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ENTROPY SOLUTION FOR A NONLINEAR DEGENERATE ELLIPTIC PROBLEM WITH DIRICHLET-TYPE BOUNDARY CONDITION IN WEIGHTED SOBOLEV SPACES

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In this paper, we prove the existence and uniqueness results of an entropy solution to a class of nonlinear degenerate elliptic problem with Dirichlet-type boundary condition and L^1 data. The main tool used here is the regularization approach combined with the theory of weighted Sobolev spaces.

1. Introduction

Let $\Omega \subset \mathbb{R}^N, (N \geq 2)$ be an open bounded domain and let $p \in (1,\infty)$. In this paper we study the existence and uniqueness question of entropy solution for the nonlinear degenerate elliptic problem

$$\begin{cases} -\operatorname{div}\Big(\omega(x)|\nabla u-\Theta(u)|^{p-2}\big(\nabla u-\Theta(u)\big)\Big)+\alpha(u)=f \text{ in } \Omega, \\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where ω is a measurable positive function defined on $\mathbb{R}^N$, α is a non decreasing continuous real function defined on $\mathbb{R}$ and Θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$, the datum f is in L^1.

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The study of partial differential equations and variational problems has received considerable attention in many models coming from various branches of mathematical physics, such as elastic mechanics, electrorheological fluid dynamics and image processing, etc. Degenerate phenomena appear in area of oceanography, turbulent fluid flows, induction heating and electrochemical problems (cf. e.g. [8, 11, 14]). The problem (1) is modeling several natural phenomena, we cite for example the following two parabolic models.

• *Model 1. Filtration in a porous medium.* The filtration phenomena of fluids in porous media are modeled by the following equation,

$$\frac{\partial c(p)}{\partial t} = \nabla a[k(c(p))(\nabla p + e)], \tag{2}$$

where p is the unknown pressure, c volumetric moisture content, k the hydraulic conductivity of the porous medium, a the heterogeneity matrix and $-e$ is the direction of gravity.

• *Model 2. Fluid flow through porous media.* This model is governed by the following equation,

$$\frac{\partial \theta}{\partial t} - div\left(|\nabla \varphi(\theta) - K(\theta)e|^{p-2}(\nabla \varphi(\theta) - K(\theta)e)\right) = 0, \tag{3}$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.

The problem (1) or some particular cases of it have recently been considered by several authors, for example, in the case when $\omega \equiv 1$, the existence and uniqueness of weak or entropy solution for the problem (1) are already proven (cf. e.g. [1] and [9]). Many authors have considered the problem (1) in the case when $\Theta = 0$ and especially the study of questions of existence and uniqueness of entropy solution to the problem (1) (cf. e.g. [7]).

In this paper and by using the regularization approach, we prove in the first step existence of a sequence of weak solutions to approximate problems (7), we apply here the variational method combined with a special type of operators (operator of type (M), see definition 2.6 below). In the second step, we will prove that the sequence of weak solutions converges to some function u and by using some a priori estimates, we will show that this function u is an entropy solution of nonlinear elliptic problem (1). We recall that the notion of entropy solutions was introduced by Ph. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, J.L. Vazquez in [2] and adapted by many authors to study some nonlinear elliptic and parabolic problems (cf. e.g. [1, 3–5, 13]).

The plan of our paper is divided into four sections and organized as follows, in section 2, we present some preliminaries on weighted Sobolev spaces and some basic tools to prove our main result of this paper, in section 3, we introduce some assumptions, and we give the definition of entropy solutions of problem (1), we finish this paper by proving the main result of this paper.

2. Preliminaries and notations

In this section, we give some notations and definitions and we state some results which will be used in this work.
Let ω be a measurable positive and a.e finite function defined on $\mathbb{R}^N$, further, we suppose that the following integrability conditions are satisfied:

(H_1) $\omega \in L^1_{loc}(\Omega)$ and $\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$,

(H_2) $\omega^{-s} \in L^1_{loc}(\Omega)$ where $s \in \left(\frac{N}{p}, \infty\right) \cap \left[\frac{1}{p-1}, \infty\right)$.

The weighted Lebesgue space $L^p(\Omega, \omega)$ is defined by

$$L^p(\Omega,\omega) = \{u\ \Omega \to \mathbb{R},\ u \text{ is measurable and } \int_\Omega \omega(x)|u|^p dx < \infty\},$$

endowed with the norm

$$\|u\|_{p,\omega} := \|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega \omega(x)|u|^p dx\right)^{\frac{1}{p}}.$$

The weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined by

$$W^{1,p}(\Omega,\omega) = \{u \in L^p(\Omega,\omega) \text{ and } |\nabla u| \in L^p(\Omega,\omega)\},$$

with the norm

$$\|u\|_{1,p,\omega} = \|u\|_p + \|\nabla u\|_{p,\omega},\ \forall\ u \in W^{1,p}(\Omega,\omega).$$

In the following, the space $W_0^{1,p}(\Omega, \omega)$ denote the closure of C_0^∞ in $W^{1,p}(\Omega, \omega)$ endowed by the norm

$$\|u\|_{W_0^{1,p}(\Omega,\omega)} = \left(\int_\Omega |\nabla u|^p \omega(x) dx\right)^{\frac{1}{p}}.$$

Let s be a real number satisfying hypothesis (H_2), we define the following critical exponents

$$p^* = \frac{Np}{N-p} \ for\ \ p < N,$$

$$p_s = \frac{ps}{1+s} < p,$$

$$p_s^* = \begin{cases} \dfrac{ps}{(1+s)N - ps} & \text{if } N > p_s, \\ +\infty & \text{if } N \leq p_s. \end{cases}$$

In the following of this work, we need to following results

Proposition 2.1 ([10])**.** *Let $\Omega \subset \mathbb{R}^N$ be an open set of $\mathbb{R}^N$ and let hypothesis (H_1) be satisfied, we have*

$$L^p(\Omega, \omega) \hookrightarrow L^1_{Loc}(\Omega).$$

Proposition 2.2 ([10])**.** *Let hypothesis (H_1) be satisfied, the space $\left(W^{1,p}(\Omega, \omega), \|u\|_{1,p,\omega}\right)$ is a separable and reflexive Banach space.*

Proposition 2.3 ([10])**.** *Assume that hypotheses (H_1) and (H_2) hold, we have the continuous embedding*

$$W^{1,p}(\Omega, \omega) \hookrightarrow W^{1,p_s}(\Omega, \omega).$$

Moreover, we have the compact embedding

$$W^{1,p}(\Omega, \omega) \hookrightarrow\hookrightarrow L^r(\Omega),$$

where $1 \leq r < p_s^*$.

Proposition 2.4 ([10])**.** *(Hardy-type inequality) There exist a weight function ω defined on Ω and a parameter $q,\ 1 < q < \infty$ such that the inequality*

$$\left(\int_\Omega \omega(x)|u(x)|^q dx\right)^{\frac{1}{q}} \leqslant C_0 \left(\int_\Omega \omega(x)|\nabla u|^p dx\right)^{\frac{1}{p}} \tag{4}$$

holds for every $u \in W_0^{1,p}(\Omega, \omega)$, C_0 is a strictly positive constant independent of u. Moreover, the embedding

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow L^q(\Omega, \omega)$$

expressed by inequality (4) *is compact.*

Let k be a strictly positive real, we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \min(k, \max(s, -k)) = \begin{cases} s & \text{if } |s| \leq k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω, we define the truncated function $T_k u$ as follows, for every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$.
We define also the space

$$\mathcal{T}_0^{1,p}(\Omega, \omega) =$$
$$\Big\{u : \Omega \to \mathbb{R},\ \ u \text{ is measurable and } \ T_k(u) \in W_0^{1,p}(\Omega, \omega) \ \text{ for all } \ k > 0\Big\}.$$

By [2, lemma 2.1], the weak gradient of a measurable function $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$ is defined as

Proposition 2.5. *For every function $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$, there exists a unique measurable function $v :\ \Omega \to \mathbb{R}^N$, which we call the very weak gradient of u (if there is any confusion, we denote $v = \nabla u$)such that*

$$\nabla T_k(u) = v\chi_{\{|u| \le k\}} \text{ for a.e } x \in \Omega \text{ and for all } k > 0,$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^N$. Moreover, if u belongs to $W_0^{1,p}(\Omega, \omega)$, the very weak gradient of u coincides to its weak gradient.

Definition 2.6 ([12]). Let Y be a reflexive Banach space and let P be an operator from Y to its dual Y'. We say that P is of type (M) if and only if

$$\left.\begin{array}{l} u_n \rightharpoonup u \ \ weakly \ \ in \ \ Y \\ Pu_n \rightharpoonup \chi \ \ weakly \ \ in \ \ Y' \\ \limsup\limits_{n \to +\infty} \langle Pu_n, u_n \rangle \le \langle \chi, u \rangle \end{array}\right\} \text{Then } Pu = \chi.$$

Theorem 2.7 ([12]). *Let Y be a reflexive real Banach space and $P : Y \longrightarrow Y'$ be a bounded operator, hemi-continuous, coercive and of type (M) on space Y, the equation $Pu = h$ has at least one solution $u \in Y$ for each $h \in Y'$.*

Lemma 2.8 ([1]). *For $\xi,\ \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \le |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2.9. *For $a \ge 0$, $b \ge 0$ and $1 \le p < +\infty$, we have*

$$(a+b)^p \le 2^{p-1}(a^p + b^p).$$

Lemma 2.10 ([9]). *Let $p,\ p'$ two reals numbers such that $p > 1,\ p' > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$, we have*

$$\big||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big|^{p'} \le C\left\{(\xi - \eta)(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\right\}^{\frac{\beta}{2}} \left\{|\xi|^p + |\eta|^p\right\}^{1-\frac{\beta}{2}},$$
$$\forall\ \xi, \eta \in \mathbb{R}^N, \text{ where } \beta = 2 \text{ if } 1 < p \le 2 \text{ and } \beta = p' \text{ if } p \ge 2.$$

Remark 2.11. Hereinafter, $C_i, i \in \{1; 2; ...\}$ is a positive constant and $meas\{A\}$ denotes the measure of the measurable set $A \subset \mathbb{R}^N$.

3. Assumptions and main result

In this section, we will introduce the concept of entropy solution for problem (1) and we will state the existence and the uniqueness results for this type of solution. Firstly and in addition to hypotheses (H_1) and (H_2) listed earlier, we suppose the following assumptions.

(H_3) α is a non-decreasing continuous real function defined on $\mathbb{R}$, surjective such that $\alpha(0) = 0$.

(H_4) Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^N$ such that $\Theta(0) = 0$, and for all real numbers x, y, we have $|\Theta(x) - \Theta(y)| \leq \lambda |x - y|$, where λ is a real constant such that $0 < \lambda < \dfrac{1}{2C_0}$, and C_0 is the constant given in Proposition 2.4.

(H_5) $f \in L^1(\Omega)$.

Definition 3.1. A function $u \in \mathcal{T}_0^{1,p}(\Omega, \omega)$ is an entropy solution of degenerate elliptic problem (1) if and only if

$$\int_\Omega \omega\Phi\big(\nabla u - \Theta(u)\big)\nabla T_k(u-\varphi) + \int_\Omega \alpha(u) T_k(u-\varphi) \leq \int_\Omega f T_k(u-\varphi) \tag{5}$$

for all $k > 0$ and $\varphi \in W_0^{1,p}(\Omega, \omega) \cap L^\infty(\Omega)$,
where

$$\Phi(\xi) = |\xi|^{p-2}\xi, \quad \forall\, \xi \in \mathbb{R}^N.$$

Our main result of this work is the following Theorem

Theorem 3.2. *Let hypotheses* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *be satisfied, then the problem* (1) *has a unique entropy solution.*

4. Proof of the main result

The proof of our main result is divided into three steps, in the first one and by using the regularization approach, we regularize the problem (1) and study the existence of weak solutions to approximate problems (7). In the second step, we give some a priori estimates which will be used to prove the existence of an entropy solution for problem (1). We finish this section by proving the uniqueness of the entropy solution.

4.1. The approximate problem

Let the operator $B_n : W_0^{1,p}(\Omega,\omega) \longrightarrow (W_0^{1,p}(\Omega,\omega))'$ (where $(W_0^{1,p}(\Omega,\omega))'$ is the dual space of $W_0^{1,p}(\Omega,\omega)$) and let

$$B_n = A_n - L_n,$$

where for $u_n, v \in W_0^{1,p}(\Omega,\omega)$

$$\langle A_n u_n, v\rangle = \int_\Omega \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla v\,dx + \int_\Omega T_n(\alpha(u_n))v\,dx$$

and

$$\langle L_n, v\rangle = \int_\Omega T_n(f)v\,dx.$$

We will prove that B_n satisfies the assertions of Theorem 2.7. Firstly, we prove that B_n is of type (M) and coercive, for that, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,p}(\Omega,\omega)$ such that

$$\begin{cases} u_k \rightharpoonup u \;\; weakly \;\; in \;\; W_0^{1,p}(\Omega,\omega), \\ B_n u_k \rightharpoonup \chi \;\; weakly \;\; in \;\; (W_0^{1,p}(\Omega,\omega))', \\ \limsup\limits_{k\to+\infty}\langle B_n u_k, u_k\rangle \le \langle \chi, u\rangle. \end{cases}$$

We will prove that $\chi = B_n u$, indeed, the sequence $(u_k)_{k\in\mathbb{N}}$ converges weakly to u in $W_0^{1,p}(\Omega,\omega)$, so, there exists a subsequence, still denoted $(u_k)_{k\in\mathbb{N}}$ such that $u_k \longrightarrow u$ in $L^p(\Omega,\omega)$. Since $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,p}(\Omega,\omega)$, then

$$\Big(|\nabla u_k - \Theta(u_k)|^{p-2}\big(\nabla u_k - \Theta(u_k)\big)\Big)_{k\in\mathbb{N}} \text{ is bounded in } \left(L^{p'}(\Omega,\omega)\right)^N.$$

Consequently

$$|\nabla u_k - \Theta(u_k)|^{p-2}\big(\nabla u_k - \Theta(u_k)\big) \rightharpoonup |\nabla u - \Theta(u)|^{p-2}\big(\nabla u - \Theta(u)\big)$$

in $\left(L^{p'}(\Omega,\omega)\right)^N$ as $k \to +\infty$.
According to the above result, we deduce that, for all $v \in W_0^{1,p}(\Omega,\omega)$, that

$$\begin{aligned} \langle \chi, v\rangle &= \lim_{k\to+\infty}\langle B_n u_k, v\rangle \\ &= \lim_{k\to+\infty}\left(\int_\Omega \omega\Phi\left(\nabla u_k - \Theta(u_k)\right)\nabla v\,dx + \int_\Omega T_n(\alpha(u_k))v\,dx - \int_\Omega T_n(f)v\,dx\right) \\ &= \int_\Omega \omega\Phi\left(\nabla u - \Theta(u)\right)\nabla v\,dx + \int_\Omega T_n(\alpha(u))v\,dx - \int_\Omega T_n(f)v\,dx \\ &= \langle B_n u, v\rangle. \end{aligned}$$

This implies that $\chi = B_n u$. Therefore B_n is of type (M).
Let $u_n \in W_0^{1,p}(\Omega,\omega)$, we have

$$\langle B_n u_n, u_n\rangle =$$
$$\int_\Omega \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla u_n\,dx + \int_\Omega T_n(\alpha(u_n))u_n\,dx - \int_\Omega T_n(f)u_n\,dx.$$

On the one hand, we have by application of hypothesis (H_3) that

$$\int_\Omega T_n(\alpha(u_n))u_n\,dx \geq 0.$$

And, by Hölder inequality and Proposition 2.3, there exists a positive constant C_1 such that

$$\int_\Omega T_n(f)u_n\,dx \leq C_1\|f\|_{p'}\|u_n\|_{W_0^{1,p}(\Omega,\omega)}.$$

This implies that

$$\langle B_n u_n, u_n\rangle \geq \int_\Omega \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla u_n\,dx - C_1\|f\|_{p'}\|u_n\|_{W_0^{1,p}(\Omega,\omega)}. \tag{6}$$

On the other hand, using Lemma 2.8 and Lemma 2.9, we obtain that

$$\begin{aligned}
\int_\Omega \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla u_n\,dx &\geq \frac{1}{p}\int_\Omega \omega|\nabla u_n - \Theta(u_n)|^p\,dx - \frac{1}{p}\int_\Omega \omega|\Theta(u_n)|^p\,dx\\
&\geq \int_\Omega \frac{1}{p}\,\omega\left[\frac{1}{2^{p-1}}|\nabla u_n|^p - 2|\Theta(u_n)|^p\right]dx\\
&\geq \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega \omega|\nabla u_n|^p\,dx - \frac{2\lambda^p}{p}\int_\Omega \omega|u_n|^p\,dx\\
&\geq \frac{1}{p}\frac{1}{2^{p-1}}\int_\Omega \omega|\nabla u_n|^p\,dx - \frac{2\lambda^p}{p}C_0^p\int_\Omega \omega|\nabla u_n|^p\,dx\\
&\geq \frac{1}{p}\left(\frac{1}{2^{p-1}} - 2\lambda^p C_0^p\right)\|u_n\|^p_{W_0^{1,p}(\Omega,\omega)}.
\end{aligned}$$

So, the choice of λ in (H_4) gives the existence of a positive constant C_2 such that

$$\int_\Omega \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla u_n\,dx \geq C_2\|u_n\|^p_{W_0^{1,p}(\Omega,\omega)}.$$

Then, inequality (6) becomes

$$\langle B_n u_n, u_n\rangle \geq C_2\|u_n\|^p_{W_0^{1,p}(\Omega,\omega)} - C_1\|f\|_{p'}\|u_n\|_{W_0^{1,p}(\Omega,\omega)}.$$

Therefore

$$\frac{\langle B_n u_n, u_n\rangle}{\|u_n\|_{W_0^{1,p}(\Omega,\omega)}} \to +\infty \text{ as } \|u_n\|_{W_0^{1,p}(\Omega,\omega)} \to +\infty.$$

Then we conclude that B_n is coercive. The operator B_n is hemi-continuous, then by Theorem 2.7, there exists $u_n \in W_0^{1,p}(\Omega,\omega)$ such that

$$\int_\Omega \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla v\,dx + \int_\Omega T_n(\alpha(u_n))v\,dx = \int_\Omega T_n(f)v\,dx \tag{7}$$

for all $v \in W_0^{1,p}(\Omega,\omega)$.

4.2. *A priori estimates*

In this section, all the proofs of a priori estimates are inspired by the outline of the Boccardo-Gallouet proof (see [2, 3]).

Lemma 4.1. *Let hypotheses* $(H_1),(H_2),(H_3),(H_4)$ *and* (H_5) *be satisfied, then* $(\nabla T_k(u_n))_{n\in\mathbb{N}}$ *is bounded in* $\left(L^p(\Omega,\omega)\right)^N$.

Proof. Taking $v = T_k(u_n)$ in equality (7), we have by hypothesis (H_3) that

$$\int_{\Omega_k(n)} \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla u_n \leq k\|f\|_1,$$

where $\Omega_k(n) = \{|u_n| \leq k\}$. So, by using the same arguments used to prove the coercivity of B_n, we obtain that

$$\int_\Omega \omega|\nabla u_n|^p \leq kC_3,$$

where C_3 is a positive constant.
Therefore,

$$\|T_k(u_n)\|_{W_0^{1,p}(\Omega,\omega)} \leq (kC_3)^{\frac{1}{p}}. \tag{8}$$

Then, for any $k > 0$, $(T_k(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,p}(\Omega,\omega)$. □

Lemma 4.2. *Let hypotheses* $(H_1),(H_2),(H_3),(H_4)$ *and* (H_5) *be satisfied, the sequence* $(u_n)_{n\in\mathbb{N}}$ *converges in measure to some measurable function u.*

Proof. To prove this, we show that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Let $k > 0$ be large enough positive number. Noting that $\{|u_n| > k\} = \{|T_k(u_n)| > k\}$, then by inequality (8) and Markov inequality, we have

$$meas\{|u_n| > k\} \leq \left(\frac{\|T_k(u_n)\|_{W_0^{1,p}(\Omega,\omega)}}{k}\right)^p \leq \frac{C_3}{k^{p-1}}.$$

Therefore

$$meas\{|u_n| > k\} \to 0 \text{ as } k \to \infty, \text{uniformly with respect to n.} \tag{9}$$

Moreover, for every fixed $t>0$ and every real positive k, we know that

$$\{|u_n-u_m|>t\}\subset\{|u_n|>k\}\cup\{|u_m|>k\}\cup\{|T_k(u_n)-T_k(u_m)|>t\}$$

and hence

$$meas\{|u_n-u_m|>t\}\leq$$
$$meas\{|u_n|>k\}+meas\{|u_m|>k\}+meas\{|T_k(u_n)-T_k(u_m)|>t\} \tag{10}$$

Let $\varepsilon>0$, we have by (9) that

$$\text{meas}\left(\{|u_n|>k\}\right)\leq\frac{\varepsilon}{3}\quad\text{and}\quad\text{meas}\left(\{|u_m|>k\}\right)\leq\frac{\varepsilon}{3}. \tag{11}$$

Since $T_k(u_n)$ converges strongly in $L^p(\Omega,\omega)$, then it is a Cauchy sequence in $L^p(\Omega,\omega)$, thus implies by Markov inequality that

$$meas\left(\{|T_k(u_n)-T_k(u_m)|>t\}\right)\leq\frac{1}{t^p}\int_\Omega\omega\,|T_k(u_n)-T_k(u_m)|^p\,dx\leq\frac{\varepsilon}{3} \tag{12}$$

for all $n,m\geq n_0(t,\varepsilon)$. Finally, from (10), (11) and (12) we obtain

$$\text{meas}\left(\{|u_n-u_m|>t\}\right)\leq\varepsilon\ \text{ for all } n,m\geq n_0(t,\varepsilon). \tag{13}$$

This proves that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in measure and then it converges almost everywhere to some measurable function u.
Therefore

$$\begin{array}{ll} T_k(u_n)\rightharpoonup T_k(u) & \text{in } W_0^{1,p}(\Omega,\omega)\\ T_k(u_n)\to T_k(u) & \text{in } L^p(\Omega,\omega)\text{ and a.e. in }\Omega.\end{array} \tag{14}$$

□

Lemma 4.3. *Let hypotheses* $(H_1),(H_2),(H_3),(H_4)$ *and* (H_5) *be satisfied, the sequence* $(\nabla u_n)_{n\in\mathbb{N}}$ *converges in measure to* ∇u.

Proof. Let ε,t,k,μ are positive real numbers and let $n\in\mathbb{N}$, we have the following inclusion

$$\{|\nabla u_n-\nabla u|>t\}\subset$$
$$\{|u_n|>k\}\cup\{|u|>k\}\cup\{|\nabla T_k(u_n)|>k\}\cup\{|\nabla T_k(u)|>k\}\cup\{|u_n-u|>\mu\}\cup G,$$

where

$$G=$$
$$\{|\nabla u_n-\nabla u|>t,|u_n|\leq k,|u|\leq k,|\nabla T_k(u_n)|\leq k,|\nabla T_k(u)|\leq k,|u_n-u|\leq\mu\}.$$

The same method used in the proof of Lemma 4.2, enable us to obtain, for k sufficiently large, that

$$meas\left(\{|u_n|>k\}\cup\{|u|>k\}\cup\{|\nabla T_k(u_n)|>k\}\right)\leq\frac{\varepsilon}{4}. \tag{15}$$

This implies by (14) that

$$\nabla T_k(u_n) \textit{ converges weakly to } \nabla T_k(u) \textit{ in } \left(L^p(\Omega,\omega)\right)^N. \tag{16}$$

Then, we have for k sufficiently large that

$$meas\left(\{|\nabla T_k(u)|>k\}\right)\leq \frac{\varepsilon}{4}. \tag{17}$$

On the other hand, by using the Lemma 4.2, we deduce the existence of $n_1 \in \mathbb{N}$, such that

$$meas\left(\{|u_n-u|>\mu\}\right)\leq \frac{\varepsilon}{4} \text{ for } n\geq n_1. \tag{18}$$

Now, the application

$$\mathcal{A}:(s,\xi_1,\xi_2)\to \omega\Big(\Phi(\xi_1-\Theta(s))-\Phi(\xi_2-\Theta(s))\Big)\Big(\xi_1-\xi_2\Big)$$

is continuous, and the set

$$\mathbf{K}:=\left\{(s,\xi_1,\xi_2)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N, |s|\leq k, |\xi_1|\leq k, |\xi_2|\leq k, |\xi_1-\xi_2|>t\right\}$$

is compact. Moreover, we have

$$\omega\Big(\Phi(\xi_1-\Theta(s))-\Phi(\xi_2-\Theta(s))\Big)\Big(\xi_1-\xi_2\Big)>0,\ \ \forall\, \xi_1\neq\xi_2.$$

Then, the application $\mathcal{A}$ attains its minimum on $\mathbf{K}$, we shall note it by β, we have easily that $\beta>0$ and

$$\begin{aligned}\int_G \beta\, dx &\leq \int_G \omega\left[\Phi(\nabla u_n-\Theta(u_n))-\Phi(\nabla u-\Theta(u_n))\right]\left[\nabla u_n-\nabla u\right]\, dx\\ &\leq \int_\Omega \omega\left[\Phi(\nabla u_n-\Theta(u_n))-\Phi(\nabla T_k(u)-\Theta(T_{k+\mu}(u_n)))\right]\nabla T_\mu\left(T_{k+\mu}(u_n)-T_k(u)\right)\, dx.\end{aligned}$$

By taking $v=T_\mu\left(T_{k+\mu}(u_n)-T_k(u)\right)$ in equality (7), we get

$$\int_\Omega \omega\Phi(\nabla u_n-\Theta(u_n))\nabla T_\mu\left(T_{k+\mu}(u_n)-T_k(u)\right)\, dx\leq \mu\left(\|f\|_1+\|T_n(\alpha(u_n))\|_1\right). \tag{19}$$

However, by hypothesis (H_3), we have

$$sign(u_n)=sign(T_n(\alpha(u_n))),$$

where

$$sign(s):=\begin{cases}1 & \text{if } s>0,\\ 0 & \text{if } s=0,\\ -1 & \text{if } s<0.\end{cases}$$

Then, by taking $v = sign(u_n)$ in equality (7), we obtain

$$\|T_n(\alpha(u_n))\|_1 \leq \|f\|_1.$$

Consequently

$$\left| \int_\Omega \omega\Phi(\nabla u_n - \Theta(u_n))\nabla T_\mu \left(T_{k+\mu}(u_n) - T_k(u)\right) dx \right| \leq \mu C_4. \tag{20}$$

On the other hand, we have

$$T_{k+\mu}(u_n) \textit{ converges weakly to } T_{k+\mu}(u) \textit{ in } W_0^{1,p}(\Omega,\omega). \tag{21}$$

This implies, by using hypothesis (H_4) that

$$\Theta(T_{k+\mu}(u_n)) \textit{ converges to } \Theta(T_{k+\mu}(u)) \textit{ in} \left(L^p(\Omega,\omega)\right)^N, \tag{22}$$

and that

$$\nabla T_\mu \left(T_{k+\mu}(u_n) - T_k(u)\right) \rightharpoonup \nabla T_\mu \left(T_{k+\mu}(u) - T_k(u)\right) \textit{ in} \left(L^p(\Omega,\omega)\right)^N. \tag{23}$$

Then, by (21), (22), (23), we deduce that

$$\lim_{n\to\infty} \int_\Omega \omega\Phi(\nabla T_k(u) - \Theta(T_{k+\mu}(u_n)))\nabla T_\mu \left(T_{k+\mu}(u_n) - T_k(u)\right) dx$$

$$= \int_\Omega \omega\Phi(\nabla T_k(u) - \Theta(T_{k+\mu}(u)))\nabla T_\mu \left(T_{k+\mu}(u) - T_k(u)\right) dx.$$

However,

$$\lim_{\mu\to 0} \nabla T_\mu \left(T_{k+\mu}(u) - T_k(u)\right) = 0.$$

Let $\mu < 1$, we have from hypothesis (H_4) that

$$\Phi(\nabla T_k(u) - \Theta(T_{k+\mu}(u_n)))\nabla T_\mu \left(T_{k+\mu}(u_n) - T_k(u)\right)$$

$$\leq C_5 \left(|T_{k+1}(u)|^{p-1} + |\nabla T_k(u)|^{p-1}\right) |\nabla T_1 \left(T_{k+1}(u) - T_k(u)\right)|.$$

Now, as

$$\left(|T_{k+1}(u)|^{p-1} + |\nabla T_k(u)|^{p-1}\right) |\nabla T_1 \left(T_{k+1}(u) - T_k(u)\right)| \in L^1(\Omega).$$

Then, we get by using the Dominated Convergence Theorem that

$$\lim_{\mu\to 0} \int_\Omega \omega\Phi(\nabla T_k(u) - \Theta(T_{k+\mu}(u)))\nabla T_\mu \left(T_{k+\mu}(u) - T_k(u)\right) dx = 0.$$

Let δ be a strictly positive number such that $\mu < \frac{\delta}{C_4}$, there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have

$$\int_\Omega \omega\Phi(\nabla T_k(u) - \Theta(T_{k+\mu}(u_n)))\nabla T_\mu \left(T_{k+\mu}(u_n) - T_k(u)\right)\, dx \leq \frac{\delta}{2}. \tag{24}$$

Therefore, by (20) and (24), we deduct that

$$\int_\Omega \beta\, dx \leq \delta.$$

This implies that

$$meas(G) \leq \frac{\varepsilon}{4}. \tag{25}$$

Consequently, from (15), (17), (18) and (25), we conclude that

$$meas\{|\nabla u_n - \nabla u| > t\} \leq \varepsilon.$$

This implies that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges in measure to ∇u. $\square$

Now, we shall prove that limit function u is an entropy solution of problem (1). Let $\varphi \in W_0^{1,p}(\Omega,\omega) \cap L^\infty(\Omega)$ and take $v = T_k(u_n - \varphi)$ in equality (7), we get

$$\int_\Omega \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla T_k(u_n-\varphi)\, dx + \int_\Omega T_n(\alpha(u_n))T_k(u_n-\varphi)\, dx$$
$$= \int_\Omega T_n(f)T_k(u_n-\varphi)\, dx. \tag{26}$$

Let $\overline{k} = k + \|\varphi\|_\infty$, we have

$$\int_\Omega \omega\Phi\left(\nabla u_n - \Theta(u_n)\right)\nabla T_k(u_n-\varphi)\, dx$$
$$= \int_\Omega \omega\Phi\left(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n))\right)\nabla T_k(T_{\overline{k}}(u_n)-\varphi)\, dx$$
$$= \int_\Omega \omega\Phi\left(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n))\right)\nabla T_{\overline{k}}(u_n)\chi_{\Omega(n,\overline{k})}\, dx$$
$$- \int_\Omega \omega\Phi\left(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n))\right)\nabla\varphi\chi_{\Omega(n,\overline{k})}\, dx,$$

where $\Omega(n,\overline{k}) = \{|T_{\overline{k}}(u_n) - \varphi| \leq k\}$ and χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^N$.

The above equality implies that

$$\int_\Omega \omega\left(\Phi\left(\nabla T_{\overline{k}}(u_n) - \Theta(T_{\overline{k}}(u_n))\right)\nabla T_{\overline{k}}(u_n) + \frac{2}{p}|\Theta(T_{\overline{k}}(u_n))|^p\right)\chi_{\Omega(n,\overline{k})}\, dx$$

$$-\int_\Omega \omega\Phi\left(\nabla T_{\bar{k}}(u_n)-\Theta(T_{\bar{k}}(u_n))\right)\nabla\varphi\chi_{\Omega(n,\bar{k})}\,dx+\int_\Omega T_n(\alpha(u_n))T_k(u_n-\varphi)\,dx$$
$$=\int_\Omega T_n(f)T_k(u_n-\varphi)\,dx+\frac{2}{p}\int_\Omega \omega|\Theta(T_{\bar{k}}(u_n))|^p\chi_{\Omega(n,\bar{k})}\,dx. \tag{27}$$

We know that the function $T_{\bar{k}}(u_n)$ is bounded in $W_0^{1,p}(\Omega,\omega)$, then by hypothesis (H_4), $\Theta(T_{\bar{k}}(u_n))$ is also bounded in $(L^p(\Omega,\omega))^N$, this implies that
$\Phi\left(\nabla T_{\bar{k}}(u_n)-\Theta(T_{\bar{k}}(u_n))\right)$ is bounded in $\left(L^p(\Omega,\omega)\right)'$, (where $\left(L^p(\Omega,\omega)\right)'$ is the dual space of $L^p(\Omega,\omega)$) and weakly converges.
However, we have

$$u_n\to u\ \ a.e.\ in\ \Omega, \tag{28}$$

and

$$\nabla u_n\to\nabla u\ \ a.e.\ in\ \Omega.$$

Hence follows that

$$\Theta(T_{\bar{k}}(u_n))\to\Theta(T_{\bar{k}}(u))\ \ a.e.\ in\ \Omega, \tag{29}$$

and

$$\nabla T_{\bar{k}}(u_n)\to\nabla T_{\bar{k}}(u))\ \ a.e.\ in\ \Omega. \tag{30}$$

This implies that

$$\Phi\left(\nabla T_{\bar{k}}(u_n)-\Theta(T_{\bar{k}}(u_n))\right)\to\Phi\left(\nabla T_{\bar{k}}(u)-\Theta(T_{\bar{k}}(u))\right)\quad in\ \left(L^p(\Omega,\omega)\right)'.$$

Now, as

$$\nabla\varphi\chi_{\Omega(n,\bar{k})}\ converges\ in\,(L^p(\Omega,\omega))^N.$$

Then

$$\int_\Omega \omega\Phi\left(\nabla T_{\bar{k}}(u_n)-\Theta(T_{\bar{k}}(u_n))\right)\nabla\varphi\chi_{\Omega(n,\bar{k})}dx\to\int_\Omega \omega\Phi\left(\nabla T_{\bar{k}}(u)-\Theta(T_{\bar{k}}(u))\right)\nabla\varphi\chi_{\Omega(\bar{k})}dx$$

where $\Omega(\bar{k})=\{|T_{\bar{k}}(u)-\varphi|\le k\}$.
By hypothesis (H_4) and properties of the truncated function, we have

$$|\Theta(T_{\bar{k}}(u_n))|^p\le\left(C_6\bar{k}\right)^p.$$

This implies by using (29) and Dominated Convergence Theorem that

$$\frac{2}{p}\int_\Omega \omega|\Theta(T_{\bar{k}}(u_n))|^p\chi_{\Omega(n,\bar{k})}\,dx\longrightarrow\frac{2}{p}\int_\Omega \omega|\Theta(T_{\bar{k}}(u))|^p\chi_{\Omega(\bar{k})}\,dx.$$

On the other hand, we have by using Lemma 2.8 that

$$\left(\Phi\left(\nabla T_{\bar{k}}(u_n)-\Theta(T_{\bar{k}}(u_n))\right)\nabla T_{\bar{k}}(u_n)+\frac{2}{p}|\Theta(T_{\bar{k}}(u_n))|^p\right)\chi_{\Omega(n,\bar{k})}\ge 0\ \ a.e.\ in\ \Omega.$$

Therefore, by (29), (30) and Fatou's lemma, we have

$$\int_\Omega \omega \left(\Phi\left(\nabla T_{\bar{k}}(u) - \Theta(T_{\bar{k}}(u))\right) \nabla T_{\bar{k}}(u) + \frac{2}{p}|\Theta(T_{\bar{k}}(u))|^p \right) \chi_{\Omega(\bar{k})}\, dx$$

$$\leq \liminf \int_\Omega \omega \left(\Phi\left(\nabla T_{\bar{k}}(u_n) - \Theta(T_{\bar{k}}(u_n))\right) \nabla T_{\bar{k}}(u_n) + \frac{2}{p}|\Theta(T_{\bar{k}}(u_n))|^p \right) \chi_{\Omega(n,\bar{k})}.$$

Finally, taking limits as n goes to infinity in (27) and using the above results to conclude that u satisfies the entropy inequality (5).

4.3. Uniqueness

The proof of uniqueness part of Theorem 3.2 is inspired by the ideas found in [6]. Firstly, we need the following Lemma.

Lemma 4.4. *Let hypotheses* (H_1), (H_2), (H_3), (H_4) *and* (H_5) *be satisfied, if* u *is an entropy solution of problem* (1)*, then*

1. $\displaystyle\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} \omega\,|\nabla u - \Theta(u)|^{p-2}\,|\nabla u - \Theta(u)|\,\nabla u\, dx = 0.$
2. $\displaystyle\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} \omega|\nabla u|^p\, dx = 0.$
3. $\displaystyle\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}} \omega|\nabla u - \Theta(u)|^p\, dx = 0.$

Proof. **1.** Let k and h be two real numbers such that $1 < k < h$. Taking $\varphi = T_h(u)$ in inequality (5), we get

$$\int_\Omega \omega \left(|\nabla u - \Theta(u)|^{p-2}\,(\nabla u - \Theta(u)) \right) \nabla T_k(u - T_h(u))\, dx$$

$$+ \int_\Omega \alpha(u) T_k(u - T_h(u))\, dx \leq \int_\Omega f T_k(u - T_h(u))\, dx. \tag{31}$$

Firstly, we have

$$\int_\Omega \alpha(u) T_k(u - T_h(u))\, dx = \int_{\{|u|>h\}} \alpha(u) T_k(u - h sign(u))\, dx,$$

and

$$sign(u)\chi_{\{|u|>h\}} = sign(u - h sign(u))\chi_{\{|u|>h\}} = sign(T_k(u - h sign(u)))\chi_{\{|u|>h\}}.$$

Then

$$\int_\Omega \alpha(u) T_k(u - T_h(u))\, dx \geq 0.$$

Therefore, inequality (31) becomes

$$\int_{\{h\leq|u|\leq h+k\}} \omega \left(|\nabla u - \Theta(u)|^{p-2}\,(\nabla u - \Theta(u)) \right) \nabla T_k(u - T_h(u))\, dx$$

$$\leq k \int_{\{|u|>h\}} |f|\,dx.$$

By (9), we deduce that $meas\{|u|>h\}$ converges to 0 as h go to infinity, then, we conclude that

$$\lim_{h\to\infty} \int_{\{|u|>h\}} |f|dx = 0.$$

This implies that

$$\lim_{h\to\infty}\lim_{k\to 0} \frac{1}{k} \int_{\{h<|u|<k+h\}} \omega\,|\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u\,dx = 0.$$

2. By using Lemma 2.8 and Lemma 2.9, we have

$$\frac{1}{p2^{p-1}}\,|\nabla u|^p - \frac{2}{p}|\Theta(u)|^p \leq |\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u.$$

We use hypothesis (H_4), we get

$$\frac{1}{p2^{p-1}}\,|\nabla u|^p - \frac{2\lambda^p}{p}|u|^p \leq |\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u.$$

This implies that

$$\frac{1}{p2^{p-1}} \int_{\Omega_k^h} \omega\,|\nabla u|^p\,dx - \frac{2\lambda^p}{p} \int_{\Omega_k^h} \omega|u|^p\,dx$$
$$\leq \int_{\Omega_k^h} \omega\,|\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u\,dx,$$

where $\Omega_k^h = \{h \leq |u| \leq h+k\}$.
Then, by using Proposition 2.4, we get that

$$\frac{1}{p2^{p-1}} \int_{\Omega_k^h} \omega\,|\nabla u|^p\,dx \;-\; \frac{2\lambda^p C_0^p}{p} \int_{\Omega_k^h} \omega|\nabla u|^p\,dx$$
$$\leq \int_{\Omega_k^h} \omega\,|\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u\,dx.$$

Then, by hypothesis (H_4), there exists a positive constant C_6 such that

$$\int_{\Omega_k^h} \omega|\nabla u|^p\,dx \leq C_7 \int_{\Omega_k^h} \omega\,|\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u\,dx.$$

This, we allow to deduce that

$$\lim_{h\to\infty}\lim_{k\to 0} \frac{1}{k} \int_{\{h<|u|<k+h\}} \omega|\nabla u|^p\,dx = 0.$$

3. We have by Lemma 2.8 that

$$\frac{1}{p}\,|\nabla u-\Theta(u)|^p - \frac{1}{p}|\Theta(u)|^p \leq |\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u.$$

This implies that

$$\begin{aligned}
\frac{1}{p}\int_{\Omega_k^h}\omega\,|\nabla u-\Theta(u)|^p\,dx &\leq \int_{\Omega_k^h}\omega\,|\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u\,dx+\frac{1}{p}\int_{\Omega_k^h}\omega|\Theta(u)|^p\,dx\\
&\leq \int_{\Omega_k^h}\omega\,|\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u\,dx+\frac{\lambda^p}{p}\int_{\Omega_k^h}\omega|u|^p\,dx\\
&\leq \int_{\Omega_k^h}\omega\,|\nabla u-\Theta(u)|^{p-2}\,|\nabla u-\Theta(u)|\,\nabla u\,dx+\frac{\lambda^pC_0^p}{p}\int_{\Omega_k^h}\omega|\nabla u|^p\,dx.
\end{aligned}$$

We apply the previous results 1 and 2, we get that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\int_{\{h<|u|<k+h\}}\omega|\nabla u-\Theta(u)|^p\,dx=0.$$

□

Now, let u and v are two entropy solutions of degenerate elliptic problem (1) and let h,k two positive real numbers such that $1<k<h$. In inequality (5), we take for the solution u, $\varphi=T_h(v)$ and for the solution v, we take $\varphi=T_h(u)$ as a test function, we have

$$\int_\Omega \alpha(u)T_k(u-T_h(v))dx+\int_\Omega\omega\left(|\nabla u-\Theta(u)|^{p-2}\,(\nabla u-\Theta(u))\right)\nabla T_k(u-T_h(v))\,dx$$

$$\leq\int_\Omega fT_k(u-T_h(v))dx$$

and

$$\int_\Omega \alpha(v)T_k(v-T_h(u))dx+\int_\Omega\omega\left(|\nabla v-\Theta(v)|^{p-2}\,(\nabla v-\Theta(v))\right)\nabla T_k(v-T_h(u))\,dx$$

$$\leq\int_\Omega fT_k(v-T_h(u))dx.$$

We divide the two above inequalities by k and we pass to limit when $k\to 0$ and $h\to\infty$, we find by applying Dominated Convergence Theorem, hypotheses (H_3) and (H_5) that

$$\|\alpha(u)-\alpha(v)\|_1+\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}(k;h)\leq 0, \tag{32}$$

where

$$\begin{aligned}
&\mathcal{I}(k;h)=\\
&\int_\Omega\omega\left(|\nabla u-\Theta(u)|^{p-2}\,(\nabla u-\Theta(u))\right)\nabla T_k(u-T_h(v))\\
+\;&\left(|\nabla v-\Theta(v)|^{p-2}\,(\nabla v-\Theta(v))\right)\nabla T_k(v-T_h(u))\,dx.
\end{aligned}$$

We will prove that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}(k;h)\geq 0.$$

For that, we consider the following decomposition

$$\Omega_h^1 = \{|u| \le h; |v| \le h\}; \quad \Omega_h^2 = \{|u| \le h; |v| > h\},$$

$$\Omega_h^1 = \{|u| > h; |v| \le h\}; \quad \Omega_h^2 = \{|u| > h; |v| > h\},$$

and for $i = 1;...;4$

$$\mathcal{I}_i(k;h) =$$

$$\int_{\Omega_h^i} \omega \left|\nabla u - \Theta(u)\right|^{p-2} (\nabla u - \Theta(u)) \nabla T_k(u - T_h(v))$$

$$+ \left|\nabla v - \Theta(v)\right|^{p-2} (\nabla v - \Theta(v)) \nabla T_k(v - T_h(u))\, dx.$$

Firstly, we pose

$$\mathcal{I}_1(k;h) = \mathcal{I}_1^1(k;h) + \mathcal{I}_1^2(k;h),$$

where

$$\mathcal{I}_1^1(k;h) =$$

$$\int_{\Omega_h^k(1)} \omega \Big(\left|\nabla u - \Theta(u)\right|^{p-2} (\nabla u - \Theta(u)) - \left|\nabla v - \Theta(v)\right|^{p-2} (\nabla v - \Theta(v))\Big) \psi_\theta(u;v)\, dx$$

$$\mathcal{I}_1^2(k;h) =$$

$$\int_{\Omega_h^k(1)} \omega \Big(\left|\nabla u - \Theta(u)\right|^{p-2} (\nabla u - \Theta(u)) - \left|\nabla v - \Theta(v)\right|^{p-2} (\nabla v - \Theta(v))\Big) \Psi_\theta(u;v)\, dx$$

$$\Omega_h^k(1) = \{|u - v| \le k; |u| \le h; |v| \le h\}$$

and

$$\psi_\theta(u;v) = (\nabla u - \Theta(u)) - (\nabla v - \Theta(v)); \quad \Psi_\theta(u;v) = \Theta(u) - \Theta(v).$$

To show that

$$\lim_{h\to\infty} \lim_{k\to 0} \frac{1}{k} \mathcal{I}_1(k;h) = 0,$$

we consider two cases according to the value of p.
• *First case,* $1 < p \le 2$. Let $\varepsilon > 0$, we apply Young's inequality, we find

$$\begin{aligned} \mathcal{I}_1^2(k;h) \quad \le \quad & \frac{\varepsilon}{p'} \int_{\Omega_h^k(1)} \omega \left| \Big(\left|\nabla u - \Theta(u)\right|^{p-2} (\nabla u - \Theta(u))\Big) - \Big(\left|\nabla v - \Theta(v)\right|^{p-2} (\nabla v - \Theta(v))\Big) \right|^{p'} dx \\ & + \frac{1}{\varepsilon p} \int_{\Omega_h^k(1)} \omega \left|\Theta(u) - \Theta(v)\right|^p dx. \end{aligned}$$

We apply Lemma 2.10 and hypothesis (H_4), we get

$$\left|\mathcal{I}_1^2(k;h)\right| \le \varepsilon C_8 \mathcal{I}_1^1(k;h) + \frac{C_9}{\varepsilon} k^p.$$

This implies that

$$\lim_{k\to 0} \frac{1}{k} \left|\mathcal{I}_1^2(k;h)\right| \le \varepsilon C_8 \lim_{k\to 0} \frac{1}{k} \mathcal{I}_1^1(k;h). \tag{33}$$

If $\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(k,h)=0$, the above inequality (33) becomes

$$\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(k,h)=0.$$

i.e.

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(k,h)=0.$$

If $0<\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(k,h)<\infty$, we take $\varepsilon=\frac{1}{h\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(k,h)}$ in (33), we deduce that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(k,h)=0.$$

It follows that

$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(k,h)\geq 0.$$

If $\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(k,h)=+\infty$, we have by using hypothesis (H_4) that

$$\begin{aligned}\left|\mathcal{I}_1^2(k;h)\right| &\leq k\lambda\int_{\Omega_h^k(1)}\omega\left||\nabla u-\Theta(u)|^{p-2}(\nabla u-\Theta(u))-|\nabla v-\Theta(v)|^{p-2}(\nabla v-\Theta(v))\right|\\ &\leq k\lambda\int_{\Omega_h^k(1)}\omega\Big(|\nabla u-\Theta(u)|^{p-1}+|\nabla v-\Theta(v)|^{p-1}\Big)\,dx.\end{aligned}$$

Consequently

$$\frac{1}{k}\left|\mathcal{I}_1^2(k;h)\right|\leq\lambda\int_{\Omega_h^k(1)}\omega\Big(|\nabla u-\Theta(u)|^{p-1}+|\nabla v-\Theta(v)|^{p-1}\Big)\,dx.$$

On the other hand, for the solution u, we take $\varphi=0$ in inequality (5), we find

$$\int_{\{|u|\leq k\}}\omega\left(|\nabla u-\Theta(u)|^{p-2}(\nabla u-\Theta(u))\right)\nabla u\,dx\leq kC_{10}$$

This implies that

$$\begin{aligned}\int_{\{|u|\leq k\}}\omega\,|\nabla u-\Theta(u)|^p\,dx &\leq kC_{10}+C_{11}\int_{\{|u|\leq k\}}\omega|\Theta(u)|^p\,dx\\ &\leq kC_{10}+C_{12}k^p\\ &\leq C_{13}k^p.\end{aligned}$$

Similarly, we prove that

$$\int_{\{|u|\leq k\}}\omega\,|\nabla v-\Theta(v)|^p\,dx\leq C_{14}k^p.$$

Therefore

$$\frac{1}{k}\left|\mathcal{I}_1^2(k;h)\right|\leq\lambda C_{15}(h+k)^p,$$

i.e.
$$\lim_{k\to 0}\frac{1}{k}\left|\mathcal{I}_1^2(k;h)\right| \le \lambda C_{16} h^p.$$
Thus, it follows that
$$\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^1(k,h)+\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(k,h)=+\infty.$$
Then
$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(k,h)=+\infty.$$
• *Second case,* $p>2$. We use Young's inequality to deduce
$$\frac{1}{k}\left|\mathcal{I}_1^2(k,h)\right| \le \frac{C_{17}\varepsilon(k+h)}{p'k}+\frac{C_{18}}{p\varepsilon}k^{p-1} \quad \forall\, \varepsilon>0.$$
Then, we take $\varepsilon=\dfrac{k}{h^2}$, we obtain
$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1^2(k,h)=0.$$
Consequently
$$\lim_{h\to\infty}\lim_{k\to 0}\frac{1}{k}\mathcal{I}_1(k,h)\ge 0.$$
Secondly, we pose
$$\mathcal{I}_2(k;h)=\mathcal{I}_2^1(k;h)+\mathcal{I}_2^2(k;h),$$
where
$$\begin{aligned}
&\mathcal{I}_2^1(k;h)\\
=\;&\int_{\Omega_2(h)}\omega\left|\nabla v-\Theta(v)\right|^{p-2}(\nabla v-\Theta(v))\nabla T_k(v-u)\,dx\\
=\;&\int_{\Omega_{h,k}^{2,1}}\omega\left|\nabla v-\Theta(v)\right|^{p-2}(\nabla v-\Theta(v))\nabla v\,dx-\int_{\Omega_{h,k}^{2,1}}\omega\left|\nabla v-\Theta(v)\right|^{p-2}(\nabla v-\Theta(v))\nabla u\,dx,
\end{aligned}$$
$$\begin{aligned}
\mathcal{I}_2^2(k;h) &= \int_{\Omega_2(h)}\omega\left|\nabla u-\Theta(u)\right|^{p-2}(\nabla u-\Theta(u))\nabla T_k(u-h\,\mathrm{sign}(v))\,dx\\
&= \int_{\Omega_{h,k}^{2,2}}\omega\left|\nabla u-\Theta(u)\right|^{p-2}(\nabla u-\Theta(u))\nabla u\,dx,
\end{aligned}$$
and
$$\Omega_{h,k}^{2,1}=\{|u|\le h;|v|>h;|v-u|\le k\}$$
$$\Omega_{h,k}^{2,2}=\{|u|\le h;|v|>h;|u-hsign(v)|\le k\}.$$
On the one hand, since ω is a positive function, then, by application of Lemmas 2.8 and 2.9, we get
$$\mathcal{I}_2^2(k;h)\ge 0.$$

In the same manner, we prove that

$$\int_{\Omega_{h,k}^{2,1}} \omega \left|\nabla v-\Theta(v)\right|^{p-2}(\nabla v-\Theta(u)) \nabla v \, dx \geq 0.$$

On the other hand, by Hölder inequality, we have

$$\left|\int_{\Omega_{h,k}^{2,1}} \omega \left|\nabla v-\Theta(v)\right|^{p-2}(\nabla v-\Theta(u)) \nabla u \, dx\right|$$

$$\leq\left(\int_{\Omega_{h,k}^{2,1}} \omega|\nabla v-\Theta(v)|^{p} \, dx\right)^{\frac{p-1}{p}}\left(\int_{\Omega_{h,k}^{2,1}} \omega|\nabla u|^{p} \, dx\right)^{\frac{1}{p}}$$

Hence, by application of Lemma 4.4, we get

$$\lim_{h \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \int_{\Omega_{h,k}^{2,1}} \omega \left|\nabla v-\Theta(v)\right|^{p-2}(\nabla v-\Theta(u)) \nabla u \, dx=0.$$

Then

$$\lim_{h \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \mathcal{I}_{2}^{1}(k;h) \geq 0.$$

Therefore

$$\lim_{h \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \mathcal{I}_{2}(k;h) \geq 0.$$

Finally, in the same manner, we show that

$$\lim_{h \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k}\left(\mathcal{I}_{3}(k;h)+\mathcal{I}_{4}(k;h)\right) \geq 0.$$

Hence

$$\lim_{h \rightarrow \infty} \lim_{k \rightarrow 0} \frac{1}{k} \mathcal{I}(k;h) \geq 0.$$

Therefore, inequality (32) becomes

$$\|\alpha(u)-\alpha(v)\|_{1} \leq 0.$$

This implies that

$$u=v \text{ a. e. in } \Omega.$$

Hence the uniqueness of entropy solution of the problem (1).

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