LE MATEMATICHE
Vol. LVII (2002) – Fasc. II, pp. 205–215

EXISTENCE OF THREE PERIODIC SOLUTIONS FOR A NON-AUTONOMOUS SECOND ORDER SYSTEM

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1. Introduction.

The purpose of the present paper is to establish a multiplicity result for the following problem

$$(P_\lambda) \qquad \begin{cases} \ddot{u} = A(t)u - \lambda b(t)\nabla G(u) & \text{a. e. in } [0, T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

where T and λ are positive numbers, k is a positive integer, $A(t)$ is a $k \times k$ matrix valued function defined in $[0, T]$, b is an a. e. nonnegative function belonging to $L^1([0, T]) \setminus \{0\}$ and G is a real continuously differentiable function in $\mathbb{R}^k$.

As far as we know, many multiplicity results concerning the existence of at least three periodic solutions for a non-autonomous system of the form

$$(P) \qquad \begin{cases} \ddot{u} = \nabla_u F(t, u) & \text{a. e. in } [0, T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

are available in [4], [5], [9], [10] and the solutions are obtained as critical points of a suitable functional.

Entrato in redazione il 10 Ottobre 2002.

2000 Methematics Subject Classification: 34C25, 35A15.

Key words: Second order non-autonomous systems, Multiple solutions, Critical points.

In particular, in order to guarantee the existence of solutions to (P), in [4], [9], [10], in addition to a coercivity condition, a suitable behaviour of $F(t, \cdot)$ near to 0 is required: there exists $\delta > 0$ and an integer $p \geq 0$ such that

$$(1) \qquad -\frac{1}{2}(p+1)^2\omega^2|x|^2 \leq F(t,x) - F(t,0) \leq -\frac{1}{2}p^2\omega^2|x|^2$$

for all $|x| \leq \delta$ and a. e. $t \in [0, T]$, where $\omega = \frac{2\pi}{T}$.
Problem (P_1) is treated in [5] under assumptions which are different from ours.

Our main result is Theorem 1, given in Section 3. In particular, still using a variational technique, without assuming assumption (1) (see Remark 1), but requiring novel assumptions on G (see (ii)) , we are able to state that there exists a computable real number λ^* such that, for every $b \in L^1([0, T]) \setminus \{0\}$ that is a. e. nonnegative, there exist an open interval $\Lambda \subseteq \left[0, \frac{\lambda^*}{\|b\|_1}\right]$ and a positive real number ρ such that for every $\lambda \in \Lambda$ problem (P_λ) admits at least three solutions in $\mathbb{H}^1_T$ whose norms are less than ρ.

Finally, we want to stress that our main tool is a result, recently obtained by G. Bonanno ([3], Theorem 2.1), whose proof is based on a three critical points theorem due to B. Ricceri ([8], Theorem 3).

2. Preliminaries.

Let $T > 0$ and $k \in \mathbf{N}$. Let $A : [0, T] \to \mathbb{R}^{k\times k}$ a matrix valued function, $A = [a_{ij}]_{i,j=1,2,\dots,k}$, such that $a_{ij} \in L^\infty([0, T])$, $a_{ij} = a_{ji}$ for each $i,\ j = 1,\ 2, \dots,\ k$; suppose, moreover, that there exists $\mu > 0$ such that

$$(2) \qquad A(t)\xi \cdot \xi \geq \mu|\xi|^2 \qquad \text{for each } \xi \in \mathbb{R}^k, \text{ a. e. in } [0, T].$$

It is easy to check that

$$(3) \qquad A(t)\xi \cdot \xi \leq \sum_{i,j=1}^{k} \|a_{ij}\|_\infty |\xi|^2 \qquad \text{for each } \xi \in \mathbb{R}^k .$$

In according to the notations of [7], denote by $\mathbb{H}^1_T$ the space of functions $u \in L^2([0, T], \mathbb{R}^k)$ having a weak derivative $\dot{u} \in L^2([0, T], \mathbb{R}^k)$. We want explicitly recall that, for this kind of derivative, the test functions belong to the space C^∞_T of infinitely differentiable T-periodic functions from $\mathbb{R}$ into $\mathbb{R}^k$. Moreover, if $u \in \mathbb{H}^1_T$ then $\int_0^T \dot{u}(t)dt = 0$ and u is absolutely continuous (see [7], pp. 6-7). On $\mathbb{H}^1_T$ we consider the norm

$$\|u\| = \left(\int_0^T |\dot{u}(t)|^2 dt + \int_0^T A(t)u(t) \cdot u(t) dt \right)^{\frac{1}{2}},$$

which, owing to (2) and (3), is equivalent to the usual one, given by

$$\|u\|_* = \left(\int_0^T |\dot{u}(t)|^2 dt + \int_0^T |u(t)|^2 dt \right)^{\frac{1}{2}}.$$

It is well known that $\mathbb{H}_T^1$, equipped with the norm $\|\cdot\|_*$, or an equivalent one, is a Banach space, while the Rellich-Kondrachov theorem (see [1], p. 144) guarantees that it is compactly embedded in $C^0([0,T],\mathbb{R}^k)$. Write c for the smallest constant such that

$$\|u\|_{C^0} = \max_{t\in[0,T]} |u(t)| \le c\|u\| \qquad \text{for every } u \in \mathbb{H}_T^1. \tag{4}$$

A simple computation (see [6]) shows that

$$c \le \sqrt{\frac{2}{\min\{1,\ \mu\}}} \cdot \max\left\{\sqrt{T},\ \frac{1}{\sqrt{T}}\right\}. \tag{5}$$

We now introduce the functionals that will be used in the sequel. For the sake of completeness, we recall their basic properties too.

Let $\Phi,\ \Psi : \mathbb{H}_T^1 \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\|u\|^2, \qquad \Psi(u) = -\int_0^T b(t)G(u(t))dt,$$

for every $u \in \mathbb{H}_T^1$, where $b \in L^1([0,T],\mathbb{R}^k) \setminus \{0\}$ is such that $b(t) \ge 0$ a. e. in $[0,T]$ and $G : \mathbb{R}^k \to \mathbb{R}$ is a continuously differentiable function.

It is clear that Φ is a continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse. Moreover, since it is continuous and convex, it is also a sequentially weakly lower semicontinuous functional. Thanks to the Rellich-Kondrachov theorem, Ψ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. In particular, one has

$$\Phi'(u)(v) = \int_0^T \dot{u}(t)\cdot\dot{v}(t)dt + \int_0^T A(t)u(t)\cdot v(t)dt$$

and

$$\Psi'(u)(v) = -\int_0^T b(t)\nabla G(u(t))\cdot v(t)dt$$

for every $u,\ v \in \mathbb{H}_T^1$.

We recall that $u \in \mathbb{H}^1_T$ is a critical point for the functional $\Phi + \lambda\Psi$ if $\Phi'(u) + \lambda\Psi'(u) = 0$, while a solution to (P_λ) is a function $u \in C^1([0,T],\mathbb{R}^k)$ with $\dot{u}$ absolutely continuous, such that

$$\begin{cases} \ddot{u} = A(t)u - \lambda b(t)\nabla G(u) & \text{a. e. in } [0,T] \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases}$$

Finally, for the reader convenience, let us recall the theorem that play a basic role in the proof of our main result. It is due to G. Bonanno ([3], Theorem 2.1), and its proof is based on a three critical points theorem established by B. Ricceri ([8], Theorem 3).

Theorem A. *Let X be a separable and reflexive real Banach space, and let $\Phi,\ J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and $\Phi(x) \geq 0$ for every $x \in X$ and that there exist $x_1 \in X,\ r > 0$ such that*

(j) $r < \Phi(x_1)$.

(jj) $\sup_{\Phi(x)<r} J(x) < r\frac{J(x_1)}{\Phi(x_1)}$.
Further, put $\bar{a} = \frac{hr}{r\frac{J(x_1)}{\Phi(x_1)} - \sup_{\Phi(x)<r} J(x)}$ with $h > 1$, assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(jjj) $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$

for every $\lambda \in [0,\bar{a}]$.

Then, there exist an open interval $\Lambda \subseteq [0,\bar{a}]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(x) - \lambda J'(x) = 0$$

admits at least three solutions in X whose norms are less than ρ.

3. Main results.

In this section we will prove our main theorem that can be stated as follows

Theorem 1. *Assume that $G(0) \geq 0$ and that there exist a positive constant d and $\hat{\xi} \in \mathbb{R}^k$ such that*

(i) $|\hat{\xi}| > \dfrac{d}{c\sqrt{\mu T}}$.

(ii) $\dfrac{\max_{|\xi|\leq d} G(\xi)}{d^2} < \dfrac{1}{c^2 T \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} \cdot \dfrac{G(\hat{\xi})}{|\hat{\xi}|^2}$,

where c is the constant given in (4).

Put

$$\lambda^* = \frac{p\, d^2}{\frac{d^2}{T \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} \cdot \frac{G(\hat{\xi})}{|\hat{\xi}|^2} - c^2 \max_{|\xi|\leq d} G(\xi)}$$

with $p > \frac{1}{2}$ and suppose that

(iii) $\limsup_{|\xi|\to+\infty} \dfrac{G(\xi)}{|\xi|^2} < \dfrac{1}{2c^2\lambda^*}$.

Then, for every function $b \in L^1([0,T]) \setminus \{0\}$ that is a. e. nonnegative, there exist an open interval $\Lambda \subseteq \left[0, \frac{\lambda^*}{\|b\|_{L^1}}\right]$ and a positive real number ρ such that for every $\lambda \in \Lambda$ problem (P_λ) admits at least three solutions in $\mathbb{H}_T^1$ whose norms are less than ρ.

Proof. We want to apply Theorem A, where X is $\mathbb{H}_T^1$, $J = -\Psi - \|b\|_{L^1} G(0)$, being Φ and Ψ the functionals introduced in Section 2.

Put $r = \frac{d^2}{2c^2}$, $u_0 = 0$, $u_1(t) = \hat{\xi}$ for every $t \in [0,T]$. Clearly $\Phi(u_0) = J(u_0) = 0$ and $\Phi(u) \geq 0$ for every $u \in X$. Moreover

$$\begin{aligned}\Phi(u_1) &= \frac{1}{2}\int_0^T A(t)\hat{\xi}\cdot\hat{\xi}\, dt \geq \frac{1}{2}\mu|\hat{\xi}|^2 \int_0^T dt \\ &= \frac{1}{2}\mu T|\hat{\xi}|^2 > \frac{1}{2}\mu T \frac{d^2}{c^2\mu T} = r.\end{aligned} \tag{6}$$

By the fact that

$$\Phi^{-1}(]-\infty, r]) \subseteq \left\{u \in X : \|u\|_{C^0} \leq c\sqrt{2r}\right\}$$

one has

$$\begin{aligned}\sup_{\frac{1}{2}\|u\|^2\leq r} \int_0^T b(t)G(u(t))\, dt &\leq \sup_{\|u\|_{C^0}\leq c\sqrt{2r}} \int_0^T b(t)G(u(t))\, dt \\ &= \sup_{\|u\|_{C^0}\leq d} \int_0^T b(t)G(u(t))\, dt \\ &\leq \sup_{|\xi|\leq d} \int_0^T b(t)G(\xi)\, dt \\ &= \|b\|_1 \cdot \sup_{|\xi|\leq d} G(\xi).\end{aligned} \tag{7}$$

Moreover, taking in mind condition (3) and observing that from assumption (ii) and $G(0) \geq 0$ follows that $G(\hat{\xi}) > 0$, we can write

$$
\begin{aligned}
(8)\quad \|b\|_1 \frac{d^2}{c^2} \frac{1}{T \cdot \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} \frac{G(\hat{\xi})}{|\hat{\xi}|^2} &= \|b\|_1 \frac{2r}{T \cdot \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} \frac{G(\hat{\xi})}{|\hat{\xi}|^2} \\
&= 2r \frac{[-\Psi(u_1)]}{T \cdot \sum_{i,j=1}^{k} \|a_{ij}\|_\infty \cdot |\hat{\xi}|^2} \\
&\leq r \frac{[-\Psi(u_1)]}{\frac{1}{2}\int_0^T A(t)\hat{\xi} \cdot \hat{\xi} dt} = r \frac{[-\Psi(u_1)]}{\Phi(u_1)}.
\end{aligned}
$$

At this point, thanks to (7), (ii), (8) and (6), one has that

$$
\begin{aligned}
(9)\qquad \sup_{\Phi(u)<r} J(u) &\leq \|b\|_1 \sup_{|\xi| \leq d} G(\xi) - \|b\|_1 G(0) \\
&< \|b\|_1 \frac{d^2}{c^2} \cdot \frac{1}{T \cdot \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} \cdot \frac{G(\hat{\xi})}{|\hat{\xi}|^2} - \|b\|_1 G(0) \\
&\leq r \frac{[-\Psi(u_1)]}{\Phi(u_1)} - \|b\|_1 G(0) \\
&\leq r \frac{[J(u_1)]}{\Phi(u_1)}.
\end{aligned}
$$

Put

$$\delta_1 = \sup_{\Phi(u)<r} J(u),$$

$$\delta_2 = r \frac{J(u_1)}{\Phi(u_1)},$$

$$\alpha = \|b\|_1 \cdot \sup_{|\xi| \leq d} G(\xi) - \|b\|_1 G(0),$$

$$\beta = \|b\|_1 \frac{d^2}{c^2} \cdot \frac{1}{T \cdot \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} \cdot \frac{G(\hat{\xi})}{|\hat{\xi}|^2} - \|b\|_1 G(0).$$

We have proved that

$$\delta_1 \leq \alpha < \beta \leq \delta_2.$$

Hence, if we put $h = 2p$, one has

$$0 < \bar{a} = \frac{hr}{\delta_2 - \delta_1} \leq \frac{\lambda^*}{\|b\|_1}.$$

Let us now prove that for each $\lambda \in \left[0, \frac{\lambda^*}{\|b\|_1}\right]$ one has

$$\lim_{\|u\|\to+\infty} (\Phi(u) - \lambda J(u)) = +\infty.$$

If $\lambda = 0$ the conclusion if obvious. Fix $\lambda \in \left]0, \frac{\lambda^*}{\|b\|_1}\right]$. Thanks to assumption (iii) we can consider a real positive number γ such that

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2} < \gamma < \frac{1}{2c^2\lambda^*}.$$

Hence, there exists a positive real number γ' such that

$$G(\xi) < \gamma|\xi|^2 + \gamma' \tag{10}$$

for every $\xi \in \mathbb{R}^k$. Hence, by (4) one has that

$$\Phi(u) - \lambda J(u) > \left(\frac{1}{2} - \lambda\gamma c^2\|b\|_1\right)\|u\|^2 - \lambda\gamma'\|b\|_1 + \lambda\|b\|_1 G(0)$$

for every $u \in X$. At this point the coercivity of $\Phi - \lambda J$ follows observing that, from condition (10) one has

$$\frac{1}{2} - \lambda\gamma c^2\|b\|_1 \geq \frac{1}{2} - \lambda^*\gamma c^2 > 0.$$

Finally, taking in mind that Φ' is a continuous operator that admits a continuous inverse and that J' is a continuous and compact operator, from the coercivity of the functional $\Phi - \lambda J$, for every $\lambda \in \left[0, \frac{\lambda^*}{\|b\|_1}\right]$ follows that the Palais-Smale condition is verified. All the assumptions of Theorem A are satisfied, so there exists an open interval $\Lambda \subseteq [0, \bar{a}] \subseteq \left[0, \frac{\lambda^*}{\|b\|_1}\right]$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, the equation

$$\Phi'(u) - \lambda J'(u) = 0 \tag{11}$$

admits at least three solutions in X whose norms are less than ρ. At this point, we can observe that, since $C_T^\infty \subset X$, if u is a solution of (11), then $\dot{u} \in X$ and, in particular, $\ddot{u} = A(t)u - \lambda b(t)\nabla G(u)$ a. e. in $[0, T]$, $\dot{u}$ is absolutely continuous and $\int_0^T \ddot{u}(t)dt = 0$ hence u is a solution of problem (P_λ) and the proof is complete. □

4. Examples.

This section is devoted to stress some examples of applications of Theorem 1.

Example 1. Let k be a positive integer, let $A : [0,1] \to \mathbb{R}^{k\times k}$ be a matrix valued function satisfying the assumptions made in Section 2. Moreover, suppose that $\|a_{ii}\|_1 < 9\mu$ for some $i = 1, 2, \dots, k$ and

$$\sum_{i,j=1}^{k} \|a_{ij}\|_\infty < \frac{10^5}{9} \min\{1, \mu\}. \tag{12}$$

Put

$$G(\xi) = \frac{|\xi|^{20}}{e^{|\xi|^2}}$$

for every $\xi \in \mathbb{R}^k$. Then, for every a. e. nonnegative function $b \in L^1([0,1]) \setminus \{0\}$ there exist an open interval $\Lambda \subseteq \left[0, \frac{\lambda^*}{\|b\|_1}\right]$, where λ^* is given by

$$\frac{p}{\frac{3^{18}}{e^9 \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} - c^2 \max_{|\xi|\le 1} G(\xi)}$$

with $p > \frac{1}{2}$ and a positive real number ρ such that, for every $\lambda \in \Lambda$ problem (P_λ) admits at least three solutions in $\mathbb{H}^1_T$ whose norms are less than ρ.

First of all, we observe that, obviously, G is a continuously differentiable function and $G(0) = 0$. Fix $T = 1$, $d = 1$ and any $\hat{\xi} \in \mathbb{R}^k$ such that $|\hat{\xi}| = 3$. From condition (4), if we consider the functions $u_i(t) = e_i$, with $i = 1, 2, \dots, k$, one has

$$\frac{1}{c} \le \sqrt{\|a_{ii}\|_1} \tag{13}$$

for every $i = 1, 2, \dots, k$. So, since $\|a_{ii}\|_1 < 9\mu$ for some $i = 1, 2, \dots, k$, it is easy to check that assumption (i) holds. After that, taking in mind conditions (5) and (12), one has that

$$\frac{1}{c^2 \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} \frac{4\cdot 10^5}{9} \ge \frac{2\cdot 10^5}{9} \frac{\min\{1, \mu\}}{\sum_{i,j=1}^{k} \|a_{ij}\|_\infty} > 2.$$

A simple computation shows that if $\tilde{\xi} \in \mathbb{R}^k$ is such that $|\tilde{\xi}| = 1$ then

$$\max_{|\xi|\le 1} G(\xi) = G(\tilde{\xi})$$

and

$$G(\tilde{\xi}) < 2 < \frac{2 \cdot 10^5}{9} \frac{\min\{1,\ \mu\}}{\sum_{i,j=1}^{k} \|a_{ij}\|_\infty} < \frac{1}{c^2 \sum_{i,j=1}^{k} \|a_{ij}\|_\infty} \frac{G(\hat{\xi})}{9}.$$

Hence assumption (ii) holds. Finally, since $\lim_{|\xi| \to +\infty} \frac{G(\xi)}{|\xi|^2} = 0$, also assumption (iii) is verified and the conclusion follows.

Remark 1. Let A and G be as in Example 1. Fix $b \in C^0([0, 1], \mathbb{R}^+)$ and $\lambda > 0$. Clearly, if we put

$$F(t, \xi) = \frac{1}{2} A(t)\xi \cdot \xi - \lambda b(t)G(\xi)$$

for every $(t, \xi) \in [0, 1] \times \mathbb{R}^k$ one has that problem (P) becomes problem (P_λ). At this point, it is easy to check that $\liminf_{|\xi| \to 0} \frac{F(t,\xi)}{|\xi|^2} \geq \frac{\mu}{2}$ uniformly respect to t. Hence, since $F(t, 0) = 0$, condition (1) is not verified and no one of the results in [4, 8, 9] can be applied to problem (P_λ).

Example 2. Let k be a positive integer such that $k \leq 4$ and $p \in \left]\frac{1}{2}, 1\right]$. Put

$$G(\xi) = e^{\frac{10|\xi|^2}{1+|\xi|^4}} \cdot |\xi|^2 + \arctan(1 + |\xi|^2)$$

for every $\xi \in \mathbb{R}^k$. Then, for every function $b \in L^1([0, 1]) \setminus \{0\}$ that is a. e. nonnegative, there exist an open interval $\Lambda \subseteq \left[0, \frac{p}{\|b\|_1}\right]$ and a positive real number ρ such that for every $\lambda \in \Lambda$ the problem

$$\begin{cases} \ddot{u} = u - \lambda b(t)\nabla G(u) & \text{a. e. in } [0,1] \\ u(0) - u(1) = \dot{u}(0) - \dot{u}(1) = 0. \end{cases}$$

admits at least three solutions in H_T^1 whose norms are less than ρ.

First of all, we observe that, obviously, G is a continuously differentiable function and $G(0) = \frac{\pi}{4}$. Consider $A \equiv I_{k \times k}$ and fix $T = 1$, $d = \frac{1}{2}$ and $\hat{\xi} \in \mathbb{R}^k$ such that $|\hat{\xi}| = 1$. In this situation it is simple to see that $\mu = 1$, hence, by (13) and (5), one has $1 \leq c \leq \sqrt{2}$. So $\frac{1}{2c} \leq \frac{1}{2} < |\hat{\xi}|$ and assumption (i) holds. Fix now $\tilde{\xi} \in \mathbb{R}^k$ such that $|\tilde{\xi}| = \frac{1}{2}$. It is easy to verify that $\max_{|\xi| \leq d} G(\xi) = G(\tilde{\xi})$ and that

$$4 \cdot G(\tilde{\xi}) < \frac{1}{2k} G(\hat{\xi}) \leq \frac{1}{c^2\, k} G(\hat{\xi})$$

so that also assumption (ii) is verified. Now, we can observe that

$$\frac{G(\hat{\xi})}{4k} - 2\,G(\tilde{\xi}) > 1,$$

hence, if $p \in \left]\frac{1}{2}, 1\right]$ one has

$$\lambda^* = \frac{p}{4\left(\frac{G(\hat{\xi})}{4k} - c^2\,G(\tilde{\xi})\right)} \leq \frac{p}{\frac{G(\hat{\xi})}{4k} - 2\,G(\tilde{\xi})} < p$$

and

$$\frac{1}{2c^2\lambda^*} = \frac{2}{p\,c^2}\left(\frac{G(\hat{\xi})}{4k} - c^2\,G(\tilde{\xi})\right) \geq \frac{1}{p}\left(\frac{G(\hat{\xi})}{4k} - 2\,G(\tilde{\xi})\right) > \frac{1}{p} \geq 1.$$

Moreover, it is easy to verify that

$$\lim_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2} = 1$$

and all the assumptions of Theorem 1 are satisfied.

Remark 2. A classical growth condition on G to realize the coercivity of the energy functional related to problem (P_λ) is

$$|G(\xi)| \leq a|\xi|^s + b \tag{14}$$

for every $\xi \in \mathbb{R}^k$, where $a,\ b$ and s are three real positive numbers with $s < 2$. We want to emphasize that our assumption (iii) generalizes (14). In fact, when (14) holds one has that $\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^2} = 0$, so that (iii) is verified, but the converse is not, in general, true as examples 1 and 2 show.

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