the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^5} \longrightarrow \mathcal{O}_Q \longrightarrow 0$$

tensoring by 3

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^5}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^5}(3) \longrightarrow \mathcal{O}_Q(3) \longrightarrow 0$$

and passing in cohomology, since $H^1(\mathcal{O}_{\mathbb{P}^5}(1)) = 0$, we obtain the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^5}(3)) \longrightarrow H^0(\mathcal{O}_Q(3)) \longrightarrow 0$$

on the other hand $\mathbb{P}(H^0(\mathcal{O}_Q(3))) \cong \pi^{-1}(Q)$, Then: $\dim \pi^{-1}(Q) = 49$. Now we can conclude that

$$\dim \mathbb{Q} = \dim \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))) + \dim \pi^{-1}(Q) = 69.$$

To determine the dimension of the scheme that parametrises the lines it is necessary to consider the fibers of the projections pr_1 and pr_2 naturally restricted to the first and second factor respectively of the incidence variety $\mathbb{I} = \{(\ell, X) \in G(2, 6) \times \mathbb{Q}/\ell \subseteq X\} \subseteq G(2, 6) \times \mathbb{Q}$.
Since pr_1 has irreducible fibers of constant dimension and the Grassmannian is an irreducible variety, for [12], Theorem 8 Chapter 1 Section 5, we have that $\mathbb{I}$ is irreducible and that

$$\dim \mathbb{I} = \dim G(2, 6) + \dim pr_1^{-1}(\ell_0) = 70.$$

Let us remark that $\dim pr_1^{-1}(\ell_0) = 62$ and we get that $\dim pr_2^{-1}(V(2, 3)) = 1$.

Example 2. For the study of the family of the lines of the complete intersection $V(2, 2, 2)$ in $\mathbb{P}^6$, we consider the Hilbert scheme $\mathbb{Q}$ that parametrises all smooth complete intersections of type $V(2, 2, 2) \subseteq \mathbb{P}^6$. It is clear that $\dim \mathbb{Q} = \dim G(3, H^0\mathcal{O}_{\mathbb{P}^6}(2)) = 75$. In the same way of the previous example we obtain that $\dim pr_1^{-1}(\ell_0) = 66$ and also $\dim pr_2^{-1}(V(2, 2, 2)) = 1$.

Remark 3. In both examples we have used the surjectivity of pr_2, essential point to determine the result of the following

Theorem 1. *If X is a Fano n-fold ($n \geq 2$) complete intersection of r generic hypersurfaces of degrees $n_1, \dots, n_r$ respectively, then the scheme $\mathcal{F}(X)$ that parametrises the lines on X has dimension $n - 3 + index(X)$.*

Proof. Let us consider the scheme $\mathbb{Q}$ parametrising the smooth complete intersections of type $X = V(n_1, ..., n_r)$ in $\mathbb{P}^{r+n}$. Since $\mathbb{Q}$ is dominated by a dense open set $\mathcal{P}$ of the variety

$$\mathbb{P}(H^0(\mathbb{P}^{r+n}, \mathcal{O}_{\mathbb{P}^{r+n}}(n_1))) \times \ldots \times \mathbb{P}(H^0(\mathbb{P}^{r+n}, \mathcal{O}_{\mathbb{P}^{r+n}}(n_r))),$$

and knowing that

$$\dim \mathbb{P}(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(s))) = \binom{n+s}{n} - 1$$

clearly:

$$d_{\mathbb{Q}} := \dim \mathbb{Q} \leq \sum_{i=1}^{r} \binom{r+n+n_i}{r+n} - r.$$

Let us consider the incidence varieties:

$$\mathbb{I}' = \{(\ell, (V(n_1), \ldots, V(n_r))) \in G(2, r+n+1) \times \mathcal{P}/\ell \subseteq V(n_1) \cap \cdots \cap V(n_r)\}$$

$$\subseteq G(2, r+n+1) \times \mathcal{P}$$

and

$$\mathbb{I} = \{(\ell, V(n_1, \ldots, n_r)) \in G(2, r+n+1) \times \mathbb{Q}/\ell \subseteq V(n_1, \ldots, n_r)\}$$

$$\subseteq G(2, r+n+1) \times \mathbb{Q}$$

and the projections pr_1', pr_2', pr_1, pr_2 onto the first and the second factor respectively. In order to complete the proof of the Theorem we demonstrate some Lemmas.

Lemma 1. *The variety $\mathbb{I}'$ is an irreducible variety of dimension*

$$2n - 2 + \sum_{i=1}^{r} \left[\binom{r+n+n_i}{r+n} - n_i \right].$$

Proof. We have the following diagram

$$\begin{tikzcd} \mathbb{I}' \arrow[r, "pr_2'"] \arrow[d, "pr_1'"'] & \mathcal{P} \\ G(2, r+n+1) & \end{tikzcd}$$

If $\ell_0 \in G(2, r+n+1)$, then the fiber can be described as

$$pr_1'^{-1}(\ell_0) = \{(\ell_0, Y) \in \{\ell_0\} \times \mathcal{P}/\ell_0 \subseteq Y\} \cong \{Y \in \mathcal{P}/Y \supseteq \ell_0\}.$$

However, the fact that $\ell_0 \subseteq Y$, means that the line ℓ_0 must be contained in each of the r hypersurfaces $V(n_i)$, with $i = 1, \ldots, r$. Hence $\ell_0 \subseteq Y$ imposes $n_i + 1$ linearly independent conditions, for every $i = 1, \ldots, r$. Therefore $pr_1'^{-1}(\ell_0)$ is isomorphic to a product of linear subspaces in $\mathcal{P}$ of codimension $n_1 + \ldots + n_r + r$, so that

$$\dim pr_1'^{-1}(\ell_0) = \sum_{i=1}^{r} \binom{r+n+n_i}{r+n} - \sum_{i=1}^{r} n_i - 2r.$$

Since $G(2, r+n+1)$ is an irreducible variety and pr_1' has irreducible fibers of constant dimension, we obtain the irreducibility of $\mathbb{I}'$. Moreover:

$$\begin{aligned}\dim \mathbb{I}' &= \dim G(2, r+n+1) + \dim pr_1'^{-1}(\ell_0)\\ &= 2n - 2 + \sum_{i=1}^{r}\left[\binom{r+n+n_i}{r+n} - n_i\right]. \qquad \square\end{aligned}$$

We can prove also the

Lemma 2. *The variety $\mathbb{I}$ is an irreducible variety of dimension*

$$2n - 2 + r + d_{\mathbb{Q}} - \sum_{i=1}^{r} n_i.$$

Proof. We now proceed in the same way as in Lemma 1, the problem is to guarantee that all conditions imposed by the line are independent. If we consider the following commutative diagram of the natural morphisms

$$\begin{tikzcd} \mathbb{I}' \arrow[r, "pr_2'"] \arrow[d, "\pi'"'] & \mathcal{P} \arrow[d, "\pi"] \\ \mathbb{I} \arrow[r, "pr_2"] & \mathbf{Q} \end{tikzcd}$$

we can compare these conditions with those of Lemma 1. In fact, since we know that the generic fiber of π has constant dimension, that is

$$\dim \pi^{-1}(X_0) = \sum_{i=1}^{r} \binom{r+n+n_i}{r+n} - r - d_{\mathbb{Q}}, \quad X_0 \in \mathbb{Q},$$

we obtain automatically that $\pi^{-1}(X_0)$ has the same dimension of the fiber $\pi^{'-1}(\ell, X_0)$. Then:

$$\dim \mathbb{I} = \dim \mathbb{I}' - \dim \pi^{'-1}(\ell, X_0) = 2n - 2 + r + d_{\mathbb{Q}} - \sum_{i=1}^{r} n_i. \qquad \square$$

The following Lemma is the answer to Fano's Conjecture for this type of varieties.

Lemma 3. *The projection pr_2 is surjective.*

Proof. Suppose that pr_2 is not surjective and that $Im(pr_2)$ is denoted by $\mathbb{Q}_1$, then $\dim \mathbb{Q}_1 = \dim \mathbb{Q} - \varepsilon$, with $\varepsilon > 0$. We have the diagram

$$\begin{tikzcd}
\mathbb{I} \arrow[r, "pr_1"] \arrow[d] \arrow[dr, "pr_2"] & G(2, r+n+1) \\
\mathbb{Q}_1 \arrow[r, hook] & \mathbb{Q}
\end{tikzcd}$$

where the dimension of the generic fiber of X_0 is:

$$\dim pr_2^{-1}(X_0) = \dim \mathbb{I} - \dim \mathbb{Q}_1 = 2n - 2 - \sum_{i=1}^{r} n_i + r + \varepsilon.$$

Now, we consider a fixed line $\ell_1 \in G(2, r+n+1)$, and the schemes:

$$\mathbb{Q}' = \{X \in \mathbb{Q}/\ell_1 \subseteq X\}$$

and

$$\mathbb{J}' = \{(\ell, X) \in G(2, r+n+1) \times \mathbb{Q}'/\ell \subseteq X\} \subseteq \mathbb{I}.$$

We can observe that $\dim \mathbb{Q}' = d_{\mathbb{Q}} - \sum_{i=1}^{r} n_i - r$, because $\mathbb{Q}' \cong pr_1^{-1}(\ell_1)$. Furthermore, the projection $t : \mathbb{J}' \longrightarrow \mathbb{Q}'$, with $t = pr_2 \mid_{\mathbb{J}'}$, has the fibers of dimension $2n - 2 - \sum_{i=1}^{r} n_i + r + \varepsilon$, because $t^{-1}(X_0) = \{(\ell, X_0) \in G(2, r+n+1) \times \mathbb{Q}'/\ell \subseteq X_0\} \cong \{\ell/\ell \subseteq X_0\} = pr_2^{-1}(X_0)$. Then:

$$(*) \qquad \begin{aligned} \dim \mathbb{J}' &= 2n - 2 - \sum_{i=1}^{r} n_i + r + \varepsilon + d_{\mathbb{Q}} - \sum_{i=1}^{r} n_i - r \\ &> 2n - 2 + d_{\mathbb{Q}} - 2\sum_{i=1}^{r} n_i. \end{aligned}$$

Let $\mathbb{J}$ be an irreducible component of maximal dimension of $\mathbb{J}'$ that dominates $\mathbb{Q}'$ under t, then $\dim \mathbb{J} = \dim \mathbb{J}'$. We consider the diagram

$$\begin{tikzcd} & \mathbb{J} \arrow[r, "p_2"] \arrow[dl, "p_1'"'] \arrow[d, "p_1"] & \mathbb{Q} \\ Im(p_1) = G_1 \arrow[r, hook] & G(2, r+n+1) & \end{tikzcd}$$

where p_1 e p_2 are the corresponding projections, p_1' is the induced morphism and:

$$\ell \in G_1 \Leftrightarrow \exists X \in \mathbb{Q}/X \supseteq \ell \cup \ell_1 \quad \text{and} \quad (\ell, X) \in \mathbb{J}.$$

We have the following three possible cases:

1) $G_1 = \{\ell_1\}$. Then:

$$p_1'^{-1}(\ell_1) = \{(\ell_1, X) \in \mathbb{J}/X \supseteq \ell_1\} \subseteq \{(\ell_1, X) \in G(2, r+n+1) \times \mathbb{Q}/X \supseteq \ell_1\}$$

$$\cong \{X \in \mathbb{Q}/X \supseteq \ell_1\}$$

with dimension: $d_{\mathbb{Q}} - \sum_{i=1}^r n_i - r$, therefore

$$\begin{aligned} \dim \mathbb{J} &\leq \dim G_1 + \dim p_1'^{-1}(\ell_1) \\ &\leq 0 + d_{\mathbb{Q}} - \sum_{i=1}^r n_i - r \\ &\leq 2n - 2 + d_{\mathbb{Q}} - 2\sum_{i=1}^r n_i \\ &< \dim \mathbb{J}'. \end{aligned}$$

Using the Fano condition we have a contradiction! (with $(*)$.)

2) $G_1 \neq \{\ell_1\}$ and for any $\ell \in G_1$, $\ell_1 \cap \ell \neq \emptyset$, then:

$$G_1 \subseteq \{\ell \in G(2, r+n+1)/\ell_1 \cap \ell \neq \emptyset\}$$

corresponding to the Schubert cycle σ_{r+n-2} of codimension $r+n-2$ in $G(2, r+n+1)$ and

$$p_1'^{-1}(\ell) \subseteq \{(\ell, X) \in G(2, r+n+1) \times \mathbb{Q}/X \supseteq \ell \cup \ell_1\}$$

Since $X \supseteq \ell$, and $V(n_i) \supseteq \ell$ then for every i $(i = 1, \dots, r)$ there are $n_i + 1$ independent conditions. On the other hand since $X \supseteq \ell_1$ and $V(n_i) \supseteq \ell_1$ then for every i $(i = 1, \dots, r)$, there are only n_i independent conditions, in fact ℓ and ℓ_1 have a common point. For example, if X is a quadric, there are 3 independent conditions because it contains a line, moreover since the two lines have a common point then the conditions imposed by the other line are only 2. It is clear that analogous considerations can be repeated for intersections of hypersurfaces of higher degree. Then as in Lemma 1 and Lemma 2 we have that all conditions are independent; it follows that:

$$\dim p_1^{'-1}(\ell) \leq d_{\mathbb{Q}} - 2\sum_{i=1}^{r} n_i - r$$

therefore,

$$\begin{array}{rcl}\dim \mathbb{J} & \leq & \dim G_1 + \dim p_1^{'-1}(\ell) \\ & \leq & 2r + 2n - 2 - r - n + 2 + d_{\mathbb{Q}} - 2\sum_{i=1}^{r} n_i - r \\ & = & n + d_{\mathbb{Q}} - 2\sum_{i=1}^{r} n_i \\ & < & \dim \mathbb{J}'.\end{array}$$

Contradiction!

3) There exists $\ell \in G_1$, such that $\ell \cap \ell_1 = \emptyset$. Remember that $\{\ell \in G(2, r + n + 1)/\ell \cap \ell_1 = \emptyset\}$ corrispond to the Schubert cycle of codimension 0. In this case we have that $\dim G_1 \leq 2n - 2 + 2r$. Now,

$$\dim p_1^{'-1}(\ell) \leq d_{\mathbb{Q}} - 2\sum_{i=1}^{r} n_i - 2r$$

In fact, $X \supseteq \ell$ imposes $\sum_{i=1}^{r} n_i + r$ conditions, and $X \supseteq \ell_1$ also imposes $\sum_{i=1}^{r} n_i + r$ conditions. They are all independent among them because we have skew lines, but we can repeat the same consideration as before; therefore,

$$\begin{array}{rcl}\dim \mathbb{J} & \leq & \dim G_1 + \dim p_1^{'-1}(\ell) \\ & \leq & \dim G(2, r + n + 1) + d_{\mathbb{Q}} - 2\sum_{i=1}^{r} n_i - 2r \\ & = & 2(r + n - 1) + d_{\mathbb{Q}} - 2\sum_{i=1}^{r} n_i - 2r \\ & = & 2n - 2 + d_{\mathbb{Q}} - 2\sum_{i=1}^{r} n_i \\ & < & \dim \mathbb{J}'.\end{array}$$

Contradiction!

And then $\varepsilon = 0$. $\square$

Now, using that pr_2 is surjective we can conclude the proof of the Theorem, in fact

$$\dim pr_2^{-1}(X_0) = 2n - 2 + r - \sum_{i=1}^{r} n_i = n - 3 + \mathrm{index}(X). \qquad \square$$

Corollary 2. *If $X \subseteq \mathbb{P}^n$ is a Fano 3-fold complete intersection of a finite number of generic hypersurfaces then* $\dim \mathcal{F}(X) = $ *index*(X).

Remark 4. We obtain, in particular, that if $\sum_{i=1}^{r} n_i = r + 3$ then $\dim \mathcal{F}(X) = 1$.

This holds true for:

- $V(4) \subseteq \mathbb{P}^4$;
- $V(2, 3) \subseteq \mathbb{P}^5$;
- $V(2, 2, 2) \subseteq \mathbb{P}^6$.

that is for the ones having index 1.

When $\sum_{i=1}^{r} n_i < r + 3$, then $\dim \mathcal{F}(X) \geq 1$ and we have that

- For $V(1)$ the hyperplane (a $\mathbb{P}^3$) of $\mathbb{P}^4$, having index 4, one has $\dim \mathcal{F}(X) = 4$.
- For the quadric hypersurface $V(2) \subseteq \mathbb{P}^4$, having index 3, one has $\dim \mathcal{F}(X) = 3$.
- For cubic hypersurface $V(3) \subseteq \mathbb{P}^4$ and for $V(2, 2)$, the intersection of 2 quadrics of $\mathbb{P}^5$, both of index 2, one has $\dim \mathcal{F}(X) = 2$.

4. The family of lines in Fano varieties sections of Grassmannians.

Theorem 2. *Let X be a n-dimensional Fano variety ($n \geq 2$) which is the intersection of $G(k, m) \subseteq \mathbb{P}^{\binom{m}{k}-1}$ with a finite number of generic hypersurfaces $V(n_1), \ldots, V(n_r)$, that is a complete intersection in the Grassmannian $G(k, m)$, and assume that X contains at least one line. Then the Hilbert scheme $\mathcal{F}(X)$ that parametrises its lines has dimension greater than or equal to $n - 3 +$ index(X).*

Proof. Let $X = G(k, m) \cap V(n_1) \cap \ldots \cap V(n_r) \subseteq \mathbb{P}^{\binom{m}{k}-1}$, with $r = \dim G(k, m) - n$ and let n_i be the degree of each hypersurface $V(n_i)$. Now, let $\ell_0 \in G(2, \binom{m}{k})$ such that $\ell_0 \subseteq X$. Then:

$$\ell_0 \subseteq X \quad \Leftrightarrow \quad \ell_0 \subseteq V(n_1) \cap \ldots \cap V(n_r) \quad \wedge \quad \ell_0 \subseteq G(k, m)$$

Let $\tilde{\mathbb{S}}_1$ and $\tilde{\mathbb{S}}_2$ be the two cycles in the cohomology ring of the Grassmannian $G(2, \binom{m}{k})$ corresponding respectively to the subvariety $\mathbb{S}_1$ of the lines in $\mathbb{P}^{\binom{m}{k}-1}$ contained in the intersection of r fixed generic hypersurfaces and the subvariety $\mathbb{S}_2$ of the lines in $\mathbb{P}^{\binom{m}{k}-1}$ that are contained in $G(k, m)$.
The intersection $\mathbb{S}_1 \cap \mathbb{S}_2$ represents the cycle $\tilde{\mathbb{S}}_1 \cdot \tilde{\mathbb{S}}_2$ corresponding to the lines in $\mathbb{P}^{\binom{m}{k}-1}$ contained in X. If a_1, a_2 are the codimensions of $\mathbb{S}_1$ and $\mathbb{S}_2$ in $G(2, \binom{m}{k})$ respectively, then we can calculate this codimensions, in fact, since $V(n_1) \cap \ldots \cap V(n_r)$ is a Fano variety (see Corollary 1), then by Theorem 1 we obtain immediately that $a_1 = \sum_{i=1}^{r} n_i + r$.

Now let us compute directly also the codimension of $\mathbb{S}_2$. We know ([3], page 207) that if $\phi : S \longrightarrow G(k, m)$ is a morphism, it induces a surjective morphism $\mathcal{O}_S^{\oplus m} \longrightarrow \mathcal{F}$, with $\mathcal{F}$ a bundle of rank $m - k$ on S. In fact, we can consider the exact sequence of bundles over $G(k, m)$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T} \arrow[r] \arrow[dr] & G(k,m)\times\mathbb{C}^m \arrow[r] \arrow[d] & \mathcal{Q} \arrow[r] \arrow[dl] & 0 \\
& S \arrow[r, "\phi"] & G(k,m) & &
\end{tikzcd}$$

where $\mathcal{T}$ is the tautological bundle and $\mathcal{Q}$ is the universal quotient bundle. We remember that $\mathcal{T} = \{(\Lambda, u) \in G(k, m) \times \mathbb{C}^m / u \in \Lambda\}$. Taking the pull-back via ϕ, we obtain:

$$\phi^*(G(k, m) \times \mathbb{C}^m) \longrightarrow \phi^*(\mathcal{Q}) \longrightarrow 0$$

where $\phi^*(G(k, m) \times \mathbb{C}^m) = \mathcal{O}_S^m$ e $\phi^*(\mathcal{Q}) = \mathcal{F}$. By the universal property of the quotient bundle the vice versa holds too. As $G(k, m) \cong G(m - k, m)$, then we have the identification

$$Mor(S, G(m - k, m)) \cong Epi(\mathcal{O}_S^m, \mathcal{F}).$$

where here $\mathcal{F}$ is a bundle on S of rank k. But, $Epi(\mathcal{O}_S^m, \mathcal{F})$ is an open set in $Hom(\mathcal{O}_S^m, \mathcal{F})$, which is isomorphic to $H^0(S, \mathcal{F} \otimes (\mathcal{O}_S^m)^\vee)$.
In our case, for $S = \mathbb{P}^1$, we have that the bundle $\mathcal{F}$ is a vector bundle of rank k and that on $\mathbb{P}^1$ it has the form $\mathcal{O}_{\mathbb{P}^1}(\alpha_1) \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha_2) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(\alpha_k)$ (by Grothendieck Theorem). As the image must be a line, then up to order the integers α_i

$$\begin{aligned} \alpha_1 + \alpha_2 + \ldots + \alpha_k = 1; \quad & \alpha_1, \alpha_2, \ldots, \alpha_k \geq 0 \quad \Rightarrow \\ & \alpha_1 = \alpha_2 = \cdots = \alpha_{k-1} = 0, \quad \alpha_k = 1. \end{aligned}$$

Then:

$$\mathcal{F} = \bigoplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

Using the additive property of the cohomology, we can calculate

$$\begin{aligned} h^0((\bigoplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes (\mathcal{O}_{\mathbb{P}^1}^m)^\vee) &= h^0(\bigoplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}^1}^m \oplus \mathcal{O}_{\mathbb{P}^1}(1)^m) \\ &= (k-1)h^0(\mathcal{O}_{\mathbb{P}^1}^m) + h^0(\mathcal{O}_{\mathbb{P}^1}(1)^m) \\ &= (k+1)m. \end{aligned}$$

However, to each surjective morphism corresponds more than one morphism from $\mathbb{P}^1$ to $G(k,m)$. We need to identify those equivalents. First of all there is a three dimensional group acting on $\mathbb{P}^1$, so that the dimension drops by three. Furthermore, there is the action of the linear group $GL(k,k)$ on the Grassmannian, so the dimension diminishes by k^2. Hence, the dimension of $\mathbb{S}_2$ is

$$h^0((\bigoplus_{i=1}^{k-1} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}^m) - (\dim Aut(\mathbb{P}^1) + k^2).$$

Therefore, $a_2 = \dim G(2, \binom{m}{k}) - (k+1)m + k^2 + 3$.

Finally:

$$\dim \mathcal{F}(X) \geq -\sum_{i=1}^{r} n_i - r + (k+1)m - k^2 - 3 = n - 3 + \mathrm{index}(X). \ \square$$

Corollary 3. *Let X be a Fano surface complete intersection of generic hypersurfaces in a Grassmannian, then* $\dim \mathcal{F}(X) = index(X) - 1$.

Proof. It is a straightforward computation. $\square$

Corollary 4. *Let X be a Fano threefold complete intersection of generic hypersurfaces in a Grassmannian, then* $\dim \mathcal{F}(X) = index(X)$.

Proof. Concerning threefolds, Shŏkurov guarantees the existence of at least one line for any Fano variety of type $\mathbb{V}_{2g-2}$ in $\mathbb{P}^{g+1}$ and of the first kind ([13],[5]). In fact for the varieties $\mathbb{V}_3^{14}$ and $\mathbb{V}_3^{10}$ which are the two last varieties of the list given in the classification of the threefolds, we have that $\dim \mathcal{F}(\mathbb{V}_3^{14}) = \dim \mathcal{F}(\mathbb{V}_3^{10}) = 1$ ([10], page 84).

Variaties X which are the intersection of the Grassmannian $G(2,5)$ with 3 generic hyperplanes H_1, H_2, H_3 in $\mathbb{P}^9$ contain a family of lines parametrised by a 2-dimensional scheme. In fact, since X is a Fano threefold of index 2 and degree 5, then it contains at least one line ([7], page 64).

The corollary is proved, in fact there are only these three varieties to study, as the others are isomorphic to complete intersections in a projective space so that from Theorem 1 it easily follows that the dimension of $\mathcal{F}(X)$ is equal to the index of X. $\square$

Example 3. If X is the fourfold intersection of the Grassmannian $G(2, 6)$ with 4 generic hyperplanes H_1, H_2, H_3, H_4 in $\mathbb{P}^{14}$ or if X is the intersection of $G(2, 5)$ with a generic hyperplane H_1 and a generic quadric Q in $\mathbb{P}^9$, then $\dim \mathcal{F}(X) \geq 3$. If X is the intersection of $G(2, 5)$ with 2 generic hyperplanes H_1, H_2 in $\mathbb{P}^9$, then $\dim \mathcal{F}(X) \geq 4$.

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