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$L^{p,\lambda}$ REGULARITY FOR DIVERGENCE FORM ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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We will prove $L^{p,\lambda}$ regularity results for the gradient of the solution to Dirichlet problem concerning the equation

$$-\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j}-\sum_{i=1}^{n}(d_i u)_{x_i}+cu=f_0-\sum_{i=1}^{n}(f_i)_{x_i}$$

with coefficients in $VMO\cap L^\infty$ and Morrey spaces.

1. Introduction.

Let Ω be a bounded open set of $\mathbb{R}^n$, $n>2$, with smooth boundary $\partial\Omega$. In Ω we shall consider the following linear elliptic equation of second order in divergence form

$$-\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j}-\sum_{i=1}^{n}(d_i u)_{x_i}+cu=f_0-\sum_{i=1}^{n}(f_i)_{x_i}, \tag{1.1}$$

where the coefficients a_{ij} are in VMO (see Section 2 for definitions) and the other coefficients are in suitable Morrey spaces $L^{p,\lambda}$. Several authors have

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studied linear elliptic equations of second order with coefficients in $VMO \cap L^\infty$ both in the variational and nonvariational case. These studies began with the papers [3] and [4] by F. Chiarenza, M. Frasca and P. Longo, where the authors proved the well-posedness of the Dirichlet Problem for the equation $\sum_{i,j=1}^n a_{ij} u_{x_i x_j} = f$ in the class $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. These results were further extended to equations containing lower order terms (see [13] and [14]) as well as to the case of oblique derivative boundary conditions (see [7]) and quasilinear equations (see [8]). The study of linear elliptic equations of second order in divergence form with coefficients in VMO began with the paper [6] of Di Fazio who proved L^p estimates for the solution to the Dirichlet problem for equation (1.1) with $c = 0$, $d = 0$, and $f_0 = 0$. Further M.A. Ragusa has continued in [10] and [11] the study of the equations of type (1.1) (still under the assumptions $c = 0$, $d = 0$, $f_0 = 0$) obtaining $L^{p,\lambda}$ regularity results.

The general aim of the present paper is to extend the $L^{p,\lambda}$ regularity results of [10] and [11] to the case when lower order terms are present. More precisely, under the following assumptions

$$a_{ij} \in VMO \cap L^\infty(\Omega),\, d_i \in L^{p,\eta}(\Omega),\, c \in L^{p,\mu}(\Omega),\, f_0 \in L^{p_*,\lambda_*}(\Omega),\, f_i \in L^{p,\lambda}(\Omega),$$

$$i = 1, 2, \dots, n,\, 2 < p < n,\, n - p < \eta, \mu < n,\, \frac{1}{p_*} = \frac{1}{p} + \frac{1}{n},\, \lambda_* =$$

$$= \lambda \frac{p_*}{p},\, 0 < \lambda < n,$$

we shall prove that the gradient ∇u of the solution u to the Dirichlet problem for equation (1.1), for each value of ε in the range $]0, n - p[$, belongs to the space $L^{p,\lambda_\varepsilon}(\Omega)$, $\lambda_\varepsilon = \min\{\lambda, \eta - \varepsilon, \mu - \varepsilon\}$, and the relative inequality holds (see Sections 3 and 4).

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2. Some definitions and known results.

For reader's convenience we recall some definitions. A functional space we shall use throught this paper is the John-Nirenberg space BMO of the functions of bounded mean oscillation and its subspace VMO introduced in [9] and [12] respectively. We say that a locally integrable function f on $\mathbb{R}^n$ is in the space BMO if

$$\|f\|_* := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B|\, dx < +\infty, \tag{2.1}$$

where B ranges in the class of the balls in $\mathbb{R}^n$ and f_B is the integral average $\fint_B f(x)\,dx = \frac{1}{|B|}\int_B f(x)\,dx$. For $f \in BMO$ and $r > 0$, we set

$$\eta(r) = \sup_{\rho \le r} \frac{1}{|B|}\int_B |f(x) - f_B|\,dx\,, \tag{2.2}$$

where B ranges in the class of the balls with radius ρ less than or equal to r. We will say that a function $f \in BMO$ is in the space VMO if $\lim_{r\to 0}\eta(r) = 0$ and we will call $\eta(r)$ the VMO modulus of the function f.

If Ω is a bounded open set of $\mathbb{R}^n$ and if $1 \le p < +\infty$, and $0 \le \lambda \le n$, $L^{p,\lambda}(\Omega)$ denotes the space of the functions $u \in L^p(\Omega)$ such that

$$\|u\|_{L^{p,\lambda}(\Omega)} = \left(\sup_{(x,r)\in\Omega_\delta} \frac{1}{r^\lambda}\int_{\Omega(x,r)} |u(y)|^p\,dy\right)^{\frac{1}{p}} < +\infty,$$

where $\Omega(x,r) = \{y \in \Omega \,:\, |x-y| < r\}$, $\Omega_\delta = \Omega \times]0,\delta]$, and $\delta =$ diam Ω.

Lemma 2.1. ([1]) *Let* $1 \le q \le p < +\infty$ *and* $0 \le \lambda, \lambda_1 \le n$. *If* $q(n-\lambda) \le p(n-\lambda_1)$, *then* $L^{p,\lambda}(\Omega)$ *is continuously imbedded in* $L^{q,\lambda_1}(\Omega)$.

Lemma 2.2. ([2]) *If* $u \in W^{1,p}(\Omega)$, $1 \le p < +\infty$, *and* $u_{x_i} \in L^{p,\lambda}(\Omega)$, $i = 1, 2, \ldots, n$, $0 \le \lambda < n - p$, *then* $u \in L^{p,\lambda+p}(\Omega)$ *and moreover there exists a constant* k, *independent of* u, *such that*

$$\|u\|_{L^{p,\lambda+p}(\Omega)} \le k(\|\nabla u\|_{L^{p,\lambda}(\Omega)} + \|u\|_{L^p(\Omega)}). \tag{2.3}$$

Let us give a result that will be useful later on. It could be proved by a technique similar to that one used in [5], Lemma 4.1, in the case $p = 2$.

Lemma 2.3. *Let* $u \in W^{1,p}(\Omega)$ *and* $g \in L^{p,\eta}(\Omega)$, *with* $2 \le p < n$, $n - p < \eta < n$. *If* $u_{x_i} \in L^{p,\nu}(\Omega)$, $i = 1, 2, \ldots, n$, *for same* $\nu \in [0, n-p[$, *then*

$$gu \in L^{p,\eta+\nu-n+p}(\Omega).$$

Moreover there exists a constant k, *that does not depend on* u *and* g, *such that*

$$\|gu\|_{L^{p,\eta+\nu-n+p}(\Omega)} \le k\|g\|_{L^{p,\eta}(\Omega)}(\|\nabla u\|_{L^{p,\nu}(\Omega)} + \|u\|_{L^p(\Omega)}).$$

Let Ω be a bounded open set of $\mathbb{R}^n$, $n > 2$, of generic point $x = (x_1, x_2, \ldots, x_n)$, with smooth boundary, say $C^{1,1}$. Let us consider the following Dirichlet problem

$$\begin{cases} -\sum\limits_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j} - \sum\limits_{i=1}^{n}(d_i u)_{x_i} + cu = f_0 - \sum\limits_{i=1}^{n}(f_i)_{x_i} & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega) \quad (1 < p < \infty), \end{cases} \tag{2.4}$$

where we assume ([1]),

$$\begin{cases} i)\ a_{ij} \in VMO \cap L^\infty(\Omega), \quad i, j = 1, 2, \ldots, n; \\ ii)\ a_{ij} = a_{ji}, \ \text{ and } \exists\, \nu > 0 \text{ such that } \nu^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \le \nu|\xi|^2, \\ \qquad\qquad\qquad\qquad \forall \xi \in \mathbb{R}^n, \ \text{a.e. } x \in \Omega, i, j = 1, 2, \ldots, n; \\ iii)\ d = (d_1, d_2, \ldots, d_n) \in [L^{p,\eta}(\Omega)]^n \ c \in L^{p,\mu}(\Omega), \\ \quad f = (f_1, f_2, \ldots, f_n) \in [L^{p,\lambda}(\Omega)]^n, \ f_0 \in L^{p_*,\lambda_*}(\Omega), \ 2 < p < n, \\ \quad n - p < \eta, \mu < n, \ 0 < \lambda < n, \ \frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}, \ \lambda_* = \lambda \frac{p_*}{p}. \end{cases} \tag{2.5}$$

Solution of Problem (2.4) will be a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij}u_{x_i}\varphi_{x_j} + \sum_{i=1}^{n} d_i u\, \varphi_{x_i} + cu\, \varphi \Big) dx = \tag{2.6}$$

$$= \int_\Omega \Big(f_0\varphi + \sum_{i=1}^{n} f_i\, \varphi_{x_i} \Big)\, dx\,, \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Our technique is the same introduced in [3] and [4] for non divergence form equations. We estabilish interior and boundary $L^{p,\lambda}$ estimates for the gradient of u in "small" balls, using a suitable representation formula. The representation formula expresses locally the gradient of u by means of singular integral operators and commutators of the kind already considered in [3] and [4].
In the sequel we shall set, for the sake of brevity

$$Lu = -\sum_{i,j=1}^{n}(a_{ij}u_{x_i})_{x_j}.$$

Lemma 2.4. ([6]) *Let $i)$ and $ii)$ in* (2.5) *hold true and let v be a solution of the equation*

$$Lv = div\, F + F_0,$$

([1]) If φ is a function which maps Ω in $\mathbb{R}^n$, we often shall set, for the sake of brevity, $\varphi \in L^{p,\eta}(\Omega)$, instead of $\varphi \in [L^{p,\eta}(\Omega)]^n$.

whose support is contained in a ball $B_\sigma \subset\subset \Omega$*. Let us assume that* $F = (F_1, F_2, \ldots, F_n)$ *and* F_0 *are supported in* B_σ*,* $F \in [L^{p,\lambda}(B_\sigma)]^n$*,* $2 < p < n$*,* $0 < \lambda < n$*, and* $F_0 \in L^{p_*,\lambda_*}(B_\sigma)$*,* $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$*,* $\lambda_* = \lambda \frac{p_*}{p}$*. Then*

$$v_{x_i}(x) = \sum_{h,j=1}^{n} P.V. \int_{B_\sigma} \Gamma_{ij}(x, x-y)\big\{[a_{hj}(x) - a_{hj}(y)]\, v_{x_h}(y) - \tag{2.7}$$

$$- F_j(y)\big\}\, dy - \int_{B_\sigma} \Gamma_i(x, x-y) F_0(y) dy + \sum_{h=1}^{n} c_{ih}(x) F_h(x)\,, \quad \forall x \in B_\sigma,$$

where

$$c_{ih}(x) = \int_{|t|=1} \Gamma_i(x,t) t_h d\sigma_t,\ \Gamma_i(x,t) = \frac{\partial}{\partial t_i}\Gamma(x,t),\ \Gamma_{ij}(x,t) = \frac{\partial^2}{\partial t_i \partial t_j}\Gamma(x,t),$$

and

$$\Gamma(x,t) = \frac{1}{(n-2)\omega_n (det\, a_{ij}(x))^{\frac{1}{2}}} \left(\sum_{i,j=1}^{n} A_{ij}(x) t_i t_j \right)^{\frac{2-n}{2}} a.e.\ x \in \Omega,\ t \neq 0,$$

with A_{ij} *cofactor of* a_{ij} *in the matrix* (a_{ij}) *and* ω_n *surface area of the unit ball.*

It is a well known fact that Γ_{ij} are Calderon–Zygmund kernel in the t variable for a.a. $x \in \Omega$.

We conclude this section recalling two known existence and regularity results for Problem (2.4), in the case $d = 0$, $c = 0$, and $f_0 = 0$.

Theorem 2.1. ([6], Theorem 2.1) *Let* $i)$ *and* $ii)$ *in* (2.5) *hold true. If* $f_i \in L^p(\Omega)$*,* $i = 1, 2, \ldots, n$*, and* $1 < p < +\infty$*, then the Dirichlet problem* (2.4)*, with* $d = 0$*,* $c = 0$*, and* $f_0 = 0$*, has a unique solution* u *and moreover there exists a constant* k*, that does not depend on* u *and* f*, such that*

$$\|\nabla u\|_{L^p(\Omega)} \leq k \|f\|_{L^p(\Omega)}. \tag{2.8}$$

Theorem 2.2. ([11], Theorem 4.3) *Let* $i)$ *and* $ii)$ *in* (2.5) *hold true. If* $f_i \in L^{p,\lambda}(\Omega)$*,* $i = 1, 2, \ldots, n$*,* $2 < p < +\infty$*, and* $0 < \lambda < n$*, then the gradient of the solution* u *of Dirichlet problem* (2.4)*, with* $d = 0$*,* $c = 0$*, and* $f_0 = 0$ (2)*, belongs to* $L^{p,\lambda}(\Omega)$*. Moreover there exists a constant* k *that does not depend on* u *and* f *such that*

$$\|\nabla u\|_{L^{p,\lambda}(\Omega)} \leq k \|f\|_{L^{p,\lambda}(\Omega)}. \tag{2.9}$$

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(2) The existence and uniqueness of the solution u of Dirichlet problem (2.4) are assured by Theorem 2.1.

3. $L^{p,\lambda}$ regularity: the case $d = f = 0$.

In this section and in the next one we shall show regularity results for (2.4). We shall study the effect of lower order terms looking at them one by one. The first term we study is the one concerned with the potential $c(x)$. The study is splitted into two parts. In the first one we prove a regularity result assuming some extra technical hypotheses. Namely we assume that the term cu belongs to a convenient Morrey space. These assumption will be removed later. Once we get the result for the potential $c(x)$, we shall sketch the proof of the case when the other lower order terms are present. Let us start with the following lemma.

Lemma 3.1. *Let $i)$ and $ii)$ in (2.5) hold true and let $u \in W^{1,p}(B_\sigma)$, $2 < p < n$, be a solution in the ball $B_\sigma \subset\subset \Omega$ of the equation*

$$Lu + cu = f_0,$$

where $f_0 \in L^{p_*,\lambda_*}(B_\sigma)$, $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$, $\lambda_* = \lambda \frac{p_*}{p}$, $0 < \lambda < n$. Let us suppose that $u \in L^{p,\alpha}(B_\sigma)$, $\nabla u \in L^{p_*,\alpha_*}(B_\sigma)$, and $cu \in L^{p_*,\beta_*}(B_\sigma)$, with $\alpha_* = \alpha \frac{p_*}{p}$, $\beta_* = \beta \frac{p_*}{p}$, and $0 < \alpha, \beta < n$. Then there exists $\bar{\sigma} \in]0, \sigma[$ such that, for every ball B_ρ concentric to B_σ with $\rho \leq \bar{\sigma}$, we have

a) $\nabla u \in L^{p,\delta}(B_{\frac{\rho}{2}})$;
b) $\|\nabla u\|_{L^{p,\delta}(B_{\frac{\rho}{2}})} \leq k(\|u\|_{L^{p,\alpha}(B_\rho)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\rho)} + \|cu\|_{L^{p_*,\beta_*}(B_\rho)} +$
$\|f_0\|_{L^{p_*,\lambda_*}(B_\rho)})$,

where $\delta = \min\{\alpha, \beta, \lambda\}$.

Proof. We localize the solution. Fixed a ball B_ρ concentric to B_σ with $\rho < \sigma$, let $\theta \in C_0^\infty(B_\rho)$ a standard cut-off function identically 1 in $B_{\frac{\rho}{2}}$, $0 \leq \theta \leq 1$ and $|\nabla\theta| < \frac{2c}{\rho}$.
The function $v = \theta u$ is supported in B_ρ and it is a solution of the equation

$$Lv = \operatorname{div} F + F_0,$$

where $F = (F_1, F_2, \ldots, F_n)$, $F_j = -\sum_{i=1}^n (a_{ij}\theta_{x_i} u)$, $F_0 = -\sum_{i,j=1}^n (a_{ij}\theta_{x_j} u_{x_i})$ $+\theta(f_0 - cu)$. Moreover we have $F \in [L^{p,\delta}(B_\rho)]^n$, $F_0 \in L^{p_*,\delta_*}(B_\rho)$, with $\delta_* = \delta \frac{p_*}{p}$. Therefore the functions θu, F_0 and F fulfill the hypotheses of Lemma 2.4, and we have

$$(\theta u)_{x_i}(x) = \sum_{h,j=1}^n P.V. \int_{B_\rho} \Gamma_{ij}(x, x-y)\{[a_{hj}(x) - a_{hj}(y)](\theta u)_{x_h}(y) - F_j(y)\}dy-$$

$$-\int_{B_\rho} \Gamma_i(x, x-y)F_0(y)dy + \sum_{h=1}^{n} c_{ih}(x)F_h(x), \quad x \in B_\rho.$$

By virtue of the previous representation formula and by a similar argument of that one used in [10], Theorem 4.2, based on the uniqueness of the fixed point of a contraction, it is possible to prove the existence of a $\bar{\sigma} \in]0, \sigma[$ such that, for every ball B_ρ concentric to B_σ with $\rho \le \bar{\sigma}$, one has

$$\left(\frac{1}{\rho^\delta}\int_{B_\rho} |(\theta u)_{x_i}|^p\, dx\right)^{\frac{1}{p}} \le k(\|F\|_{L^{p,\delta}(B_\rho)} + \|F_0\|_{L^{p_*,\delta_*}(B_\rho)}),$$

from which $a)$ and $b)$ follow easily. □

The next lemma removes the extra assumption on the potential term $c(x)u(x)$. Namely we have

Lemma 3.2. *Let $i)$ and $ii)$ in (2.5) hold true and let $u \in W^{1,p}(B_\sigma)$, $2 < p < n$, be a solution in the ball $B_\sigma \subset\subset \Omega$ of the equation*

$$Lu + cu = f_0,$$

where $c \in L^{p,\mu}(B_\sigma)$, $f_0 \in L^{p_*,\lambda_*}(B_\sigma)$, $n - p < \mu < n$, $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$, $\lambda_* = \lambda\frac{p_*}{p}$, $0 < \lambda < n$. Let us suppose that $u \in L^{p,\alpha}(B_\sigma)$, $\nabla u \in L^{p_*,\alpha_*}(B_\sigma)$, with $0 < \alpha < n$, and $\alpha_* = \alpha\frac{p_*}{p}$. Let $\varepsilon > 0$ such that $\varepsilon < n - p$. Then there exists $\sigma_1 \in]0, \sigma[$ such that, for every ball B_ρ concentric to B_σ with $\rho \le \sigma_1$, we have:

$j)$ $\nabla u \in L^{p,\tilde{\lambda}^{(\varepsilon)}}(B_\rho)$;
$jj)$ $\|\nabla u\|_{L^{p,\tilde{\lambda}^{(\varepsilon)}}(B_\rho)} \le k(\|u\|_{L^{p,\alpha}(B_\sigma)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)})$,

where $\tilde{\lambda}^{(\varepsilon)} = \min\{\alpha, \lambda, \mu - \varepsilon\}$.

Proof. Since $c \in L^{p,\mu}(B_\sigma)$ and $\nabla u \in L^p(B_\sigma)$, thanks to Lemma 2.3 (with $\eta = \mu$ and $\nu = 0$), if $\bar{\mu} = \mu - n + p$, $\bar{\mu}_* = \bar{\mu}\frac{p_*}{p}$, it results

$$cu \in L^{p,\bar{\mu}}(B_\sigma), \tag{3.1}$$

and, for every ball B_r concentric to B_σ with $r \le \sigma$, we have

$$\|cu\|_{L^{p,\bar{\mu}}(B_r)} \le k\|c\|_{L^{p,\mu}(B_r)}(\|\nabla u\|_{L^p(B_r)} + \|u\|_{L^p(B_r)}). \tag{3.2}$$

From (3.1) and (3.2), thanks to the imbedding $L^{p,\bar{\mu}}(B_\sigma) \subset L^{p_*,\bar{\mu}_*}(B_\sigma)$, it follows that $cu \in L^{p_*,\bar{\mu}_*}(B_\sigma)$ and, for every ball B_r concentric to B_σ with $r \leq \sigma$, we have

$$\|cu\|_{L^{p_*,\bar{\mu}_*}(B_r)} \leq k\|c\|_{L^{p,\mu}(B_r)}(\|\nabla u\|_{L^p(B_r)} + \|u\|_{L^p(B_r)}). \tag{3.3}$$

Let us consider now the following cases.

1) If $\bar{\mu} \geq \tilde{\lambda}^{(\varepsilon)}$, one has $\delta = \min\{\alpha, \lambda, \bar{\mu}\} \geq \tilde{\lambda}^{(\varepsilon)}$. On the other hand Lemma 3.1 (with $\beta = \bar{\mu}$) ensures that, there exists $\bar{\sigma} \in]0, \sigma[$ such that, for every ball B_ρ concentric to B_σ with $\rho \leq \bar{\sigma}$, it results

$$\nabla u \in L^{p,\delta}(B_{\frac{\rho}{2}}), \tag{3.4}$$

and the following estimate holds

$$\|\nabla u\|_{L^{p,\delta}(B_{\frac{\rho}{2}})} \leq k(\|u\|_{L^{p,\alpha}(B_\rho)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\rho)} + \|cu\|_{L^{p_*,\bar{\mu}_*}(B_\rho)} + \\ + \|f_0\|_{L^{p_*,\lambda_*}(B_\rho)}). \tag{3.5}$$

$j)$ and $jj)$ are consequence of the inequality $\delta \geq \tilde{\lambda}^{(\varepsilon)}$ and of (3.4), (3.5), and (3.3) (with $r = \rho$). In this case it is possible to assume $\sigma_1 = \frac{\bar{\sigma}}{2}$.

2) If $\bar{\mu} < \tilde{\lambda}^{(\varepsilon)}$, and $\bar{\mu} \geq n - p$, one has $\delta = \min\{\alpha, \lambda, \bar{\mu}\} = \bar{\mu} \geq n - p > n - p - \varepsilon$. By Lemma 3.1 (with $\beta = \bar{\mu}$) we have $\nabla u \in L^{p,\delta}(B_{\frac{\bar{\sigma}}{2}})$ and estimate (3.5) holds. Then, thanks to the inequality $\delta > n - p - \varepsilon$ we have, for every ball B_ρ concentric to B_σ with $\rho \leq \bar{\sigma}$, $\nabla u \in L^{p,n-p-\varepsilon}(B_{\frac{\rho}{2}})$ and

$$\|\nabla u\|_{L^{p,n-p-\varepsilon}(B_{\frac{\rho}{2}})} \leq k(\|u\|_{L^{p,\alpha}(B_\rho)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\rho)} + \\ + \|cu\|_{L^{p_*,\bar{\mu}_*}(B_\rho)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\rho)}) \leq \\ \leq k(\|u\|_{L^{p,\alpha}(B_\sigma)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}). \tag{3.6}$$

By Lemma 2.3 (with $\Omega = B_{\frac{\bar{\sigma}}{2}}$, $\eta = \mu$, and $\nu = n - p - \varepsilon$) one has $cu \in L^{p,\mu-\varepsilon}(B_{\frac{\bar{\sigma}}{2}})$ and

$$\|cu\|_{L^{p,\mu-\varepsilon}(B_{\frac{\bar{\sigma}}{2}})} \leq k\|c\|_{L^{p,\mu}(B_{\frac{\bar{\sigma}}{2}})}(\|\nabla u\|_{L^{p,n-p-\varepsilon}(B_{\frac{\bar{\sigma}}{2}})} + \|u\|_{L^p(B_{\frac{\bar{\sigma}}{2}})}). \tag{3.7}$$

Thanks to the imbedding $L^{p,\mu-\varepsilon}(B_{\frac{\bar{\sigma}}{2}}) \subset L^{p_*,(\mu-\varepsilon)_*}(B_{\frac{\bar{\sigma}}{2}})$, $(\mu - \varepsilon)_* = (\mu - \varepsilon)\frac{p_*}{p}$, and from (3.6) (with $\rho = \bar{\sigma}$) and (3.7), we have $cu \in L^{p_*,(\mu-\varepsilon)_*}(B_{\frac{\bar{\sigma}}{2}})$ and

$$\|cu\|_{L^{p_*,(\mu-\varepsilon)_*}(B_{\frac{\bar{\sigma}}{2}})} \leq k(\|u\|_{L^{p,\alpha}(B_\sigma)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\sigma)} + \tag{3.8}$$

$$+ \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}).$$

Lemma 3.1 (with $\beta = \mu - \varepsilon$) ensures that there exists $\bar\sigma_1 \in]0, \frac{\bar\sigma}{2}[$ such that, for every ball B_ρ concentric to B_σ with $\rho \le \bar\sigma_1$, it results $\nabla u \in L^{p,\tilde\lambda^{(\varepsilon)}}(B_{\frac{\rho}{2}})$ and the following inequalities hold

$$\|\nabla u\|_{L^{p,\tilde\lambda^{(\varepsilon)}}(B_{\frac{\rho}{2}})} \le k(\|u\|_{L^{p,\alpha}(B_\rho)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\rho)} + \tag{3.9}$$

$$+\|cu\|_{L^{p_*,(\mu-\varepsilon)_*}(B_\rho)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\rho)}) \le$$

$$\le k(\|u\|_{L^{p,\alpha}(B_\sigma)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\sigma)} + \|cu\|_{L^{p_*,(\mu-\varepsilon)_*}(B_{\frac{\bar\sigma}{2}})} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}).$$

From (3.9) and (3.8) it follows that

$$\|\nabla u\|_{L^{p,\tilde\lambda^{(\varepsilon)}}(B_{\frac{\rho}{2}})} \le k(\|u\|_{L^{p,\alpha}(B_\sigma)} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)})$$

and then $jj)$ with $\sigma_1 = \frac{\bar\sigma_1}{2}$.

3) If $\bar\mu < \tilde\lambda^{(\varepsilon)}$ and $\bar\mu < n - p$, one has $\delta = \min\{\alpha, \lambda, \bar\mu\} = \bar\mu$. By Lemma 3.1 (with $\beta = \bar\mu$) we have $\nabla u \in L^{p,\bar\mu}(B_{\frac{\bar\sigma}{2}})$ and

$$\|\nabla u\|_{L^{p,\bar\mu}(B_{\frac{\bar\sigma}{2}})} \le k(\|u\|_{L^{p,\alpha}(B_{\bar\sigma})} + \|\nabla u\|_{L^{p_*,\alpha_*}(B_{\bar\sigma})} + \|\nabla u\|_{L^p(B_{\bar\sigma})} + \|f_0\|_{L^{p_*,\lambda_*}(B_{\bar\sigma})}).$$

Thanks again to Lemma 2.3 (with $\Omega = B_{\frac{\bar\sigma}{2}}$, $\eta = \mu$, and $\nu = \bar\mu$) one has $cu \in L^{p,2\bar\mu}(B_{\frac{\bar\sigma}{2}}) \subset L^{p_*,(2\bar\mu)_*}(B_{\frac{\bar\sigma}{2}})$ ($(2\bar\mu)_* = 2\bar\mu\frac{p_*}{p}$) and the relative inequalities hold. If $2\bar\mu \ge \tilde\lambda^{(\varepsilon)}$ we may proceed as in 1). Otherwise iterating this procedure: if h is the greatest positive integer such that $h\bar\mu < \tilde\lambda^{(\varepsilon)}$ and $h\bar\mu < n - p$, then there exists $\bar\sigma_{h-1} \in]0, \sigma[$ such that $\nabla u \in L^{p,h\bar\mu}(B_{\bar\sigma_{h-1}})$. Thanks to Lemma 2.3 (with $\Omega = B_{\bar\sigma_{h-1}}$, $\eta = \mu$, and $\nu = h\bar\mu$) we have $cu \in L^{p,(h+1)\bar\mu}(B_{\bar\sigma_{h-1}}) \subset L^{p_*,((h+1)\bar\mu)_*}(B_{\bar\sigma_{h-1}})$ ($((h+1)\bar\mu)_* = (h+1)\bar\mu\frac{p_*}{p}$) and the relative inequalities hold.

Now there are two possibilities

i) $(h+1)\bar\mu \ge \tilde\lambda^{(\varepsilon)}$;

ii) $(h+1)\bar\mu < \tilde\lambda^{(\varepsilon)}$.

If $i)$ is true, then we may proceed exactly as in 1); if $ii)$ is true, it results $(h+1)\bar\mu \ge n - p$, and then we may proceed as in 2). □

We can give now a first regularity result.

Theorem 3.1. *Let $i)$ and $ii)$ in (2.5) hold true and let $u \in W^{1,p}(B_\sigma)$, $2 < p < n$, be a solution in the ball $B_\sigma \subset\subset \Omega$ of the equation*

$$Lu + cu = f_0,$$

where $c \in L^{p,\mu}(B_\sigma)$, $f_0 \in L^{p_*,\lambda_*}(B_\sigma)$, $n - p < \mu < n$, $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$, $\lambda_* = \lambda \frac{p_*}{p}$, $0 < \lambda < n$. Let $\varepsilon > 0$ such that $\varepsilon < n - p$. Then there exists $\sigma^* \in]0, \sigma[$ such that, for every ball B_ρ concentric to B_σ with $\rho \leq \sigma^*$, we have

i) $\nabla u \in L^{p_*,\lambda_*^{(\varepsilon)}}(B_\rho)$;

ii) $u \in L^{p,\lambda^{(\varepsilon)}}(B_\rho)$,

where $\lambda_*^{(\varepsilon)} = \lambda^{(\varepsilon)} \frac{p_*}{p}$, $\lambda^{(\varepsilon)} = \min\{\lambda, \mu - \varepsilon\}$. Moreover the following inequalities hold

j) $\|\nabla u\|_{L^{p_*,\lambda_*^{(\varepsilon)}}(B_\rho)} \leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)})$;

jj) $\|u\|_{L^{p,\lambda^{(\varepsilon)}}(B_\rho)} \leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)})$,

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Proof. We know that $u \in L^{p^*}(B_\sigma) \subset L^{p,p}(B_\sigma)$ and $\nabla u \in L^p(B_\sigma) \subset L^{p_*,p_*}(B_\sigma)$. If $p \geq \lambda^{(\varepsilon)}$ we have finished.
Let us consider the case $p < \lambda^{(\varepsilon)}$. Thanks to Lemma 3.2, there exists $\sigma_1^{(1)} \in]0, \sigma[$ such that, for every ball B_ρ concentric to B_σ with $\rho \leq \sigma_1^{(1)}$, we have

$$\nabla u \in L^{p,p}(B_\rho) \tag{3.10}$$

and

$$\|\nabla u\|_{L^{p,p}(B_\rho)} \leq k(\|u\|_{L^{p,p}(B_\sigma)} + \|\nabla u\|_{L^{p_*,p_*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \tag{3.11}$$
$$+ \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}) \leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}).$$

From (3.10) and (3.11), since $L^{p,p}(B_\rho) \subset L^{p_*,2p_*}(B_\rho)$, for every ball B_ρ concentric to B_σ with $\rho \leq \sigma_1^{(1)}$, it follows that

$$\nabla u \in L^{p_*,2p_*}(B_\rho) \tag{3.12}$$

and

$$\|\nabla u\|_{L^{p_*,2p_*}(B_\rho)} \leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}). \tag{3.13}$$

Let us consider the following cases.

1) If $2p \geq \lambda^{(\varepsilon)}$ and $p \geq n-p$, then i) and j) are immediate consequence of (3.12) and (3.13). Moreover, from the estimate $p > n-p-\varepsilon$, we get $\nabla u \in L^{p,n-p-\varepsilon}(B_\rho)$; thus, thanks to Lemma 2.2 and (3.11) we have

$$u \in L^{p,n-\varepsilon}(B_\rho) \tag{3.14}$$

and

$$\begin{aligned} \|u\|_{L^{p,n-\varepsilon}(B_\rho)} &\leq k(\|\nabla u\|_{L^{p,n-p-\varepsilon}(B_\rho)} + \|u\|_{L^p(B_\rho)}) \leq \\ &\leq k(\|\nabla u\|_{L^{p,p}(B_\rho)} + \|u\|_{L^p(B_\rho)}) \leq \\ &\leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}). \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), since $n-\varepsilon > \lambda^{(\varepsilon)}$ we have ii) and jj).

2) If $2p \geq \lambda^{(\varepsilon)}$, $p < n-p$, as in 1), i) and j) are immediate consequence of (3.12) and (3.13), with $\sigma^* = \sigma_1^{(1)}$. Moreover, from Lemma 2.2 and (3.10), since $p < n-p$, it follows that, for every ball B_ρ concentric to B_σ with $\rho \leq \sigma_1^{(1)}$,

$$u \in L^{p,2p}(B_\rho) \tag{3.16}$$

and

$$\|u\|_{L^{p,2p}(B_\rho)} \leq k(\|\nabla u\|_{L^{p,p}(B_\rho)} + \|u\|_{L^p(B_\rho)}). \tag{3.17}$$

Thanks to (3.11), (3.16) and (3.17), and because $2p \geq \lambda^{(\varepsilon)}$, we have ii) and jj), with $\sigma^* = \sigma_1^{(1)}$.

3) If $2p < \lambda^{(\varepsilon)}$, one has $p < n-p$. Now from (3.10), taking into account Lemma 2.2 and (3.11), one has, for every ball B_ρ concentric to B_σ with $\rho \leq \sigma_1^{(1)}$, $u \in L^{p,2p}(B_\rho)$ and

$$\|u\|_{L^{p,2p}(B_\rho)} \leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)})\,. \tag{3.18}$$

We know that $u \in L^{p,2p}(B_{\sigma_1^{(1)}})$ and $\nabla u \in L^{p_*,2p_*}(B_{\sigma_1^{(1)}})$, then, Lemma 3.2 ensures the existence of $\sigma_2^{(1)} < \sigma_1^{(1)}$ such that, for every ball B_ρ concentric to B_σ with $\rho \leq \sigma_2^{(1)}$, one has

$$\nabla u \in L^{p,2p}(B_\rho) \tag{3.19}$$

and, thanks to (3.13) and (3.18), it follows

$$\|\nabla u\|_{L^{p,2p}(B_\rho)} \leq k(\|u\|_{L^{p,2p}(B_{\sigma_1^{(1)}})} + \tag{3.20}$$

$$+ \|\nabla u\|_{L^{p_*,2p_*}(B_{\sigma_1^{(1)}})} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}) \leq$$

$$\leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}).$$

From (3.19) and (3.20), because $L^{p,2p}(B_\rho) \subset L^{p_*,3p_*}(B_\rho)$, for every ball B_ρ concentric to B_σ with $\rho \leq \sigma_2^{(1)}$, it follows $\nabla u \in L^{p_*,3p_*}(B_\rho)$ and

$$\|\nabla u\|_{L^{p_*,3p_*}(B_\rho)} \leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}).$$

If $3p \geq \lambda^{(\varepsilon)}$ and $2p \geq n - p$ we may proceed as in 1).
If $3p \geq \lambda^{(\varepsilon)}$ and $2p < n - p$ we may proceed as in 2).
If $3p < \lambda^{(\varepsilon)}$, we may iterate this technique, and let h be the greatest positive integer such that $hp < \lambda^{(\varepsilon)}$. As before, it is possible to find $\sigma_{h-1}^{(1)} \in]0, \sigma[$ such that, for every $\rho \leq \sigma_{h-1}^{(1)}$, we have $u \in L^{p,hp}(B_\rho)$ and $\nabla u \in L^{p_*,hp_*}(B_\rho)$. Moreover the following inequalities hold

$$\|\nabla u\|_{L^{p_*,hp_*}(B_\rho)} \leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}) \tag{3.21}$$

and

$$\|u\|_{L^{p,hp}(B_\rho)} \leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}). \tag{3.22}$$

Thanks to Lemma 3.2 there exists $\sigma_h^{(1)} \in]0, \sigma_{h-1}^{(1)}[$ such that, for every ball B_ρ concentric to B_σ with $\rho \leq \sigma_h^{(1)}$, one has

$$\nabla u \in L^{p,hp}(B_\rho) \tag{3.23}$$

and, thanks also to (3.21) and (3.22), we have

$$\|\nabla u\|_{L^{p,hp}(B_\rho)} \leq k(\|u\|_{L^{p,hp}(B_{\sigma_{h-1}^{(1)}})} + \tag{3.24}$$

$$+ \|\nabla u\|_{L^{p_*,hp_*}(B_{\sigma_{h-1}^{(1)}})} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}) \leq$$

$$\leq k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}).$$

From (3.23) and (3.24), since $L^{p,hp}(B_\rho) \subset L^{p_*,(h+1)p_*}(B_\rho)$, it follows, for every ball B_ρ concentric to B_σ with $\rho \le \sigma_h^{(1)}$, $\nabla u \in L^{p_*,(h+1)p_*}(B_\rho)$ and

$$\|\nabla u\|_{L^{p_*,(h+1)p_*}(B_\rho)} \le k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}).$$

Finally we have to consider the cases

a) $(h+1)p \ge \lambda^{(\varepsilon)}$, $hp \ge n-p$;
b) $(h+1)p \ge \lambda^{(\varepsilon)}$, $hp < n-p$.

If *a*) is true we may proceed as in 1).
If instead *b*) is true, then we may proceed as in 2). □

Now we are able to prove the main regularity result concerning the potential term $c(x)$.

Theorem 3.2. *Let i) and ii) in* (2.5) *hold true and let* $u \in W^{1,p}(B_\sigma)$, $2 < p < n$, *be a solution in the ball* $B_\sigma \subset\subset \Omega$ *of the equation*

$$Lu + cu = f_0,$$

where $c \in L^{p,\mu}(B_\sigma)$, $f_0 \in L^{p_*,\lambda_*}(B_\sigma)$, $n-p < \mu < n$, $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$, $\lambda_* = \lambda\frac{p_*}{p}$, $0 < \lambda < n$. *Let* $\varepsilon > 0$ *such that* $\varepsilon < n-p$. *Then there exist* $\tilde\sigma \in]0,\sigma[$ *such that, for every ball* B_ρ *concentric to* B_σ *with* $\rho \le \tilde\sigma$ *we have*

i) $\nabla u \in L^{p,\lambda^{(\varepsilon)}}(B_\rho)$;
ii) $\|\nabla u\|_{L^{p,\lambda^{(\varepsilon)}}(B_\rho)} \le k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)})$,

where $\lambda^{(\varepsilon)} = \min\{\lambda, \mu-\varepsilon\}$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Proof. Thanks to Theorem 3.1 there exists $\sigma^* \in]0,\sigma[$ such that, for every ball B_ρ concentric to B_σ with $\rho \le \sigma^*$, we have $\nabla u \in L^{p_*,\lambda_*^{(\varepsilon)}}(B_\rho)$, $u \in L^{p,\lambda^{(\varepsilon)}}(B_\rho)$, $\lambda_*^{(\varepsilon)} = \lambda^{(\varepsilon)}\frac{p_*}{p}$, and the following inequalities hold

$$\|\nabla u\|_{L^{p_*,\lambda_*^{(\varepsilon)}}(B_\rho)} \le k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}) \tag{3.25}$$

and

$$\|u\|_{L^{p,\lambda^{(\varepsilon)}}(B_\rho)} \le k(\|u\|_{L^{p^*}(B_\sigma)} + \|\nabla u\|_{L^p(B_\sigma)} + \|f_0\|_{L^{p_*,\lambda_*}(B_\sigma)}). \tag{3.26}$$

Now we may use Lemma 3.2 (with $\sigma = \sigma^*$, $\alpha = \lambda^{(\varepsilon)}$), then it is possible to find $\sigma_1^* \in]0,\sigma^*[$ such that, for every ball B_ρ concentric to B_σ with $\rho \le \sigma_1^*$, one has

$$\nabla u \in L^{p,\lambda^{(\varepsilon)}}(B_\rho) \tag{3.27}$$

and

$$\|\nabla u\|_{L^{p,\lambda^{(\varepsilon)}}(B_\rho)} \leq k(\|u\|_{L^{p,\lambda^{(\varepsilon)}}(B_{\sigma^*})} + \|\nabla u\|_{L^{p_*,\lambda_*^{(\varepsilon)}}(B_{\sigma^*})} + \tag{3.28}$$

$$+ \|\nabla u\|_{L^p(B_{\sigma^*})} + \|f_0\|_{L^{p_*,\lambda_*}(B_{\sigma^*})}).$$

$i)$ and $ii)$ are consequence of (3.25)–(3.28): we can choose $\tilde{\sigma} = \sigma_1^*$. □

As an immediate consequence of Theorem 3.2 we have the following corollary:

Corollary 3.1. *Let $i)$ and $ii)$ in* (2.5) *hold true and let $u \in W^{1,p}(\Omega)$, $2 < p < n$, be a solution in Ω of the equation*

$$Lu + cu = f_0,$$

where $c \in L^{p,\mu}(\Omega)$, $f_0 \in L^{p_*,\lambda_*}(\Omega)$, $n - p < \mu < n$, $\frac{1}{p_*} = \frac{1}{p} + \frac{1}{n}$, $\lambda_* = \lambda \frac{p_*}{p}$, $0 < \lambda < n$. Let $\varepsilon > 0$ such that $\varepsilon < n - p$. Then, if $\lambda^{(\varepsilon)} = \min\{\lambda, \mu - \varepsilon\}$, it results $\nabla u \in L^{p,\lambda^{(\varepsilon)}}_{\text{loc}}(\Omega)$ and, for every ball $B \subset\subset \Omega$, we have

$$\|\nabla u\|_{L^{p,\lambda^{(\varepsilon)}}(B)} \leq k(\|u\|_{L^{p^*}(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + \|f_0\|_{L^{p_*,\lambda_*}(\Omega)}),$$

where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.

Remark 3.1. All the above results can be proved in the case when the ball B_σ intersect the boundary $\partial\Omega$ of Ω. Further, via a standard flattening of the boundary and partition of unity, we obtain global $L^{p,\lambda}$ regolarity results.

4. $L^{p,\lambda}$ regularity: the case $c = f_0 = 0$.

Let us consider now Problem (2.4) with $c = f_0 = 0$. We prove the following result.

Theorem 4.1. *Let* (2.5) *holds true (with $c = f_0 = 0$) and let $u \in W_0^{1,p}(\Omega)$, $2 < p < n$, be a solution of Problem* (2.4). *Let $\varepsilon > 0$ such that $\varepsilon < n - p$. Then we have*

a) $\nabla u \in L^{p,\lambda_\varepsilon}(\Omega)$;
b) $\|\nabla u\|_{L^{p,\lambda_\varepsilon}(\Omega)} \leq k(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)})$,

where $\lambda_\varepsilon = \min\{\lambda, \eta - \varepsilon\}$.

Proof. Since $u \in W_0^{1,p}(\Omega)$, thanks to Lemma 2.3 (with $\nu = 0$), one has $du \in L^{p,\bar{\eta}}(\Omega)$, where $\bar{\eta} = \eta - n + p$, and

$$\|du\|_{L^{p,\bar{\eta}}(\Omega)} \leq k\|d\|_{L^{p,\eta}(\Omega)}(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}) .$$

Therefore $f - du \in L^{p,\min\{\lambda,\bar{\eta}\}}(\Omega)$ and then, thanks to Theorem 2.2, we have

$$\nabla u \in L^{p,\min\{\lambda,\bar{\eta}\}}(\Omega) \tag{4.1}$$

and

$$\begin{aligned} \|\nabla u\|_{L^{p,\min\{\lambda,\bar{\eta}\}}(\Omega)} &\leq k\|f - du\|_{L^{p,\min\{\lambda,\bar{\eta}\}}(\Omega)} \leq \\ &\leq k(\|f\|_{L^{p,\lambda}(\Omega)} + \|du\|_{L^{p,\bar{\eta}}(\Omega)}) \leq \\ &\leq k(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}). \end{aligned} \tag{4.2}$$

Let us consider the following cases.

1) If $\bar{\eta} \geq \lambda$, of course $\min\{\lambda, \bar{\eta}\} = \lambda$, then *a*) and *b*) are consequences of (4.1) and (4.2).

2) If $\bar{\eta} < \lambda$ and $\bar{\eta} \geq n - p$, of course $\min\{\lambda, \bar{\eta}\} = \bar{\eta}$, then, thanks to (4.1) and (4.2), we have $\nabla u \in L^{p,\bar{\eta}}(\Omega)$ and

$$\|\nabla u\|_{L^{p,\bar{\eta}}(\Omega)} \leq k(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}) . \tag{4.3}$$

Since $\bar{\eta} \geq n - p > n - p - \varepsilon$, $\nabla u \in L^{p,n-p-\varepsilon}(\Omega)$, and

$$\|\nabla u\|_{L^{p,n-p-\varepsilon}(\Omega)} \leq k\|\nabla u\|_{L^{p,\bar{\eta}}(\Omega)}$$

from which, thanks to (4.3), it follows that

$$\|\nabla u\|_{L^{p,n-p-\varepsilon}(\Omega)} \leq k(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}). \tag{4.4}$$

Now thanks to Lemma 2.3 (with $\nu = n - p - \varepsilon$), it results $du \in L^{p,\eta-\varepsilon}(\Omega)$ and

$$\|du\|_{L^{p,\eta-\varepsilon}(\Omega)} \leq k\|d\|_{L^{p,\eta}(\Omega)}(\|\nabla u\|_{L^{p,n-p-\varepsilon}(\Omega)} + \|u\|_{L^p(\Omega)}), \tag{4.5}$$

therefore $f - du \in L^{p,\lambda_\varepsilon}(\Omega)$. Theorem 2.2 ensures *a*) and the following estimate

$$\|\nabla u\|_{L^{p,\lambda_\varepsilon}(\Omega)} \leq k\|f - du\|_{L^{p,\lambda_\varepsilon}(\Omega)} \leq k(\|f\|_{L^{p,\lambda}(\Omega)} + \|du\|_{L^{p,\eta-\varepsilon}(\Omega)}),$$

from which, thanks to (4.5), we have

$$\|\nabla u\|_{L^{p,\lambda_\varepsilon}(\Omega)} \leq k(\|f\|_{L^{p,\lambda}(\Omega)} + \|\nabla u\|_{L^{p,n-p-\varepsilon}(\Omega)} + \|u\|_{L^p(\Omega)}). \tag{4.6}$$

Inequality *b*) is consequence of (4.6) and (4.4).

3) If $\bar\eta < \lambda$ and $\bar\eta < n - p$ we have, as in 2), $\nabla u \in L^{p,\bar\eta}(\Omega)$, and estimate (4.3) holds. Now Lemma 2.3 (with $\nu = \bar\eta$) ensures that $du \in L^{p,2\bar\eta}(\Omega)$ and, thanks also to (4.3)

$$\begin{aligned}\|du\|_{L^{p,2\bar\eta}(\Omega)} &\leq k\|d\|_{L^{p,\eta}(\Omega)}(\|\nabla u\|_{L^{p,\bar\eta}(\Omega)} + \|u\|_{L^p(\Omega)}) \leq \\ &\leq k\|d\|_{L^{p,\eta}(\Omega)}(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}).\end{aligned} \tag{4.7}$$

Therefore $f - du \in L^{p,\min\{\lambda,2\bar\eta\}}(\Omega)$, then, if $2\bar\eta \geq \lambda$, *a*) and *b*) are consequence of Theorem 2.2 and of (4.7). If $2\bar\eta < \lambda$ and $2\bar\eta \geq n - p$ we may proceed as in 2). If instead $2\bar\eta < \lambda$ and $2\bar\eta < n - p$, iterating this procedure, and if h is the greatest integer such that $h\bar\eta < \lambda$ and $h\bar\eta < n - p$ one has $\nabla u \in L^{p,h\bar\eta}(\Omega)$ and

$$\|\nabla u\|_{L^{p,h\bar\eta}(\Omega)} \leq k(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}). \tag{4.8}$$

Since $h\bar\eta < n-p$, Lemma 2.3 (with $\nu = h\bar\eta$) ensures that $du \in L^{p,(h+1)\bar\eta}(\Omega)$ and, thanks also to (4.8)

$$\begin{aligned}\|du\|_{L^{p,(h+1)\bar\eta}(\Omega)} &\leq k\|d\|_{L^{p,\eta}(\Omega)}(\|\nabla u\|_{L^{p,h\bar\eta}(\Omega)} + \|u\|_{L^p(\Omega)}) \leq \\ &\leq k\|d\|_{L^{p,\eta}(\Omega)}(\|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} + \|f\|_{L^{p,\lambda}(\Omega)}).\end{aligned}$$

Then $f - du \in L^{p,\min\{\lambda,(h+1)\bar\eta\}}(\Omega)$. If $(h + 1)\bar\eta \geq \lambda$ we have finished; if instead $(h + 1)\bar\eta < \lambda$, it must result $(h + 1)\bar\eta \geq n - p$, then we may proceed as in 2). □

Remark 4.1. The techniques used in Sections 3 and 4 allow to prove, under assumptions (2.5), the $L^{p,\lambda}$ regularity for the gradient of the solution to Dirichlet problem (2.4) with $c \neq 0$, $f_0 \neq 0$, $d \neq 0$, and $f \neq 0$.

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