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# ON SOLVABILTY OF PSEUDODIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

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The paper presents a result on local solvability, in classical Sobolev spaces, of pseudodifferential operators with constant coefficients. The proof of this result is based on a new Hörmander's type inequality for such pseudodifferential operators.

## 1. Introduction

The well known Poincaré inequality in $L^2$ for bounded intervals

$$\|u\|_0 \leq (b-a)\,\|Du\|_0 \ , \forall u \in C_c^\infty(]a,b[),$$

where $\|\cdot\|_0$ denotes the $L^2$ norm, is the one dimensional precursor of the following inequality of L. Hörmander, see [4], for general linear partial differential operators with constant coefficients $P(D)$,

$$\|(\partial^\alpha P)\,(D)\,u\|_0 \leq C\,\|P\,(D)\,u\|_0 \ , \forall u \in C_c^\infty\,(\Omega)\,,$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n, \alpha \in \mathbb{N}^n$ and $C > 0$ is independent of $u$. Indeed, if $P(D) = \frac{d}{idx}$ then its complete symbol is $P(\xi) = \xi$ and $\frac{dP}{d\xi}(D)$ is the identity operator. A fundamental consequence of Hörmander's inequality is the

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$L^2-$solvability of every non null linear partial differential operators with constant complex coefficients.

The paper presents a result on local solvability, in classical Sobolev spaces, of pseudodifferential operators with constant coefficients, i.e. independent of the variable $x$, its proof is based on a new Hörmander's type inequality for pseudodifferential operators with constant coefficients. Some variants of the proposed inequality occur in the study of local solvability of pseudodifferential equations with variable coefficients, see e.g. [1], [2], [3]. The proof of the result of the paper on local solvability is more close to linear partial differential operators.

## 2. The solvability

We follow the notations and definitions of the theory of distributions and pseudodifferential operators as in [5] and [6]. Let $\Omega$ be an open set of $\mathbb{R}^n$, denote by $C_c^\infty(\Omega)$ the space of infinitely differentiable functions with compact support and by $H^s, s \in \mathbb{R}$, the Sobolev space on $\mathbb{R}^n$ with inner product and norm respectively $(\cdot,\cdot)_s$ and $\|\cdot\|_s$.

**Definition 1.** *The class $S^m, m \in \mathbb{R}$, is the space of infinitely differentiable functions P defined on $\mathbb{R}^n$ satisfying : for every multi-index $\alpha \in \mathbb{Z}_+^n$ there exists $c > 0$ such that*

$$|(\partial^\alpha P)(\xi)| \leq c(1+|\xi|)^{m-|\alpha|}, \forall \xi \in \mathbb{R}^n.$$

A pseudodifferential operator $P(D)$ of order $m$ with constant coefficients is a linear operator acting on functions $u \in H^s$ by the formula

$$P(D)u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} P(\xi)\widehat{u}(\xi)d\xi,$$

where $P \in S^m$ is called the symbol of $P(D)$, and $\widehat{u}$ denotes the Fourier transform of $u$. It holds that for every $s \in \mathbb{R}$ there exists $c > 0$ such that

$$\|P(D)u\|_s \leq c\|u\|_{s+m}, \forall u \in H^{s+m}.$$

The symbol of the pseudodifferential operators $\overline{P}(D)$ and $(\partial^\alpha P)(D)$ are respectively $\overline{P(\xi)}$ and $(\partial^\alpha P)(\xi)$. It is clear that the formal adjoint $P^*(D)$ of the operator $P(D)$ is $\overline{P}(D)$, and we have

$$\left\|\overline{P}(D)u\right\|_s = \|P(D)u\|_s, \forall u \in H^s.$$

**Remark 1.** *As we deal exclusively with local a priori estimates, and in view of the classical result on the decomposition of pseudodifferential operators, we assume that all the pseudodifferential operators are properly supported.*

Recall the following result, see [6].

**Lemma 2.1.** *Let $s,t$ be two real numbers such that $s < t$ and $t \geq -\frac{n}{2}$ , then $\forall \varepsilon > 0, \exists \rho > 0, \forall \omega$ open set of $\mathbb{R}^n, diam(\omega) < \rho, \forall u \in C_0^\infty(\omega)$,*

$$\|u\|_s \leq \varepsilon \|u\|_t .$$

The proof of the result on local solvability is based on the following Hörmander's type inequality which is proved in the next section.

**Theorem 1.** Let $P(D)$ be a pseudodifferential operator of the class $S^m, s \in \mathbb{R}, \theta \geq 1$ and $\alpha \in \mathbb{N}^n$, then $\forall \delta > 0, \exists \rho > 0, \exists c > 0, \forall u \in C_0^\infty(\Omega), diam(\Omega) < \rho$, we have

$$\|(\partial^\alpha P)(D)u\|_s \leq \delta \|P(D)u\|_s + c\|u\|_{s+m-\theta} . \qquad \text{(InApr)}$$

Now let's present the main result on local solvability.

**Theorem 2.** Let $P(D)$ be a pseudodifferential operator of the class $S^m$ such that there exist $s_0 \in \mathbb{R}, m_0 \in \mathbb{N}, c > 0, \forall \xi \in \mathbb{R}^n$,

$$\sum_{|\alpha| \leq m_0} |(\partial^\alpha P)(\xi)|^2 \geq c\left(1+|\xi|^2\right)^{s_0}, \qquad \text{(InAlg)}$$

then for every $x_0 \in \mathbb{R}^n$ and $s \leq \frac{n}{2}$, there exists a neighborhood $\omega$ of $x_0$ such that $\forall f \in H^s(\omega), \exists u \in H^{s+s_0}(\omega)$ satisfying $P(D)u = f$ .

*Proof.* From the estimate (InAlg) follows the existence of $C > 0$ such that $\forall u \in H^{s+s_0}$,

$$\|u\|_{s+s_0} \leq C \sum_{|\alpha| \leq m_0} \left\| \overline{(\partial^\alpha P)}(D)u \right\|_s$$

Let $\Omega$ be an open neighborhood of $x_0$, due to the inequality (InApr) of Theorem 1, it holds that $\forall s \in \mathbb{R}, \forall \theta \geq 1, \forall \delta > 0, \exists \rho > 0, \exists c > 0, \forall u \in C_c^\infty(\Omega), diam(\Omega) < \rho$,

$$\|u\|_{s+s_0} \leq C_{m_0}(\delta \left\| \overline{P(D)}u \right\|_s + c\|u\|_{s+m-\theta}),$$

Lemma 1 with $s \in \mathbb{R}$ and $\theta \geq 1$ such that $m - \theta < s_0$ and $s \geq -s_0 - \frac{n}{2}$, gives $\forall \varepsilon > 0$ there exists an open neighborhood $\omega$ of $x_0, \forall u \in C_c^\infty(\omega)$,

$$\|u\|_{s+m-\theta} \leq \varepsilon \|u\|_{s+s_0}$$

Consequentelly, it follows that for an open neighborhood $\omega$ of $x_0$ we have

$$\|u\|_{s+s_0} \leq C_{m_0}(\delta \left\| \overline{P(D)}u \right\|_s + \varepsilon c\|u\|_{s+s_0}), \forall u \in C_c^\infty(\omega),$$

i.e.

$$\left(1-\varepsilon c C_{m_0}\right)\|u\|_{s+s_0} \leq C_{m_0}\delta\left\|\overline{P(D)}u\right\|_s, \forall u \in C_c^\infty(\omega),$$

Taking $\varepsilon > 0$ such that $1-\varepsilon c C_{m_0} > 0$, then there exist an open neighborhood $\omega$ of $x_0$ and a constant $C > 0$ such that the following inequality holds

$$\|u\|_{s+s_0} \leq C\left\|\overline{P(D)}u\right\|_s, \forall u \in C_c^\infty(\omega).$$

Consider now for $f \in H^{-s-s_0}(\omega)$ the linear form

$$\begin{array}{llll} L: & P^*(D)\,C_c^\infty(\omega) & \to & \mathbb{C} \\ & P^*(D)\,\varphi & \to & (f,\varphi) \end{array}$$

Then

$$|(f,\varphi)| \leq \|f\|_{-s-s_0}\|\varphi\|_{s+s_0} \leq C\|f\|_{-s-s_0}\|P^*(D)\,\varphi\|_s$$

This estimate implies that the linear form $L$ can be extended to the whole Hilbert space $H_0^s(\omega) := \overline{C_c^\infty(\omega)}^{H^s}$. Consequently, by the Riesz representation Theorem, we obtain that $\forall s \in \mathbb{R}, -s-s_0 \leq \frac{n}{2}$ and $\forall f \in H^{-s-s_0}(\omega)$ there exists $u \in H^{-s}(\omega)$ such that $P(D)\,u = f$. It follows then the local solvability result of the Theorem. □

As a consequence we have the following result.

**Corollary 1.** *Let $P(D)$ be a linear partial differential operator with constant coefficients, then for every $x_0 \in \mathbb{R}^n$ and $s \leq \frac{n}{2}$ there exits an open neighborhood $\omega$ of $x_0$ such that $\forall f \in H^s(\omega), \exists u \in H^s(\omega)$ satisfying $P(D)\,u = f$.*

*Proof.* For every non null linear partial differential operator with constant coefficients $P(D)$ there exists $c > 0, \forall \xi \in \mathbb{R}^n$,

$$\sum_{|\alpha|\leq m} |(\partial^\alpha P)(\xi)|^2 \geq c,$$

where $m$ is the degree of the polynomial $P(\xi)$. Then we apply Theorem 2 with $s_0 = 0$ and $m_0 = m$. □

## 3. Proof of the inequality

The following Lemma is well known.

**Lemma 3.1.** *Let $\varphi \in C_c^\infty(\Omega)$ and $s \in \mathbb{R}$, there exists $c > 0$ such that*

$$\|\varphi u\|_s \leq \max_x |\varphi(x)|\,\|u\|_s + c\|u\|_\sigma, \forall u \in H^s,$$

*where $\sigma < s-1$.*

We recall the classical inequality $\forall s,t \in \mathbb{R}$,

$$|(u,v)_s| \leq \|u\|_{s-t}\,\|v\|_{s+t}\ ,\ \forall u \in H^{s-t}, \forall v \in H^{s+t}.$$

The open ball of center the origin and radius $\varepsilon > 0$ is denoted by $B_\varepsilon$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \varphi \leq 1, \varphi(x) = 1$ for $|x| \leq 1$, and $\varphi(x) = 0$ for $|x| > 2$. Define $\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$, and let $P(D)$ be a pseudodifferential operator of the class $S^m$, then the operator $[P(D), \varphi_\varepsilon]$ denotes the commutator of the pseudodifferential operator $P(D)$ and the operator of multiplication by the function $\varphi_\varepsilon$.

**Lemma 3.2.** *If $0 < \rho < \varepsilon$, the operator $[P(D), \varphi_\varepsilon]$ satisfies the following : for every reals $s$ and $s'$ there exists $c > 0$ such that*

$$\|[P(D), \varphi_\varepsilon]u\|_s \leq c\,\|u\|_{s'}\ ,\ \forall u \in C_0^\infty(B_\rho).$$

*Proof.* The estimate is a consequence of the fact that the pseudodifferential operator $[P(D), \varphi_\varepsilon]$ is of order $-\infty$ as its symbol is identically equals zero on a neighbourhood of $B_\rho$. □

We give the proof of Theorem 1.

*Proof.* Let $\varphi \in C_c^\infty(\mathbb{R}^n), 0 \leq \varphi \leq 1, \varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| > 2$. Define $\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon}), \varepsilon > 0$, if $0 < \rho < \varepsilon$, then we have

$$u = \varphi_\varepsilon u\ ,\ \forall u \in C_c^\infty\left(B_\rho\right).$$

Denote by $\partial_j^k$ the derivation of order $k$ with respect to the variable $\xi_j$. We have

$$P(ix_j u) = ix_j Pu + (\partial_j P)\, u, \tag{1}$$

so, $\forall u \in C_c^\infty\left(B_\rho\right)$ and $0 < \rho < \varepsilon$,

$$P(ix_j u) = ix_j \varphi_\varepsilon(x)\, Pu + (\partial_j P)\, u + T_1 u, \tag{2}$$

where

$$T_1 = ix_j[P, \varphi_\varepsilon].$$

Hence

$$\left\|(\partial_j P)\, u\right\|_s^2 = (P(ix_j u), (\partial_j P)\, u)_s - (ix_j \varphi_\varepsilon Pu, (\partial_j P)\, u)_s - (T_1 u, (\partial_j P)\, u)_s.$$

It is clear that

$$(P(ix_j u), (\partial_j P)\, u)_s = (\overline{(\partial_j P)}\,(ix_j u), \overline{P}u)_s,$$

then

$$\left\|(\partial_j P)\, u\right\|_s^2 = (\overline{(\partial_j P)}\,(ix_j u), \overline{P}u)_s - (ix_j \varphi_\varepsilon Pu, (\partial_j P)\, u)_s - (T_1 u, (\partial_j P)\, u)_s.$$

From (2), we have

$$\overline{(\partial_j P)}\,(ix_j u) = ix_j \varphi_\varepsilon \overline{(\partial_j P)} u + \overline{\left(\partial_j^2 P\right)} u + T_2 u, \tag{3}$$

where

$$T_2 = ix_j[\overline{(\partial_j P)}, \varphi_\varepsilon].$$

Consequently, we obtain the following inequality

$$\begin{aligned} \left\|(\partial_j P)\, u\right\|_s^2 \;\; \leq \;\; & \left\| ix_j \varphi_\varepsilon \overline{(\partial_j P)} u\right\|_s \|Pu\|_s + \left\|\left(\partial_j^2 P\right) u\right\|_s \|Pu\|_s + \left|\left(T_2 u, \overline{P} u\right)_s\right| + \\ & + \left\| ix_j \varphi_\varepsilon P u\right\|_s \left\|(\partial_j P)\, u\right\|_s + \left|(T_1 u, (\partial_j P)\, u)_s\right| . \end{aligned} \tag{4}$$

Lemma 2 gives

$$\begin{aligned} \left\| ix_j \varphi_\varepsilon \overline{(\partial_j P)} u\right\|_s \;\; & \leq \;\; \max_x \left| ix_j \varphi_\varepsilon (x)\right| \left\|\overline{(\partial_j P)} u\right\|_s + c_{s,\sigma}(\varepsilon) \left\|\overline{(\partial_j P)} u\right\|_\sigma , \\ & \leq \;\; 2\varepsilon \left\|(\partial_j P)\, u\right\|_s + c_{s,\sigma}(\varepsilon) \|u\|_{\sigma+m-1} \ , \end{aligned}$$

and in the same way

$$\left\| ix_j \varphi_\varepsilon P u\right\|_s \leq 2\varepsilon \|Pu\|_s + c'_{s,\sigma}(\varepsilon) \|u\|_{\sigma+m} \ .$$

For every real $t$ we have

$$\left|\left(T_2 u, \overline{P} u\right)_s\right| = \left|(P T_2 u, u)_s\right| \leq \|T_2 u\|_{s-t+m} \|u\|_{s+t} \ \ ,$$

$$\left|(T_1 u, (\partial_j P)\, u)_s\right| = \left|\left(\left(\overline{\partial_j P}\right) T_1 u, u\right)_s\right| \leq \|T_1 u\|_{s-t+m-1} \|u\|_{s+t} \ .$$

From the above inequalities it follows that

$$\begin{aligned} \left\|(\partial_j P)\, u\right\|_s^2 \;\; \leq \;\; & 4\varepsilon \|Pu\|_s \left\|(\partial_j P)\, u\right\|_s + \left\|\left(\partial_j^2 P\right) u\right\|_s \|Pu\|_s + \\ & + c'_{s,\sigma}(\varepsilon) \left\|(\partial_j P)\, u\right\|_s \|u\|_{\sigma+m} + c_{s,\sigma}(\varepsilon) \|Pu\|_s \|u\|_{\sigma+m-1} + \\ & + \|T_1 u\|_{s-t+m-1} \|u\|_{s+t} + \|T_2 u\|_{s-t+m} \|u\|_{s+t} \ . \end{aligned} \tag{5}$$

Due to the algebraic inequality $2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \forall \varepsilon > 0, \forall a, b \geq 0$, we obtain

$$\begin{aligned} \left\|(\partial_j P)\, u\right\|_s^2 \;\; \leq \;\; & 6\varepsilon \|Pu\|_s^2 + 4\varepsilon \left\|(\partial_j P)\, u\right\|_s^2 + \frac{1}{8\varepsilon} \left\|\left(\partial_j^2 P\right) u\right\|_s^2 + \\ & + \frac{\left[c_{s,\sigma}(\varepsilon)\right]^2}{8\varepsilon} \|u\|_{\sigma+m-1}^2 + \frac{\left[c'_{s,\sigma}(\varepsilon)\right]^2}{8\varepsilon} \|u\|_{\sigma+m}^2 + \|u\|_{s+t}^2 + \\ & + \frac{1}{2} \|T_1 u\|_{s-t+m-1}^2 + \frac{1}{2} \|T_2 u\|_{s-t+m}^2 \ . \end{aligned}$$

Let $\varepsilon > 0$ with $1-4\varepsilon > 0$ and $\sigma = s-\theta,\ \theta > 1$, then using Lemma 3 to the operators of infinite order $T_1$ and $T_2$ there exists a constant $c_{s,\theta}(\varepsilon) > 0$ such that

$$\left\|(\partial_j P)u\right\|_s^2 \leq \frac{6\varepsilon}{1-4\varepsilon}\left\|Pu\right\|_s^2 + \frac{1}{(1-4\varepsilon)8\varepsilon}\left\|\left(\partial_j^2 P\right)u\right\|_s^2 + c_{s,\theta}(\varepsilon)\left\|u\right\|_{s+m-\theta}^2$$

Taking $\varepsilon = \dfrac{\delta}{2(2\delta+3)}, \delta > 0$, we obtain that the following estimate holds $\forall s \in \mathbb{R}, \forall\theta > 1, \forall\delta > 0$, there exist $a(\delta) = \dfrac{(2\delta+3)^2}{12\delta} > 0$ and $b_{s,\theta}(\delta) > 0$, such that

$$\left\|(\partial_j P)u\right\|_s^2 \leq \delta\left\|Pu\right\|_s^2 + a(\delta)\left\|\left(\partial_j^2 P\right)u\right\|_s^2 + b_{s,\theta}(\delta)\left\|u\right\|_{s+m-\theta}^2\ , \qquad (6)$$

$\forall u \in C_c^\infty\left(B_\rho\right), \rho < \varepsilon(\delta) = \dfrac{\delta}{2(2\delta+3)}$ .

Now by induction we show that for arbitrary $s \in \mathbb{R}$ and $\theta > 1$ we have $\forall k \in \mathbb{N}, \forall\delta_k > 0$ there exist $a_k(\delta_k) > 0, b_k(\delta_k) > 0, \varepsilon_k(\delta_k) > 0, \forall u \in C_c^\infty\left(B_\rho\right), \rho < \varepsilon_k(\delta_k)$,

$$\left\|\left(\partial_j^k P\right)u\right\|_s^2 \leq \delta_k\left\|Pu\right\|_s^2 + a_k(\delta_k)\left\|\left(\partial_j^{k+1} P\right)u\right\|_s^2 + b_k(\delta_k)\left\|u\right\|_{s+m-\theta}^2\ , \qquad (7)$$

it is clear that only the constant $b_k(\delta_k)$ depends on $s$ and $\theta$. The case $k=1$ is true by (6), where $\delta_1 = \delta, a_1(\delta_1) = a(\delta), b_1(\delta_1) = b_{s,\theta}(\delta)$ and $\varepsilon_1(\delta_1) = \varepsilon(\delta)$. Assume the inequality true for $k$, i.e. we have (7). The inequality (6) applied to the operator $\left(\partial_j^k P\right)$ gives $\forall\delta > 0$ there exist $a_1(\delta) > 0, b_1(\delta) > 0, \varepsilon_1(\delta) > 0, \forall u \in C_c^\infty\left(B_\rho\right), \rho < \varepsilon_1(\delta)$,

$$\left\|\left(\partial_j^{k+1} P\right)u\right\|_s^2 \leq \delta\left\|\left(\partial_j^k P\right)u\right\|_s^2 + a_1(\delta)\left\|\left(\partial_j^{k+2} P\right)u\right\|_s^2 + b_1(\delta)\left\|u\right\|_{s+m-\theta}^2\ . \qquad (8)$$

Estimating $\left\|\left(\partial_j^k P\right)u\right\|_s^2$ in (8) by the inequality (7), it follows that $\forall u \in C_c^\infty\left(B_\rho\right)$, $\rho < \varepsilon \leq \min\left\{\varepsilon_1(\delta), \varepsilon_k(\delta_k)\right\}$,

$$\begin{aligned}\left\|\left(\partial_j^{k+1} P\right)u\right\|_s^2 \leq\ & \delta\delta_k\left\|Pu\right\|_s^2 + \delta a_k(\delta_k)\left\|\left(\partial_j^{k+1} P\right)u\right\|_s^2 + \\ & +\delta b_k(\delta_k)\left\|u\right\|_{s+m-\theta}^2 + a_1(\delta)\left\|\left(\partial_j^{k+2} P\right)u\right\|_s^2 + \\ & +b_1(\delta)\left\|u\right\|_{s+m-\theta}^2\ .\end{aligned}$$

Then for $\delta < \dfrac{1}{a_k(\delta_k)}$ we have that $\forall u \in C_c^\infty(B_\rho), \rho < \min\{\varepsilon_1(\delta), \varepsilon_k(\delta_k)\}$,

$$\begin{aligned}\left\|\left(\partial_j^k P\right)u\right\|_s^2 &\leq \frac{\delta\delta_k}{1-\delta a_k(\delta_k)}\|Pu\|_s^2 + \frac{a_1(\delta)}{1-\delta a_k(\delta_k)}\left\|\left(\partial_j^{k+2}P\right)u\right\|_s^2 + \\ &+\frac{b_1(\delta)+\delta b_k(\delta_k)}{1-\delta b_k(\delta_k)}\|u\|_{s+m-\theta}^2 \quad .\end{aligned}$$

Let $\delta_{k+1} > 0$ and taking

$$\begin{aligned}\delta &= \frac{\delta_{k+1}}{\delta_k + \delta_{k+1}a_k(\delta_k)}, \\ a_{k+1}(\delta_{k+1}) &= \frac{a_1(\delta)}{1-\delta a_k(\delta_k)}, \\ b_{k+1}(\delta_{k+1}) &= \frac{b_1(\delta)+\delta b_k(\delta_k)}{1-\delta a_k(\delta_k)},\end{aligned}$$

it holds that

$$\begin{aligned}\left\|\left(\partial_j^{k+1}P\right)u\right\|_s^2 \leq \delta_{k+1}\|Pu\|_s^2 + a_{k+1}(\delta_{k+1})\left\|\left(\partial_j^{k+2}P\right)u\right\|_s^2 + \\ +b_{k+1}(\delta_{k+1})\|u\|_{s+m-\theta}^2,\end{aligned}$$

$\forall u \in C_c^\infty(B_\rho), \rho < \varepsilon_{k+1}(\delta_{k+1}) = \min\left\{\varepsilon_1\left(\dfrac{\delta_{k+1}}{\delta_k+\delta_{k+1}a_k(\delta_k)}\right), \varepsilon_k(\delta_k)\right\}$. Consequently, the inequality (7) is proved.

Let $\delta_k, a_k(\delta_k), b_k(\delta_k)$ and $\varepsilon_k(\delta_k), k \in \mathbb{N}$, be the respective constants of the right member of the estimate (7). By iteration in the inequality (6), we obtain $\forall u \in C_c^\infty(B_\rho), \rho < \varepsilon \leq \min\{\varepsilon_1(\delta_1), ..., \varepsilon_k(\delta_k)\}, k \geq 2$, we have

$$\begin{aligned}\left\|(\partial_j P)u\right\|_s^2 &\leq (\delta_1 + a_1(\delta_1)\delta_2 + ... + a_1(\delta_1)...a_{k-1}(\delta_{k-1})\delta_k)\|Pu\|_s^2 + \\ &+a_1(\delta_1)....a_k(\delta_k)\left\|\left(\partial_j^{k+1}P\right)u\right\|_s^2 + \\ &+(b_1(\delta_1)+...+a_1(\delta_1)...a_{k-1}(\delta_{k-1})b_k(\delta_k))\|u\|_{s+m-\theta}^2\end{aligned} \tag{9}$$

Let $\delta > 0$ and take $\delta_1, ..., \delta_k$ as follows

$$\delta_1 = \frac{\delta}{k}, a_1(\delta_1)\delta_2 = \frac{\delta}{k}, ..., a_1(\delta_1)...a_{k-1}(\delta_{k-1})\delta_k = \frac{\delta}{k},$$

then define the constants $\widetilde{a}_k(\delta)$ and $\widetilde{b}_k(\delta)$ as respectively the coefficients of $\left\|\left(\partial_j^{k+1}P\right)u\right\|_s^2$ and $\|u\|_{s+m-\theta}^2$ in the inequality (9). Consequently, it holds that

$\forall s \in \mathbb{R}, \forall \theta > 1, \forall k \in \mathbb{N}, \forall \delta > 0$, there exist $\widetilde{a}_k(\delta) > 0, \widetilde{b}_k(\delta) > 0, \widetilde{\varepsilon}_k(\delta) > 0, \forall u \in C_c^\infty(B_\rho)$, $\rho < \varepsilon \leq \widetilde{\varepsilon}_k(\delta) = \min\{\varepsilon_1(\delta), ..., \varepsilon_k(\delta)\}$, such that

$$\left\|(\partial_j P) u\right\|_s^2 \leq \delta \|Pu\|_s^2 + \widetilde{a}_k(\delta) \left\|\left(\partial_j^{k+1} P\right) u\right\|_s^2 + \widetilde{b}_k(\delta) \|u\|_{s+m-\theta}^2 \quad . \tag{10}$$

Choose $k \in \mathbb{N}$ with $k \geq \theta - 1$, then there is $c_{s,\theta}(\delta) > 0, \widetilde{\varepsilon}(\delta) > 0, \forall u \in C_c^\infty(B_\rho)$, $\rho < \widetilde{\varepsilon}(\delta)$,

$$\left\|(\partial_j P) u\right\|_s^2 \leq \delta \|Pu\|_s^2 + c_{s,\theta}(\delta) \|u\|_{s+m-\theta}^2 \quad . \tag{11}$$

Remark that the inequality (11) is also true for $\theta = 1$, as the operator $(\partial_j P)$ is of order $m-1$. Finally, we have proved that $\forall s \in \mathbb{R}, \forall \theta \geq 1, \forall \delta > 0, \exists \rho > 0, \exists c > 0, \forall u \in C_c^\infty(\Omega), diam(\Omega) < \rho$,

$$\left\|(\partial_j P) u\right\|_s^2 \leq \delta \|Pu\|_s^2 + c \|u\|_{s+m-\theta}^2 \quad . \tag{12}$$

Let $\alpha = (\alpha_1, ..., \alpha_n)$ and $\alpha' = (\alpha_1, ..., \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1} ..., \alpha_n)$ be two given multi-indices. Assume that the hypothesis of induction : $\forall s \in \mathbb{R}, \forall \theta \geq 1, \forall \delta > 0, \exists \rho > 0, \exists c > 0, \forall u \in C_c^\infty(\Omega), diam(\Omega) < \rho$,

$$\|(\partial^\alpha P) u\|_s^2 \leq \delta \|Pu\|_s^2 + c \|u\|_{s+m-\theta}^2 \quad , \tag{13}$$

is true and apply the inequality (12) to the operator $(\partial^\alpha P)$, then we obtain $\forall \delta', \exists c' > 0, \forall u \in C_c^\infty(\Omega')$,

$$\left\|\left(\partial^{\alpha'} P\right) u\right\|_s^2 \leq \delta' \|(\partial^\alpha P) u\|_s^2 + c' \|u\|_{s+m-\theta} \ ,$$

where $\Omega'$ depends on $\delta'$. From the hypothesis of induction for $(\partial^\alpha P)$, we have for every $\delta > 0$ there is $\rho > 0$ such that

$$\left\|\left(\partial_j^{\alpha'} P\right) u\right\|_s^2 \leq \delta' \delta \|Pu\|_s^2 + \delta' c \|u\|_{s+m-\theta}^2 + c' \|u\|_{s+m-\theta}^2 \quad ,$$

$\forall u \in C_c^\infty(\Omega \cap \Omega'), diam(\Omega) < \rho$. Take $\delta' = 1$ we obtain then the inequality (13) for $\alpha'$. This ends the proof of the Theorem 1. □

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