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EQUILIBRATION IN A TWO-SPECIES–TWO-CHEMICALS CHEMOTAXIS-COMPETITION SYSTEM

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This paper is concerned with stabilization in the two-species–two-chemicals chemotaxis-competition system

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u(1 - u - a_1 w) & \text{in } \Omega \times (0,\infty), \\ 0 = \Delta v - v + w & \text{in } \Omega \times (0,\infty), \\ w_t = \Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w(1 - a_2 u - w) & \text{in } \Omega \times (0,\infty), \\ 0 = \Delta z - z + u & \text{in } \Omega \times (0,\infty), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with smooth boundary, χ_1, χ_2 and μ_1, μ_2 are constants satisfying some conditions. About this system Tu–Mu–Zheng–Lin (Discrete Contin. Dyn. Syst.;2018;38;3617–3636) showed global existence and stabilization of solutions under some smallness conditions for χ_1 and χ_2. Here energy arguments for seeing stabilization in the previous work were based on ideas in Bai–Winkler (Indiana Univ. Math. J.;2016;65;553–583); however, these ideas were recently improved by the first author (Discrete Contin. Dyn. Syst. Ser. S;2020;13;269–278), which implies that the result about stabilization in the previous work seems not to be the best. This paper gives an improvement of conditions for stabilization in the previous work. The feature of the proof is to use the Sylvester criterion in deriving energy estimates.

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1. Introduction and results

In this paper we study a two-species–two-chemicals chemotaxis-competition system. More precisely, we consider the parabolic–elliptic–parabolic–elliptic system which represents the situation that two competitive biological species diffuse randomly and move toward higher concentration of the chemical substance produced by the other species:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla\cdot(u\nabla v) + \mu_1 u(1-u-a_1 w) & \text{in } \Omega\times(0,\infty),\\ 0 = \Delta v - v + w & \text{in } \Omega\times(0,\infty),\\ w_t = \Delta w - \chi_2 \nabla\cdot(w\nabla z) + \mu_2 w(1-a_2 u-w) & \text{in } \Omega\times(0,\infty),\\ 0 = \Delta z - z + u & \text{in } \Omega\times(0,\infty),\\ \nabla u\cdot\nu = \nabla v\cdot\nu = \nabla w\cdot\nu = \nabla z\cdot\nu = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x),\ w(x,0)=w_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ $(n\geq 2)$ with smooth boundary $\partial\Omega$ and ν is the outward normal vector to $\partial\Omega$; $a_1,a_2\geq 0$, $\chi_1,\chi_2,\mu_1,\mu_2>0$ are constants; u_0,w_0 are initial data. In this setting the unknown functions $u(x,t)$ and $w(x,t)$ show the population densities of the two species; the unknown functions $v(x,t)$ and $z(x,t)$ represent the concentrations of the chemical substances produced by $w(x,t)$ and $u(x,t)$, respectively, at place x and time t. Here we note that each species moves according to the chemical stimulus produced by not itself but the other species. Moreover, we assume that these species have the Lotka–Volterra competitive kinetics.

An important theme in mathematical studies of the problem (1.1) is to show global existence and large time behavior. One of the mathematical difficulties of the problem (1.1) is to deal with the chemotaxis term (e.g., $\chi_1\nabla\cdot(u\nabla v)$) and the competition term (e.g., $\mu_1 u(1-u-a_1 w)$). In order to explain our purpose we introduce known results about the problem (1.1). In this case Tao–Winkler [12] studied boundedness vs. blow-up in the problem (1.1) with $\mu_1=\mu_2=0$ under some conditions for the initial data. In the case that $\mu_1\neq 0$ and $\mu_2\neq 0$, Zhang–Liu–Yang [18] showed global existence and boundedness under the condition that $\chi_1\chi_2<\mu_1\mu_2$, and derived stabilization under the additional conditions that $a_1,a_2\in(0,1)$, $\chi_1<a_1\mu_1$, $\chi_2<a_2\mu_2$; Tu–Mu–Zheng–Lin [14] established global existence under some conditions which partially improve those in [18]. Moreover, they showed stabilization and its convergence rate under some smallness conditions for $\frac{\chi_1^2}{\mu_1},\frac{\chi_2^2}{\mu_2}$. More related works which dealt with a two-species–one-chemical chemotaxis-competition system were in [1, 2, 5–11, 13, 15–17]; especially, Bai–Winkler [1] established energy arguments for obtaining stabilization, and these arguments were improved in [8, 10]. Here a

strategy for stabilization in [14] is based on the energy arguments in [1]; therefore, we could expect to improve the conditions for stabilization in [14] by applying ideas in [8, 10]. Thus we can come across the following natural question:

$$\text{\textit{What are the best conditions for stabilization in the case that } } \chi_1, \chi_2 \text{\textit{ are constants?}} \tag{Q}$$

The purpose of this paper is to improve the conditions for stabilization in [14].

Now the main theorems read as follows. The first one is concerned with asymptotic stability in (1.1) in the case that $a_1, a_2 \in (0,1)$.

Theorem 1.1. *Let $a_1, a_2 \in (0,1)$ and assume that $X := \frac{\mu_1}{\chi_1^2}$, $Y := \frac{\mu_2}{\chi_2^2}$ satisfy that*

$$D := \frac{XY}{256}\Bigg(256(1-a_1a_2)XY - (a_1a_2+2)u^*w^* - 16(a_1^2 w^* X + a_2^2 u^* Y) + \frac{{u^*}^2 {w^*}^2}{256XY}\Bigg) > 0 \tag{1.2}$$

and that

$$\left(8(2-a_1a_2)X - \frac{a_2^2 u^*}{2}\right)Y + 16\sqrt{D} > \frac{u^* w^*}{16}, \tag{1.3}$$

where

$$u^* := \frac{1-a_1}{1-a_1a_2}, \quad w^* := \frac{1-a_2}{1-a_1a_2}.$$

If (u,v,w,z) is a global classical solution of (1.1) and satisfies

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|v\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|w\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|z\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \le K$$

for all $t \ge 1$ with some $\theta \in (0,1)$ and $K > 0$, then there exist $C > 0$ and $\ell > 0$ such that

$$\|u(\cdot,t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot,t) - w^*\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w^*\|_{L^\infty(\Omega)} + \|z(\cdot,t) - u^*\|_{L^\infty(\Omega)} \le Ce^{-\ell t}$$

for all $t > 0$.

The second theorem derives asymptotic stability in (1.1) in the case that $a_1 \ge 1 > a_2 > 0$.

Theorem 1.2. *Let $a_1 \geq 1 > a_2 > 0$ and assume that*

$$Y := \frac{\mu_2}{\chi_2^2} > \frac{1}{16(1-a_2)}. \tag{1.4}$$

If (u,v,w,z) is a global classical solution of (1.1) and satisfies

$$\begin{aligned}\|u\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} &+ \|v\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \\ &+ \|w\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|z\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \leq K\end{aligned}$$

for all $t \geq 1$ with some $\theta \in (0,1)$ and $K > 0$, then the following properties hold:

(i) *If $a_1 > 1$, then there exist $C > 0$ and $\ell > 0$ satisfying*

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)} &+ \|v(\cdot,t)-1\|_{L^\infty(\Omega)} \\ &+ \|w(\cdot,t)-1\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \leq Ce^{-\ell t}\end{aligned}$$

for all $t > 0$.

(ii) *If $a_1 = 1$, then there exist $C > 0$ and $\ell > 0$ satisfying*

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)} &+ \|v(\cdot,t)-1\|_{L^\infty(\Omega)} \\ &+ \|w(\cdot,t)-1\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \leq C(t+1)^{-\ell}\end{aligned}$$

for all $t > 0$.

Remark 1.3. Suppose that $a_1 \geq 1 > a_2$. Then, noting that $a_2(1-a_1a_2) < 1-a_2$, we derive that the condition (1.4) is better than that assumed in [14]:

$$\frac{\mu_2}{\chi_2^2} > \frac{a_1}{16a_2(1-\ell)} \quad (\ell > a_1a_2).$$

On the other hand, in the case that $a_1, a_2 \in (0,1)$, we can also confirm that the conditions are better than those in [14]: there are $k_1, k_2 \in (0,1)$ such that

$$k_1k_2 > a_1a_2, \quad \frac{\mu_1}{\chi_1^2} > \frac{a_2u^*}{16a_1(1-k_2)}, \quad \frac{\mu_2}{\chi_2^2} > \frac{a_1w^*}{16a_2(1-k_1)}. \tag{1.5}$$

Indeed, although the conditions (1.2) and (1.3) in Theorem 1.1 cannot be explicitly compared with the condition (1.5) in [14], the viewpoint of methods in construction of energy functions tells us to improve the conditions; moreover, by using a numerical analysis, we can see that the conditions (1.2) and (1.3) improve the condition (1.5) (see Figure 1 below).

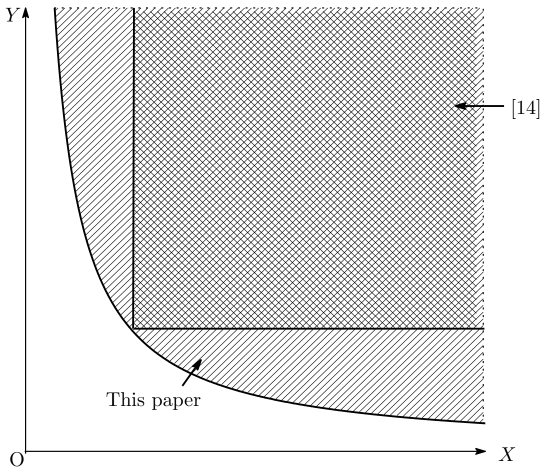

Figure 1

Remark 1.4. This result shows stabilization in (1.1) under the conditions that $X = \frac{\mu_1}{\chi_1^2}$ and $Y = \frac{\mu_2}{\chi_2^2}$ are large and that there is a global classical solution of (1.1). Here global existence of solutions to (1.1) is shown in [14] under the conditions that

$$\frac{\chi_1}{\mu_1} < \frac{a_1 n}{n-2} \quad \text{and} \quad \frac{\chi_2}{\mu_2} < \frac{a_2 n}{n-2} \tag{1.6}$$

hold. Thus, in view of Remark 1.3, these results exactly improve that in [14].

The proofs of Theorems 1.1 and 1.2 are based on a new method which is a combination of those in [10, 14]. One of the keys for the proof of Theorem 1.1 is to derive the following inequality:

$$\begin{aligned}
&\frac{d}{dt}E_1(t) \\
&\le -\varepsilon_0 \int_\Omega \left[(u(\cdot,t)-u^*)^2 + (v(\cdot,t)-w^*)^2 + (w(\cdot,t)-w^*)^2 + (z(\cdot,t)-u^*)^2 \right]
\end{aligned}$$

for all $t > 0$ and some constant $\varepsilon_0 > 0$, where

$$\begin{aligned}
E_1(t) := a_2\mu_2 \int_\Omega &\left(u(\cdot,t) - u^* - u^* \log \frac{u(\cdot,t)}{u^*} \right) \\
&+ a_1\mu_1\delta_1 \int_\Omega \left(w(\cdot,t) - w^* - w^* \log \frac{w(\cdot,t)}{w^*} \right)
\end{aligned}$$

and $(u^*, w^*, w^*, u^*) \in \mathbb{R}^4$ is a solution of (1.1). Here an energy function used in the proof of [14, Theorem 1.2] is defined as the function E_1 specialized by setting $\delta_1 = 1$, whereas the optimization by δ_1 enables us to improve conditions for χ_1, χ_2. In Section 3 we use the Sylvester criterion to prove Theorem 1.1.

This paper is organized as follows. In Section 2 we provide some lemmas which will be used later. Sections 3 and 4 are devoted to the proof of asymptotic stability (Theorems 1.1 and 1.2).

2. Preliminaries

In the following we assume that (u,v,w,z) is a classical solution of (1.1) such that

$$\|u\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|v\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|w\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} + \|z\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[t,t+1])} \le K$$

for all $t \ge 1$ with some $\theta \in (0,1)$ and $K > 0$, and will establish convergence of the solution as $t \to \infty$ in the cases that $a_1, a_2 \in (0,1)$ and $a_1 \ge 1 > a_2 > 0$. We first recall an important lemma for the proofs of Theorems 1.1 and 1.2 (see [4, Lemma 4.6]).

Lemma 2.1. *Let $f \in C^0(\overline{\Omega} \times [0,\infty))$ satisfy*

$$\|f\|_{C^{\theta^*,\frac{\theta^*}{2}}(\overline{\Omega}\times[t,t+1])} \le C^*$$

for all $t \ge 1$ with some $C^* > 0$ and $\theta^* > 0$, and assume that there exists a constant $F^* > 0$ such that

$$\int_0^\infty \int_\Omega (f(x,t) - F^*)^2 \, dxdt < \infty.$$

Then

$$\|f(\cdot,t) - F^*\|_{L^\infty(\Omega)} \to 0$$

as $t \to \infty$.

In order to upgrade the L^2-convergence rate to the L^∞-convergence rate we next give the following lemma.

Lemma 2.2. *Let $(\overline{u},\overline{w},\overline{w},\overline{u}) \in \mathbb{R}^4$ be a solution of (1.1), and assume that there exists a function $h : [0,\infty) \to \mathbb{R}$ such that*

$$\|u(\cdot,t) - \overline{u}\|_{L^2(\Omega)} + \|w(\cdot,t) - \overline{w}\|_{L^2(\Omega)} \le h(t)$$

for all $t > 0$. Then there exists a constant $C > 0$ such that

$$\|u(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} + \|v(\cdot,t) - \overline{w}\|_{L^\infty(\Omega)} + \|w(\cdot,t) - \overline{w}\|_{L^\infty(\Omega)} + \|z(\cdot,t) - \overline{u}\|_{L^\infty(\Omega)} \le C(h(t-1))^{\frac{1}{n+1}}$$

for all $t \ge 1$.

Proof. We first verify that

$$\begin{aligned}
&\|u(\cdot,t)-\overline{u}\|_{L^p(\Omega)}+\|w(\cdot,t)-\overline{w}\|_{L^p(\Omega)}\\
&\le \|u(\cdot,t)-\overline{u}\|_{L^\infty(\Omega)}^{1-\frac{2}{p}}\|u(\cdot,t)-\overline{u}\|_{L^2(\Omega)}^{\frac{2}{p}}+\|w(\cdot,t)-\overline{w}\|_{L^\infty(\Omega)}^{1-\frac{2}{p}}\|w(\cdot,t)-\overline{w}\|_{L^2(\Omega)}^{\frac{2}{p}}\\
&\le C_1^{1-\frac{2}{p}}(h(t))^{\frac{2}{p}}
\end{aligned}$$

for all $t>0$ with some $C_1>0$, where we have used the boundedness of u and w. Therefore, since the function v satisfies

$$-\Delta(v-\overline{w})+(v-\overline{w})=w-\overline{w} \quad \text{in } \Omega\times(0,\infty), \tag{2.1}$$

an elliptic regularity argument (see [3, Theorem 9.26]) derives that

$$\begin{aligned}
\|v(\cdot,t)-\overline{w}\|_{W^{2,2n+2}(\Omega)} &\le C_{\mathrm{E}}\|w(\cdot,t)-\overline{w}\|_{L^{2n+2}(\Omega)}\\
&\le C_{\mathrm{E}}C_1^{\frac{n}{n+1}}(h(t))^{\frac{1}{n+1}}
\end{aligned}$$

for all $t>0$ with some $C_{\mathrm{E}}>0$. In particular, we obtain

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^{2n+2}(\Omega)} &= \|\nabla(v(\cdot,t)-\overline{w})\|_{L^{2n+2}(\Omega)}\\
&\le \|v(\cdot,t)-\overline{w}\|_{W^{2,2n+2}(\Omega)}\\
&\le C_{\mathrm{E}}C_1^{\frac{n}{n+1}}(h(t))^{\frac{1}{n+1}}
\end{aligned}$$

for all $t>0$. Similarly we see that

$$\|\nabla z(\cdot,t)\|_{L^{2n+2}(\Omega)} \le C_{\mathrm{E}}C_1^{\frac{n}{n+1}}(h(t))^{\frac{1}{n+1}}$$

for all $t>0$. Now, applying the variation of constants formula to the first and third equations in (1.1), we obtain

$$\begin{aligned}
u(\cdot,t)-\overline{u}=e^{\Delta}(u(\cdot,t-1)-\overline{u})&\\
&-\chi_1\int_{t-1}^{t}e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\,ds\\
&+\mu_1\int_{t-1}^{t}e^{(t-s)\Delta}u(\cdot,s)(1-u(\cdot,s)-a_1w(\cdot,s))\,ds
\end{aligned}$$

and

$$\begin{aligned}
w(\cdot,t)-\overline{w}=e^{\Delta}(w(\cdot,t-1)-\overline{w})&\\
&-\chi_2\int_{t-1}^{t}e^{(t-s)\Delta}\nabla\cdot(w(\cdot,s)\nabla z(\cdot,s))\,ds\\
&+\mu_2\int_{t-1}^{t}e^{(t-s)\Delta}w(\cdot,s)(1-a_2u(\cdot,s)-w(\cdot,s))\,ds
\end{aligned}$$

for all $t \geq 1$. Then, using an argument similar to that in the proof of [1, Lemma 3.6], we infer that

$$\|u(\cdot,t)-\overline{u}\|_{L^\infty(\Omega)}+\|w(\cdot,t)-\overline{w}\|_{L^\infty(\Omega)} \leq C_2(h(t-1))^{\frac{1}{n+1}} \tag{2.2}$$

for all $t \geq 1$ with some $C_2 > 0$. Therefore, since the function $v-\overline{w}$ satisfies (2.1), the maximum principle for elliptic equations enables us to find some $C_3 > 0$ such that

$$\begin{aligned}\|v(\cdot,t)-\overline{w}\|_{L^\infty(\Omega)} &\leq C_3\|w(\cdot,t)-\overline{w}\|_{L^\infty(\Omega)}\\ &\leq C_3C_2(h(t-1))^{\frac{1}{n+1}}\end{aligned} \tag{2.3}$$

for all $t \geq 1$. Similarly we can have that

$$\|z(\cdot,t)-\overline{u}\|_{L^\infty(\Omega)} \leq C_4(h(t-1))^{\frac{1}{n+1}} \tag{2.4}$$

for all $t \geq 1$ with some $C_4 > 0$. Collecting (2.2), (2.3) and (2.4) yields the conclusion of this lemma. □

For the sake of convenience in the next two sections we present a lemma which is nothing but a special case of the Sylvester criterion for quadratic forms. The lemma is stated as follows and proved by a similar argument as in the proof of [10, Lemma 2.1].

Lemma 2.3. *Let $a,b,c,d,e,f,g,h,i,j \in \mathbb{R}$.*

(I) *Suppose that*

$$\begin{cases} a > 0, \\ ae-\dfrac{b^2}{4} > 0, \\ aeh-\dfrac{af^2}{4}-\dfrac{b^2h}{4}+\dfrac{bcf}{4}-\dfrac{c^2e}{4} > 0, \\ aehj-\dfrac{aei^2}{4}-\dfrac{af^2j}{4}+\dfrac{afgi}{4}-\dfrac{ag^2h}{4}-\dfrac{b^2hj}{4} \\ \quad +\dfrac{b^2i^2}{16}+\dfrac{bcfj}{4}-\dfrac{bdfi}{8}-\dfrac{bcgi}{8}+\dfrac{bdgh}{4} \\ \quad -\dfrac{c^2ej}{4}+\dfrac{cdei}{4}+\dfrac{c^2g^2}{16}-\dfrac{cdfg}{8}-\dfrac{d^2eh}{4}+\dfrac{d^2f^2}{16} > 0. \end{cases} \tag{2.5}$$

Then there exists $\varepsilon_0 > 0$ such that

$$\begin{aligned}&ax^2+bxy+cxz+dxw+ey^2+fyz+gyw+hz^2+izw+jw^2\\ &\geq \varepsilon_0(x^2+y^2+z^2+w^2)\end{aligned}$$

holds for all $x,y,z,w \in \mathbb{R}$.

(II) *Suppose that*

$$\begin{cases} a > 0, \\ ad - \dfrac{b^2}{4} > 0, \\ adf - \dfrac{ae^2}{4} - \dfrac{b^2 f}{4} + \dfrac{bce}{4} - \dfrac{c^2 d}{4} > 0. \end{cases} \tag{2.6}$$

Then there exists $\varepsilon_0' > 0$ *such that*

$$ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \geq \varepsilon_0'(x^2 + y^2 + z^2)$$

holds for all $x, y, z \in \mathbb{R}$.

3. Convergence. Case I: $a_1, a_2 \in (0,1)$

In this section we derive stabilization in the case that $a_1, a_2 \in (0,1)$. We first verify the following lemma which is utilized to confirm that the assumption of Lemma 2.1 is satisfied.

Lemma 3.1. *Let* $a_1, a_2 \in (0,1)$. *Assume that* (1.2) *and* (1.3) *are satisfied. Then there exists* $\delta_1 > 0$ *such that the function* $E_1 : (0,\infty) \to \mathbb{R}$ *defined as*

$$E_1 := a_2\mu_2 \int_\Omega \left(u - u^* - u^* \log \frac{u}{u^*}\right) + a_1\mu_1\delta_1 \int_\Omega \left(w - w^* - w^* \log \frac{w}{w^*}\right) \tag{3.1}$$

satisfies

$$\frac{d}{dt}E_1(t) \leq -\varepsilon_0 \int_\Omega \left[(u(\cdot,t) - u^*)^2 + (v(\cdot,t) - w^*)^2 + (w(\cdot,t) - w^*)^2 + (z(\cdot,t) - u^*)^2\right] \tag{3.2}$$

for all $t > 0$ *with some* $\varepsilon_0 > 0$, *where*

$$u^* = \frac{1 - a_1}{1 - a_1 a_2}, \quad w^* = \frac{1 - a_2}{1 - a_1 a_2}.$$

Proof. We shall show that there is a constant $\delta > 0$ such that

$$\delta > \frac{a_2 u^*}{16 a_1 X} \tag{3.3}$$

and

$$G(\delta) := X\left(\frac{a_1a_2}{4}Y + \frac{a_1w^*}{16a_2}\right)\delta^2 - \left(\frac{1}{2}(2-a_1a_2)XY + \frac{u^*w^*}{256}\right)\delta$$
$$+ \frac{a_2u^*}{16a_1}Y + \frac{a_1a_2}{4}XY < 0, \tag{3.4}$$

where $X = \frac{\mu_1}{\chi_1^2}$, $Y = \frac{\mu_2}{\chi_2^2}$. Thanks to the condition (1.2), the discriminant D of $G(\delta)$ is positive:

$$\begin{aligned}
D &:= \left(\frac{1}{2}(2-a_1a_2)XY + \frac{u^*w^*}{256}\right)^2 \\
&\quad - 4X\left(\frac{a_1a_2}{4}Y + \frac{a_1w^*}{16a_2}\right)\left(\frac{a_2u^*}{16a_1}Y + \frac{a_1a_2}{4}XY\right) \\
&= \left(1 - a_1a_2 + \frac{a_1^2a_2^2}{4}\right)X^2Y^2 + \frac{(2-a_1a_2)u^*w^*}{256}XY + \frac{u^{*2}w^{*2}}{256^2} \\
&\quad - \left(\frac{a_1^2a_2^2}{4}XY + \frac{a_2^2u^*}{16}Y + \frac{a_1^2w^*}{16}X + \frac{u^*w^*}{64}\right)XY \\
&= \frac{XY}{256}\Bigg(256(1-a_1a_2)XY - (a_1a_2+2)u^*w^* \\
&\qquad\qquad - 16(a_1^2w^*X + a_2^2u^*Y) + \frac{u^{*2}w^{*2}}{256XY}\Bigg) \\
&> 0.
\end{aligned}$$

Hence two solutions of the equation $G(\delta) = 0$ are given by

$$\delta_\pm := \frac{a_2\left(8(2-a_1a_2)XY + \frac{u^*w^*}{16} \pm 16\sqrt{D}\right)}{2a_1X(4a_2^2Y + w^*)}$$

and (3.4) holds for all $\delta \in (\delta_-, \delta_+)$. Here we can take $\delta = \delta_1$ satisfying not only (3.4) but also (3.3) if

$$\delta_+ = \frac{a_2\left(8(2-a_1a_2)XY + \frac{u^*w^*}{16} + 16\sqrt{D}\right)}{2a_1(4a_2^2Y + w^*)X} > \frac{a_2u^*}{16a_1X}. \tag{3.5}$$

This is verified by the condition (1.3). Indeed, adding $\frac{a_2^2u^*}{2}Y + \frac{u^*w^*}{16}$ to the both sides of (1.3), we infer

$$8(2-a_1a_2)XY + \frac{u^*w^*}{16} + 16\sqrt{D} > \frac{u^*(4a_2^2Y + w^*)}{8},$$

which multiplied by $\frac{a_2}{2a_1(4a_2^2Y+w^*)X}$ derives (3.5). Therefore we can choose $\delta_1 > 0$ such that $\max\left\{\delta_-, \frac{a_2u^*}{16a_1X}\right\} < \delta_1 < \delta_+$ and both (3.3) and (3.4) hold with $\delta = \delta_1$. By the first and third equations in (1.1) we have that the function E_1 defined as (3.1) satisfies

$$\begin{aligned}
\frac{d}{dt}E_1(t) &= a_2\mu_2\int_\Omega\left(u_t(\cdot,t) - \frac{u^*u_t(\cdot,t)}{u(\cdot,t)}\right) + a_1\mu_1\delta_1\int_\Omega\left(w_t(\cdot,t) - \frac{w^*w_t(\cdot,t)}{w(\cdot,t)}\right)\\
&= a_2\mu_1\mu_2\int_\Omega (u(\cdot,t)-u^*)(1-u(\cdot,t)-a_1w(\cdot,t))\\
&\quad + a_1\mu_1\mu_2\delta_1\int_\Omega (w(\cdot,t)-w^*)(1-a_2u(\cdot,t)-w(\cdot,t))\\
&\quad - a_2\mu_2u^*\int_\Omega \frac{|\nabla u(\cdot,t)|^2}{u^2(\cdot,t)} + a_2\mu_2u^*\chi_1\int_\Omega \frac{\nabla u(\cdot,t)\cdot\nabla v(\cdot,t)}{u(\cdot,t)}\\
&\quad - a_1\mu_1w^*\delta_1\int_\Omega \frac{|\nabla w(\cdot,t)|^2}{w^2(\cdot,t)} + a_1\mu_1w^*\chi_2\delta_1\int_\Omega \frac{\nabla w(\cdot,t)\cdot\nabla z(\cdot,t)}{w(\cdot,t)}.
\end{aligned} \tag{3.6}$$

Here, recalling that $u^* = \frac{1-a_1}{1-a_1a_2}$ and $w^* = \frac{1-a_2}{1-a_1a_2}$, and noting that $1-u^* = 1-\frac{1-a_1}{1-a_1a_2} = \frac{a_1(1-a_2)}{1-a_1a_2} = a_1w^*$ and $1-w^* = a_2u^*$, we have that the first and second terms of (3.6) are rewritten as

$$\begin{aligned}
(u-u^*)(1-u-a_1w) &= -(u-u^*)^2 + (u-u^*)(1-u^*-a_1w)\\
&= -(u-u^*)^2 - a_1(u-u^*)(w-w^*)
\end{aligned}$$

and

$$(w-w^*)(1-a_2u-w) = -(w-w^*)^2 - a_2(w-w^*)(u-u^*),$$

which together with (3.6) imply that

$$\begin{aligned}
\frac{d}{dt}E_1(t) &= -a_2\mu_1\mu_2\int_\Omega (u(\cdot,t)-u^*)^2 - a_1\mu_1\mu_2\delta_1\int_\Omega (w(\cdot,t)-w^*)^2\\
&\quad - a_1a_2\mu_1\mu_2(1+\delta_1)\int_\Omega (u(\cdot,t)-u^*)(w(\cdot,t)-w^*)\\
&\quad - a_2\mu_2u^*\int_\Omega \frac{|\nabla u(\cdot,t)|^2}{u^2(\cdot,t)} + a_2\mu_2u^*\chi_1\int_\Omega \frac{\nabla u(\cdot,t)\cdot\nabla v(\cdot,t)}{u(\cdot,t)}\\
&\quad - a_1\mu_1w^*\delta_1\int_\Omega \frac{|\nabla w(\cdot,t)|^2}{w^2(\cdot,t)} + a_1\mu_1w^*\chi_2\delta_1\int_\Omega \frac{\nabla w(\cdot,t)\cdot\nabla z(\cdot,t)}{w(\cdot,t)}.
\end{aligned} \tag{3.7}$$

Thus, noting from the Young inequality that

$$\chi_1 \int_\Omega \frac{\nabla u \cdot \nabla v}{u} \leq \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{\chi_1^2}{4} \int_\Omega |\nabla v|^2$$

and

$$\chi_2 \int_\Omega \frac{\nabla w \cdot \nabla z}{w} \leq \int_\Omega \frac{|\nabla w|^2}{w^2} + \frac{\chi_2^2}{4} \int_\Omega |\nabla z|^2,$$

we establish from (3.7) that

$$\begin{aligned}\frac{d}{dt}E_1(t) \leq &-a_2\mu_1\mu_2 \int_\Omega (u(\cdot,t)-u^*)^2 - a_1\mu_1\mu_2\delta_1 \int_\Omega (w(\cdot,t)-w^*)^2 \\ &- a_1a_2\mu_1\mu_2(1+\delta_1) \int_\Omega (u(\cdot,t)-u^*)(w(\cdot,t)-w^*) \\ &+ \frac{a_2\mu_2 u^* \chi_1^2}{4} \int_\Omega |\nabla v(\cdot,t)|^2 + \frac{a_1\mu_1 w^* \chi_2^2 \delta_1}{4} \int_\Omega |\nabla z(\cdot,t)|^2. \qquad (3.8)\end{aligned}$$

Here the second equation in (1.1) enables us to obtain that

$$\begin{aligned}\int_\Omega |\nabla v|^2 &= -\int_\Omega (v-w^*)\Delta v \\ &= -\int_\Omega (v-w^*)^2 + \int_\Omega (v-w^*)(w-w^*). \qquad (3.9)\end{aligned}$$

Similarly we infer that

$$\int_\Omega |\nabla z|^2 = -\int_\Omega (z-u^*)^2 + \int_\Omega (z-u^*)(u-u^*).$$

Therefore we derive from (3.8) that

$$\frac{d}{dt}E_1(t) \leq F_1(t) \qquad (3.10)$$

for all $t > 0$, where

$$\begin{aligned}F_1(t) := &-a_2\mu_1\mu_2 \int_\Omega (u(\cdot,t)-u^*)^2 - a_1\mu_1\mu_2\delta_1 \int_\Omega (w(\cdot,t)-w^*)^2 \\ &- a_1a_2\mu_1\mu_2(1+\delta_1) \int_\Omega (u(\cdot,t)-u^*)(w(\cdot,t)-w^*) \\ &+ \frac{a_2\mu_2 u^* \chi_1^2}{4} \left(\int_\Omega (v(\cdot,t)-w^*)(w(\cdot,t)-w^*) - \int_\Omega (v(\cdot,t)-w^*)^2 \right) \\ &+ \frac{a_1\mu_1 w^* \chi_2^2 \delta_1}{4} \left(\int_\Omega (z(\cdot,t)-u^*)(u(\cdot,t)-u^*) - \int_\Omega (z(\cdot,t)-u^*)^2 \right)\end{aligned}$$

for all $t > 0$. In order to see that (3.2) holds it is enough to find $\varepsilon_0 > 0$ such that

$$F_1(t) \le -\varepsilon_0 \int_\Omega \left[(u(\cdot,t)-u^*)^2 + (v(\cdot,t)-w^*)^2 + (w(\cdot,t)-w^*)^2 + (z(\cdot,t)-u^*)^2\right] \tag{3.11}$$

for all $t > 0$. To see this we shall show that the condition (2.5) in Lemma 2.3 (I) is satisfied with

$$\begin{aligned} &a = a_2\mu_1\mu_2, && b = 0,\\ &c = a_1a_2\mu_1\mu_2(1+\delta_1), && d = -\frac{a_1\mu_1 w^*\chi_2^2\delta_1}{4},\\ &e = \frac{a_2\mu_2 u^*\chi_1^2}{4}, && f = -e,\\ &g = 0, && h = a_1\mu_1\mu_2\delta_1,\\ &i = 0, && j = -d \end{aligned}$$

and

$$x = u(\cdot,t)-u^*, \quad y = v(\cdot,t)-w^*, \quad z = w(\cdot,t)-w^*, \quad w = z(\cdot,t)-u^*.$$

In order to see that the condition (2.5) holds we shall prove that

$$a = a_2\mu_1\mu_2 > 0, \tag{3.12}$$

$$ae - \frac{b^2}{4} = ae = \frac{a_2^2\mu_1\mu_2^2u^*\chi_1^2}{4} > 0, \tag{3.13}$$

$$\begin{aligned} aeh - \frac{af^2}{4} - \frac{b^2h}{4} + \frac{bcf}{4} - \frac{c^2e}{4} &= aeh - \frac{af^2}{4} - \frac{c^2e}{4}\\ &= e\left(ah - \frac{ae}{4} - \frac{c^2}{4}\right) > 0, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &aehj - \frac{aei^2}{4} - \frac{af^2j}{4} + \frac{afgi}{4} - \frac{ag^2h}{4} - \frac{b^2hj}{4} + \frac{b^2i^2}{16} + \frac{bcfj}{4} - \frac{bdfi}{8}\\ &- \frac{bcgi}{8} + \frac{bdgh}{4} - \frac{c^2ej}{4} + \frac{cdei}{4} + \frac{c^2g^2}{16} - \frac{cdfg}{8} - \frac{d^2eh}{4} + \frac{d^2f^2}{16}\\ &= aehj - \frac{af^2j}{4} - \frac{c^2ej}{4} - \frac{d^2eh}{4} + \frac{d^2f^2}{16}\\ &= ej\left[ah - \frac{ae}{4} - \frac{c^2}{4} + \frac{j}{4}\left(-h + \frac{e}{4}\right)\right] > 0. \end{aligned} \tag{3.15}$$

The conditions (3.12) and (3.13) are obvious. The conditions (3.14) and (3.15) hold if either

$$ah - \frac{ae}{4} - \frac{c^2}{4} > 0 \text{ and } -h + \frac{e}{4} \geq 0 \tag{3.16}$$

or

$$-h + \frac{e}{4} < 0 \text{ and } ah - \frac{ae}{4} - \frac{c^2}{4} + \frac{j}{4}\left(-h + \frac{e}{4}\right) > 0. \tag{3.17}$$

However, (3.16) dose not hold, since the second inequality in (3.16) implies that

$$ah - \frac{ae}{4} - \frac{c^2}{4} = -a\left(-h + \frac{e}{4}\right) - \frac{c^2}{4} < 0.$$

Recalling that $\delta = \delta_1$ satisfies (3.3) and (3.4), we have that (3.17) holds with $\delta = \delta_1$:

$$-h + \frac{e}{4} = \mu_2\chi_1^2\left(-a_1 X\delta_1 + \frac{a_2 u^*}{16}\right) < 0$$

and

$$\begin{aligned}
&ah - \frac{ae}{4} - \frac{c^2}{4} - \frac{jh}{4} + \frac{ej}{16} \\
&= -a_1a_2\mu_1\mu_2\chi_1^2\chi_2^2\left[X\left(\frac{a_1a_2}{4}Y + \frac{a_1w^*}{16a_2}\right)\delta_1^2 - \left(\frac{1}{2}(2-a_1a_2)XY + \frac{u^*w^*}{256}\right)\delta_1\right. \\
&\qquad\qquad \left. + \frac{a_2u^*}{16a_1}Y + \frac{a_1a_2}{4}XY\right] \\
&= -a_1a_2\mu_1\mu_2\chi_1^2\chi_2^2 \cdot G(\delta_1) > 0.
\end{aligned}$$

Therefore from Lemma 2.3 (I) we can find $\varepsilon_0 > 0$ such that (3.11) holds. Thus a combination of (3.10) and (3.11) yields (3.2). □

Next we will establish the convergence result for u and w in the case that $a_1, a_2 \in (0,1)$.

Lemma 3.2. *Let* $a_1, a_2 \in (0,1)$. *Assume that* (1.2) *and* (1.3) *are satisfied. Then*

$$\|u(\cdot,t) - u^*\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w^*\|_{L^\infty(\Omega)} \to 0$$

as $t \to \infty$.

Proof. Integration of (3.2) over $(0,\infty)$ together with Lemma 2.1 derives this lemma. □

We see the following lemma which gives an L^2-convergence rate for the solution.

Lemma 3.3. *Let $a_1, a_2 \in (0,1)$. Assume that* (1.2) *and* (1.3) *are satisfied. Then there exist $C > 0$ and $\ell > 0$ such that*

$$\|u(\cdot,t) - u^*\|_{L^2(\Omega)} + \|w(\cdot,t) - w^*\|_{L^2(\Omega)} \le Ce^{-\ell t}$$

for all $t > 0$.

Proof. A similar argument as in the proof of [9, Lemma 3.6] enables us to show this lemma. □

Proof of Theorem 1.1. A combination of Lemmas 2.2 and 3.3 immediately leads to the conclusion of this theorem. □

4. Convergence. Case II: $a_1 \ge 1 > a_2 > 0$

In this section we show stabilization in the case that $a_1 \ge 1 > a_2 > 0$. We first derive the following lemma which is utilized to confirm that the assumption of Lemma 2.1 is satisfied.

Lemma 4.1. *Let $a_1 \ge 1 > a_2 > 0$. Assume that* (1.4) *is satisfied. Then there exist $a_1' \in [1, a_1]$ and $\delta_2 > 0$ such that the function $E_2 : (0,\infty) \to \mathbb{R}$ defined as*

$$E_2 := a_2\mu_2 \int_\Omega u + a_1'\mu_1\delta_2 \int_\Omega (w - 1 - \log w)$$

satisfies

$$\frac{d}{dt}E_2(t) \le -a_2\mu_1\mu_2(a_1'-1)\int_\Omega u(\cdot,t) - \varepsilon_0' \int_\Omega \left[u^2(\cdot,t) + (w(\cdot,t)-1)^2 + z^2(\cdot,t)\right] \tag{4.1}$$

for all $t > 0$ with some $\varepsilon_0' > 0$.

Proof. From the condition (1.4) there exists $a_1' \in [1, a_1]$ such that

$$a_1'a_2 < 1, \quad Y = \frac{\mu_2}{\chi_2^2} > \frac{a_1'^2}{16(1-a_1'a_2)}. \tag{4.2}$$

Then we shall show that there is a constant $\delta' > 0$ such that

$$H(\delta') := \left[\left(\frac{a_1'a_2}{4}Y + \frac{a_1'}{16a_2}\right)\delta'^2 - \frac{1}{2}(2 - a_1'a_2)Y\delta' + \frac{a_1'a_2}{4}Y\right] < 0. \tag{4.3}$$

Thanks to (4.2), the discriminant D' of $H(\delta')$ is positive:

$$D' := \left(\frac{1}{2}(2-a_1'a_2)\right)^2 Y^2 - \left(\frac{a_1'a_2}{4}Y + \frac{a_1'}{16a_2}\right)\cdot a_1'a_2Y$$
$$= (1-a_1'a_2)Y\left(Y - \frac{a_1'^2}{16(1-a_1'a_2)}\right) > 0.$$

Hence two solutions of the equation $H(\delta') = 0$ are given by

$$\delta'_\pm := \frac{a_2\left(4(2-a_1'a_2)Y \pm 8\sqrt{D'}\right)}{a_1'(4a_2^2Y+1)}$$

and (4.3) holds for all $\delta' \in (\delta'_-, \delta'_+)$. Therefore we can choose $\delta_2 > 0$ such that

$$\delta'_- < \delta_2 < \delta'_+.$$

Then we infer from the fact $a_1' \in [1, a_1]$ and the Young inequality that

$$\begin{aligned}\frac{d}{dt}E_2(t) &= a_2\mu_2\int_\Omega u_t(\cdot,t) + a_1'\mu_1\delta_2\int_\Omega\left(w_t(\cdot,t) - \frac{w_t(\cdot,t)}{w(\cdot,t)}\right)\\ &\le a_2\mu_1\mu_2\int_\Omega u(\cdot,t)(1-u(\cdot,t)-a_1'w(\cdot,t))\\ &\quad + a_1'\mu_1\delta_2\int_\Omega(w(\cdot,t)-1)(1-a_2u(\cdot,t)-w(\cdot,t))\\ &\quad - a_1'\mu_1\delta_2\int_\Omega\frac{|\nabla w(\cdot,t)|^2}{w^2(\cdot,t)} + a_1'\mu_1\chi_2\delta_2\int_\Omega\frac{\nabla w(\cdot,t)\cdot\nabla z(\cdot,t)}{w(\cdot,t)}.\end{aligned} \tag{4.4}$$

Now, noticing that the first and second terms are rewritten as

$$u(1-u-a_1'w) = -(a_1'-1)u - u^2 - a_1'u(w-1),$$
$$(w-1)(1-a_2u-w) = -(w-1)^2 - a_2u(w-1),$$

we obtain from (4.4) that

$$\begin{aligned}\frac{d}{dt}E_2(t) \le{}& -a_2\mu_1\mu_2(a_1'-1)\int_\Omega u(\cdot,t) - a_2\mu_1\mu_2\int_\Omega u^2(\cdot,t)\\
&- a_1'a_2\mu_1\mu_2(1+\delta_2)\int_\Omega u(\cdot,t)(w(\cdot,t)-1)\\
&- a_1'\mu_1\mu_2\delta_2\int_\Omega (w(\cdot,t)-1)^2\\
&- a_1'\mu_1\delta_2\int_\Omega \frac{|\nabla w(\cdot,t)|^2}{w^2(\cdot,t)} + a_1'\mu_1\chi_2\delta_2\int_\Omega \frac{\nabla w(\cdot,t)\cdot\nabla z(\cdot,t)}{w(\cdot,t)}\\
\le{}& -a_2\mu_1\mu_2(a_1'-1)\int_\Omega u(\cdot,t) - a_2\mu_1\mu_2\int_\Omega u^2(\cdot,t)\\
&- a_1'a_2\mu_1\mu_2(1+\delta_2)\int_\Omega u(\cdot,t)(w(\cdot,t)-1)\\
&- a_1'\mu_1\mu_2\delta_2\int_\Omega (w(\cdot,t)-1)^2 + \frac{a_1'\mu_1\chi_2^2\delta_2}{4}\int_\Omega |\nabla z(\cdot,t)|^2.\end{aligned}$$

Here, since a similar argument as in (3.9) implies that

$$\int_\Omega |\nabla z|^2 = -\int_\Omega z^2 + \int_\Omega zu,$$

we infer that

$$\frac{d}{dt}E_2(t) \le -a_2\mu_1\mu_2(a_1'-1)\int_\Omega u(\cdot,t) + F_2(t) \tag{4.5}$$

for all $t>0$, where

$$\begin{aligned}F_2(t) :={}& -a_2\mu_1\mu_2\int_\Omega u^2(\cdot,t) - a_1'a_2\mu_1\mu_2(1+\delta_2)\int_\Omega u(\cdot,t)(w(\cdot,t)-1)\\
&- a_1'\mu_1\mu_2\delta_2\int_\Omega (w(\cdot,t)-1)^2\\
&+ \frac{a_1'\mu_1\chi_2^2\delta_2}{4}\left(\int_\Omega z(\cdot,t)u(\cdot,t) - \int_\Omega z^2(\cdot,t)\right)\end{aligned}$$

for all $t>0$. In order to show that (4.1) holds it suffices to find $\varepsilon_0'>0$ such that

$$F_2(t) \le -\varepsilon_0'\int_\Omega \left[u^2(\cdot,t) + (w(\cdot,t)-1)^2 + z^2(\cdot,t)\right] \tag{4.6}$$

for all $t>0$. To see this we shall show that the condition (2.6) in Lemma 2.3 (II) is satisfied with

$$\begin{aligned}&a = a_2\mu_1\mu_2, && b = a_1'a_2\mu_1\mu_2(1+\delta_2),\\
&c = -\frac{a_1'\mu_1\chi_2^2\delta_2}{4}, && d = a_1'\mu_1\mu_2\delta_2,\\
&e = 0, && f = -c\end{aligned}$$

and

$$x = u(\cdot,t), \qquad y = w(\cdot,t) - 1, \qquad z = z(\cdot,t).$$

In order to see that the condition (2.6) holds we shall prove that

$$a = a_2\mu_1\mu_2 > 0, \tag{4.7}$$

$$ad - \frac{b^2}{4} > 0, \tag{4.8}$$

$$adf - \frac{ae^2}{4} - \frac{b^2f}{4} + \frac{bce}{4} - \frac{c^2d}{4} = adf - \frac{b^2f}{4} - \frac{c^2d}{4} = f\left(ad - \frac{b^2}{4} - \frac{df}{4}\right) > 0. \tag{4.9}$$

The condition (4.7) is obvious. Since (4.8) is a consequence of (4.9) from the inequality

$$ad - \frac{b^2}{4} > ad - \frac{b^2}{4} - \frac{df}{4},$$

it suffices to show the relation (4.9). We infer from (4.3) with $\delta' = \delta_2$ that

$$ad - \frac{b^2}{4} - \frac{df}{4} = -a_1'a_2\mu_1^2\mu_2\chi_2^2 \cdot H(\delta_2) > 0.$$

Therefore we can find $\varepsilon_0' > 0$ such that (4.6) holds. Thus a combination of (4.5) and (4.6) completes the proof of this lemma. □

Next we will establish the convergence result for u and w in the case that $a_1 \geq 1 > a_2 > 0$.

Lemma 4.2. *Let* $a_1 \geq 1 > a_2 > 0$. *Assume that* (1.4) *is satisfied. Then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t) - 1\|_{L^\infty(\Omega)} \to 0$$

as $t \to \infty$.

Proof. Integration of (4.1) over $(0,\infty)$ together with Lemma 2.1 derives this lemma. □

Finally we shall establish a convergence rate for the solution of (1.1). We see the following lemma which gives an L^2-convergence rate for the solution.

Lemma 4.3. *Suppose that* (1.4) *holds. Then the following properties hold*:

(i) *Let* $a_1 > 1$ *and* $a_2 \in (0,1)$. *Then there exist* $C > 0$ *and* $\ell > 0$ *such that*

$$\|u(\cdot,t)\|_{L^2(\Omega)} + \|w(\cdot,t) - 1\|_{L^2(\Omega)} \leq Ce^{-\ell t}$$

for all $t > 0$.

(ii) *Let* $a_1 = 1$ *and* $a_2 \in (0,1)$. *Then there exists* $C > 0$ *such that*

$$\|u(\cdot,t)\|_{L^2(\Omega)} + \|w(\cdot,t) - 1\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{t+2}}$$

for all $t > 0$.

Proof. Similar arguments as in the proofs of [9, Lemmas 3.9 and 3.10] lead to this lemma. □

Proof of Theorem 1.2. A combination of Lemmas 2.2 and 4.3 immediately gives the conclusion of this theorem. □

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