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SUITABLE RADON MEASURE FOR NONLINEAR DIRICHLET BOUNDARY $P(U)$-LAPLACIAN PROBLEM

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This paper is devoted to the study of nonlinear homogeneous Dirichlet boundary $p(u)$-laplacian problem. Existence, uniqueness and structural stability results of weak solutions are obtained by approximation method and convergent sequences in terms of Young measures.

1. Introduction

We consider the nonlinear elliptic $p(u)-$Laplacian problem with Dirichlet boundary condition

$$\begin{cases} b(u) - \mathrm{div} a(x,u,\nabla u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is an open boundary domain of $\mathbb{R}^N$ $(N \geq 3)$, with smooth boundary and $b : \mathbb{R} \to \mathbb{R}$ is a continuous, surjective and strictly increasing function. The operator $\mathrm{div} a(x,u,\nabla u)$ is called $p(u)-$Laplacian, a prototype case is $\mathrm{div}(|\nabla u|^{p(.,u)-2}.\nabla u)$ and it is more complicated than $p(x)-$Laplacian in term

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of nonlinearity. μ is a Radon measure on Ω which does not charge the sets of zero p_- capacity.

The problem (1) is adapted into a generalized Leray-Lions framework under the assumptions that $a : \Omega \times (\mathbb{R} \times \mathbb{R}^N) \to \mathbb{R}^N$ is a Carathéodory function with

$$a(x,z,0) = 0 \text{ for all } z \in \mathbb{R}, \text{ and a.e. } x \in \Omega, \tag{2}$$

satisfying the strict monotonicity assumption

$$\big(a(x,z,\xi) - a(x,z,\eta)\big).(\xi - \eta) > 0 \text{ for all } \xi, \eta \in \mathbb{R}^N, \xi \neq \eta, \tag{3}$$

as well as the growth and the coercivity assumptions with variable exponent

$$\big|a(x,z,\xi)\big|^{p'(x,z)} \leq C_1\big(|\xi|^{p(x,z)} + \mathcal{M}(x)\big), \tag{4}$$

$$a(x,z,\xi).\xi \geq \frac{1}{C_2}|\xi|^{p(x,z)}. \tag{5}$$

Here C_1, C_2 are positive constants and $\mathcal{M}$ is a positive function such that $\mathcal{M} \in L^1(\Omega)$.
$p : \Omega \times \mathbb{R} \to [p_-, p_+]$ is a Carathéodory function, $1 < p_- \leq p_+ < +\infty$ and $p'(x,z) = \dfrac{p(x,z)}{p(x,z)-1}$ is the conjugate exponent of $p(x,z)$, with

$$p_- := ess \inf_{(x,z)\in\overline{\Omega}\times\mathbb{R}} p(x,z) \text{ and } p_+ := ess \sup_{(x,z)\in\overline{\Omega}\times\mathbb{R}} p(x,z).$$

We assume that:

$$p_- > N \text{ and } p \text{ is log-Hölder continuous in } (x,z) \text{ uniformly on } \overline{\Omega} \times [-M,M], \text{ for all } M > 0. \tag{6}$$

In the case of datum μ in $L^1(\Omega)$, Andreianov et *al.* (see [2]) established the existence results for such $p(x,u)$ variable exponent problems, the uniqueness and structural stabilty issues.
In this work, μ is a Radon diffuse measure. We define $\mathcal{M}_b(\Omega)$ as the set of bounded Radon measures in Ω.
For the variable exponent $\pi(.)$, where $\pi(.)$ is to be defined later, given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W_0^{1,\pi(.)}(\Omega)$ if $\mu(A) = 0$, for every set A such that $Cap_{\pi(.)}(A,\Omega) = 0$ (see [11]). For $A \subset \Omega$, we denote

$$S_{\pi(.)}(A) = \Big\{u \in W_0^{1,\pi(.)}(\Omega) \cap C_0(\Omega) : \ u = 1 \text{ on } A \text{ and } u \geq 0 \text{ in } \Omega\Big\}.$$

The $\pi(.)-$Capacity for every subset A with respect to Ω is defined by

$$Cap_{\pi(.)}(A,\Omega) = \inf_{u \in S_{\pi(.)}(A)} \left\{ \int_{\Omega} |\nabla u|^{\pi(.)} dx \right\}.$$

The set of bounded Radon diffuse measures in variable exponent setting $\pi(.)$ is denoted by $\mathcal{M}_b^{\pi(.)}(\Omega)$. Using the work by Boccardo et *al.* in [6], Nyanquini et *al.* in [11] proved the decomposition Theorem of measures in the context of variable exponent $p(.)$: every measure $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$ admits a decomposition in $L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.
In the present paper, as p depends on the space variable x and the unknown fonction u, it is not clear how to prove the existence and uniqueness of weak solution for the problem (1) with a measure data μ which does not charge the set of zero $\pi(.)-$ capacity. To overcome this difficulty, we extend the result of [2], taking into account the bounded Radon measure μ which is zero on the subset of zero $p_{-}-$capacity $\big($ie the capacity starting from $W_0^{1,p_-}(\Omega)\big)$. We recall the decomposition theorem of these measures (see [6], Theorem 2.1): every bounded Radon measure that is zero $p_{-}-$capacity can be splitted in the sum of an element in $W^{-1,(p_-)'}(\Omega)$ and a function in $L^1(\Omega)$, and reciprocally, every bounded measure in $L^1(\Omega) + W^{-1,(p_-)'}(\Omega)$ is zero on the set of $p_{-}-$capacity. Observe that the bounded Radon diffuse measures $\mathcal{M}_b^{p_-}(\Omega)$ is a subset of $W^{-1,(p_-)'}(\Omega)$, since for $p_- > N$, $L^1(\Omega) \subset W^{-1,(p_-)'}(\Omega)$. Therefore, we reduce the study of problem (1) to $W^{-1,(p_-)'}(\Omega)$ data. The problem (1) is also seen as a generalization of the following problem

$$\begin{cases} u - \operatorname{div}(|\nabla u|^{p(.,u)-2}.\nabla u) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

with $\mu \in W^{-1,(p_-)'}(\Omega)$, studied by M. Chipot et *al.* in [8]. Here, we use the techniques of [2], different of those used in [8].
The interest of the study of this kind of problem is due to the fact that they can model phenomena which arise in the study of elastic mechanics (see [4]), electrorheological fluid (see [13]) or image restoration (see [7]).
The goal of this study is to prove the existence, uniqueness and stability results of the weak solutions for such $p(x,u)$ variable exponent problem when the measure μ belongs to $W^{-1,(p_-)'}(\Omega)$.

The remaining part of this article is organized as follows: in the next Section, we introduce some preliminary results. In the third Section, we study the existence and uniqueness of weak solution for the problem (1). In Section 4, we end by the study of the continuous dependence of the weak solution.

2. Preliminary results

• We will use the so-called truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \leq k \\ k sign_0(s) & \text{if } |s| > k \end{cases}, \quad \text{where } sign_0(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0. \end{cases}$$

The truncation function possesses the following properties.

$$T_k(-s) = -T_k(s), |T_k(s)| = \min\{|s|, k\},$$

$$\lim_{k \to +\infty} T_k(s) = s \text{ and } \lim_{k \to 0} \frac{1}{k} T_k(s) = sign_0(s).$$

• We also need to truncate vector valued-functions with the help of the map

$$h_m : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \ h_m(\lambda) = \begin{cases} \lambda, & \text{if } |\lambda| \leq m \\ m\dfrac{\lambda}{|\lambda|} & \text{if } |\lambda| > m, \end{cases} \quad \text{where } m > 0.$$

We are looking the solutions to the problem (1) in the variable exponent Sobolev space $\dot{E}^{\pi(.)}(\Omega)$ defined below (notice that the exponent $\pi(.)$ itself is related to u by $\pi(.) := p(., u(.))$, so the solutions and different data will possess different integrability properties). For the sake of completeness, we also recall the definition of variable exponent Lebesgue and Sobolev spaces $L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$. In the sequel, we will use the same notation $L^{\pi(.)}(\Omega)$ for the space $(L^{\pi(.)}(\Omega))^N$ of vector-valued functions.

Definition 2.1. Let $\pi : \Omega \longrightarrow [1, +\infty)$ be a measurable function.
•$L^{\pi(.)}(\Omega)$ is the space of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that the modular

$$\rho_{\pi(.)}(f) := \int_\Omega |f|^{\pi(x)} dx < \infty.$$

If p_+ is finite, this space is equipped with the Luxembourg norm

$$||f||_{L^{\pi(.)}(\Omega)} := \inf\left\{\lambda > 0; \quad \rho_{\pi(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

•$W^{1,\pi(.)}(\Omega)$ is the space of all functions $f \in L^{\pi(.)}(\Omega)$ such that the gradient of f (taken in the sense of distributions) belongs to $L^{\pi(.)}(\Omega)$; the space $W^{1,\pi(.)}(\Omega)$ is equipped with the norm

$$||u||_{W^{1,\pi(.)}(\Omega)} := ||u||_{L^{\pi(.)}(\Omega)} + ||\nabla u||_{L^{\pi(.)}(\Omega)}.$$

$W_0^{1,\pi(.)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,\pi(.)}(\Omega)$.
Further, $\dot{E}^{\pi(.)}(\Omega)$ is the set of all $f \in W_0^{1,1}(\Omega)$ such that $\nabla f \in L^{\pi(.)}(\Omega)$. This space is equipped with the norm

$$||u||_{\dot{E}^{\pi(.)}(\Omega)} := ||\nabla u||_{L^{\pi(.)}(\Omega)}.$$

When $1 < p_- \leq \pi(.) \leq p_+ < +\infty$, all the above spaces are separable and reflexive Banach spaces.

Generally, $W_0^{1,\pi(.)}(\Omega) \subsetneq \dot{E}^{\pi(.)}(\Omega)$. In the present paper, we assume that $\pi(x) = p(x,u(x))$ verify the log-Hölder continuity assumption (10) below.
Furthermore, we denote $\pi_n(x) := p(x,u_n(x))$.

Proposition 2.2 (See [1], Proposition 2.3)**.** *For all measurable function $\pi : \Omega \to [p_-, p_+]$, the following properties hold.*

i) $L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$ are separable and reflexive Banach spaces.

ii) $L^{\pi'(.)}(\Omega)$ can be identified with the dual space of $L^{\pi(.)}(\Omega)$, and the following Hölder inequality holds:

$$\forall f \in L^{\pi(.)}(\Omega), g \in L^{\pi'(.)}(\Omega), \qquad \left| \int_\Omega fgdx \right| \leq 2||f||_{L^{\pi(.)}(\Omega)}||g||_{L^{\pi'(.)}(\Omega)}.$$

iii) One has $\rho_{\pi(.)}(f) = 1$ if and only if $||f||_{L^{\pi(.)}(\Omega)} = 1$; further,

$$\text{if } \rho_{\pi(.)}(f) \leq 1, \text{ then } ||f||^{p_+}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_-}_{L^{\pi(.)}(\Omega)};$$

$$\text{if } \rho_{\pi(.)}(f) \geq 1, \text{ then } ||f||^{p_-}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_+}_{L^{\pi(.)}(\Omega)}.$$

In particular, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^{\pi(.)}(\Omega)$, then $||f_n||_{L^{\pi(.)}(\Omega)}$ tends to zero (resp., to infinity) if and only if $\rho_{\pi(.)}(f_n)$ tends to zero (resp., to infinity), as $n \to \infty$.

Young measures and nonlinear weak-* convergence.
Throughout the paper, we denote by δ_c the Dirac measure on $\mathbb{R}^d$ ($d \in \mathbb{N}$), concentrated at the point $c \in \mathbb{R}^d$.
In the following theorem, we gather the results of Ball [5], Pedregal [12] and Hungerbühler [10] which are needed for our purposes (we limit the statement to the case of a bounded domain Ω). Let us underline that the results of (ii),(iii), expressed in terms of the convergence in measure, are very convenient for the applications we have in mind.

Theorem 2.3. *(i) Let $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$, and a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ -valued functions, $d \in \mathbb{N}$, such that $(v_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$ ($d \in \mathbb{N}$) , weakly measurable in x with respect to the Lebesgue measure on Ω, such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, we have*

$$\lim_{k\to\infty} \int_\Omega F(x, v_{n_k})dx = \int_\Omega \int_{\mathbb{R}^d} F(x,\lambda)d\nu_x(\lambda)dx, \tag{8}$$

whenever the sequence $(F(.,v_n(.)))_{n\in\mathbb{N}}$ is equi-integrable on Ω.
In particular,

$$v(x) := \int_{\mathbb{R}^d} \lambda d\nu_x(\lambda) \tag{9}$$

is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ in $L^1(\Omega)$.
The family $(\nu_x)_{x\in\Omega}$ is called the Young measure generated by the subsequence $(v_{n_k})_{k\in\mathbb{N}}$.

(ii) If Ω is of finite measure, and $(\nu_x)_{x\in\Omega}$ is the Young measure generated by a sequence $(v_n)_{n\in\mathbb{N}}$, then $\nu_x = \delta_{v(x)}$ for a.e. $x \in \Omega \Leftrightarrow v_n$ converges in measure on Ω to v as $n \to \infty$.

(iii) If Ω is of finite measure, $(u_n)_{n\in\mathbb{N}}$ generates a Dirac Young measure $(\delta_{u(x)})_{x\in\Omega}$ on $\mathbb{R}^{d_1}$, and $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$ on $\mathbb{R}^{d_2}$, then the sequence $(u_n, v_n)_{n\in\mathbb{N}}$ generates the Young measure $(\delta_{u(x)} \otimes \nu_x)_{x\in\Omega}$ on $\mathbb{R}^{d_1+d_2}$.

Remark 2.4. Whenever a sequence $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$, following the terminology of [9] we will say that $(v_n)_{n\in\mathbb{N}}$ nonlinear weak-* converges, and $(\nu_x)_{x\in\Omega}$ is the nonlinear weak-* limit of the sequence $(v_n)_{n\in\mathbb{N}}$. In the case $(v_n)_{n\in\mathbb{N}}$ possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact, (See [1], Theorem 2.10(i)). It means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on Ω.

The following lemma prove that the space $W_0^{1,\pi(.)}(\Omega)$ is stable by truncation. See, [1]-Lemma 2.9.

Lemma 2.5. *If $u \in W_0^{1,\pi(.)}(\Omega)$ then $T_k(u) \in W_0^{1,\pi(.)}(\Omega)$, for all $k > 0$.*

Notice that $\dot{E}^{\pi(.)}(\Omega)$ is also stable by truncation, since $W_0^{1,1}(\Omega)$ is stable by truncation and $|\nabla T_k(u)| \le |\nabla u| \in L^{\pi(.)}(\Omega)$ whenever $u \in \dot{E}^{\pi(.)}(\Omega)$.
From the results of Fan and Zhikov (See [1], Corollary 2.6), we deduce the following.

Lemma 2.6. *Assume that $\pi : \Omega \to [p_-, p_+]$ has a representative which can be extended into a continuous function up to the boundary $\partial\Omega$ and satisfying the log-Hölder continuity assumption:*

$$\exists L > 0,\ \ \forall x, y \in \overline{\Omega}, x \neq y,\ \ -\big(\log|x-y|\big)|\pi(x) - \pi(y)| \leq L. \qquad (10)$$

Then, $\mathcal{D}(\Omega)$ is dense in $\dot{E}^{\pi(.)}(\Omega)$. In particular, the spaces $\dot{E}^{\pi(.)}(\Omega)$ and $W_0^{1,\pi(.)}(\Omega)$ are Lipschitz homeomorphic and then they can be identified.

We have the following Lemma $\big($See, [1]-Theorem 3.11 and [2]- Step 2-Proof of Theorem 2.6$\big)$.

Lemma 2.7. *Assume that $(u_n)_{n\in\mathbb{N}}$ converges a.e. on Ω to some function u, then*

$$\begin{aligned} &|p(x,u_n(x)) - p(x,u(x))| \ \ \textit{converges in measure to}\ \ 0\ \ \textit{on}\ \ \Omega,\\ &\textit{and for all bounded subset}\ \ K\ \ \textit{of}\ \ \mathbb{R}^N,\\ &\sup_{\xi\in K}|a(x,u_n(x),\xi) - a(x,u(x),\xi)| \quad \textit{converges in measure to}\ \ 0\ \ \textit{on}\ \ \Omega. \end{aligned} \qquad (11)$$

3. Weak solution

We consider the following problem

$$(P_n)\begin{cases} b(u_n) - \mathrm{div} a_n(x,u_n,\nabla u_n) = \mu & \text{in} \quad \Omega \\ \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $a_n(x,z,\xi) := a(x,z,\xi) + \dfrac{1}{n}|\xi|^{p_+-2}\xi$ and $\mu \in W^{-1,(p_-)'}(\Omega)$.

Lemma 3.1. *a_n verifies the assumptions (2)-(5) with $p(x,z)$ replaced by the constant exponent p_+ and with $\mathcal{M}$, C no depending on n.*

Proof.

$$a_n(x,z,0) = 0,$$

since $a(x,z,0) = 0$.

$$(a_n(x,z,\xi) - a_n(x,z,\eta))(\xi - \eta) > 0,$$

since a verifies (3) and

$$(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)(\xi - \eta) > 0, \quad \forall \xi, \eta \in \mathbb{R}^N, \quad \xi \neq \eta.$$

Now, we prove that

$$|a_n(x,z,\xi)|^{p'_+} \leqslant C(|\xi|^{p_+} + \mathcal{M}(x)),$$

where $C > 0$ and $\mathcal{M} \in L^1(\Omega)$.

We have

$$\begin{aligned} |a_n(x,z,\xi)|^{p'_+} &\leq \left(|a(x,z,\xi)| + \frac{1}{n}|\xi|^{p_+-1}\right)^{p'_+} \\ &\leq 2^{p'_+-1}\left(|a(x,z,\xi)|^{p'_+} + \frac{1}{n^{p'_+}}|\xi|^{p'_+(p_+-1)}\right), \end{aligned}$$

since

$$\frac{1}{2^p}(a+b)^p \leqslant \frac{1}{2}(a^p + b^p),$$

for all $a, b > 0$ and $p > 1$.

Thus, we obtain

$$\begin{aligned} |a_n(x,z,\xi)|^{p'_+} &\leq 2^{p'_+-1}\left[C_1|\xi|^{p_+} + C_1\mathcal{M}(x) + \frac{1}{n^{p'_+}}|\xi|^{p_+}\right] \\ &\leq (C_1+1)2^{p'_+-1}\left(\mathcal{M}(x) + |\xi|^{p_+}\right) \end{aligned} \tag{12}$$

since a verifies (4) with $C_1 > 0$ and $p'_+ = \dfrac{p_+}{p_+ - 1}$.
Let us set

$$C = (C_1+1)2^{p'_+-1},$$

we get

$$|a_n(x,z,\xi)|^{p'_+} \leqslant C(|\xi|^{p_+} + \mathcal{M}(x)).$$

Finally, we prove that

$$a_n(x,z,\xi).\xi \geq \frac{1}{C}|\xi|^{p_+}.$$

We have

$$\begin{aligned} a_n(x,z,\xi).\xi &= a(x,z,\xi).\xi + \frac{1}{n}|\xi|^{p_+-2}\xi.\xi \\ &= a(x,z,\xi).\xi + \frac{1}{n}|\xi|^{p_+} \\ &\geq \frac{1}{C}|\xi|^{p_+} + \frac{1}{n}|\xi|^{p_+} \text{(due to the fact that } a \text{ verifies (5))} \\ &\geq \frac{1}{C}|\xi|^{p_+}. \end{aligned}$$

□

Remark 3.2. $p_+ \geq p_-$ implies that $W^{-1,(p_-)'}(\Omega) \subset W^{-1,(p_+)'}(\Omega)$ and as $\mu \in W^{-1,(p_-)'}(\Omega)$; therefore, $\mu \in W^{-1,(p_+)'}(\Omega)$.

From Lemma 3.1, the above Remark and thanks to [1]-Theorem 3.11, there exists a weak solution u_n of the problem (P_n) in $W_0^{1,p_+}(\Omega)$ in the sense

$$b(u_n) \in L^1(\Omega) \text{ and } b(u_n) - \text{div} a_n(x, u_n, \nabla u_n) = \mu \text{ is fulfilled in } \mathcal{D}'(\Omega). \quad (13)$$

Now, we introduce the notion of weak solution of the problem (1).

Definition 3.3. Let $\mu \in W^{-1,(p_-)'}(\Omega)$. A function $u \in \dot{E}^{\pi(.)}(\Omega)$ with $\pi(.) = p(., u(.))$ is called weak solution of the problem (1), if $b(u) \in L^1(\Omega)$ and

$$\int_\Omega b(u)\varphi dx + \int_\Omega a(x, u, \nabla u)\nabla\varphi dx = \int_\Omega \mu\varphi dx, \quad (14)$$

for all $\varphi \in \dot{E}^{\pi(.)}(\Omega) \cap L^\infty(\Omega)$.

Notice that the integrals are well defined. Indeed, as $\varphi \in \dot{E}^{\pi(.)}(\Omega) \cap L^\infty(\Omega)$, then, the first integral of the left hand-side and the right hand-side of (14) are well defined. The second integral of the left hand-side is also well defined thanks to (4).

We are going to prove that these approximated solutions u_n of the problem (P_n) tend, as n goes to infinity, to a measurable function u which is a weak solution of the problem (1). Now, we are going to prove the following existence result.

Theorem 3.4. *Assume that (2)-(6) hold and* $\mu \in W^{-1,(p_-)'}(\Omega)$. *Then, there exists at least one weak solution to the problem (1).*

Proof of Theorem 3.4. The proof is divided into several assertions. These assertions are based on the Young measure and nonlinear weak $-^*$ convergence results (see [5, 10, 12]).

Assertion 1. The sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges to a Young measure $\nu_x(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence and

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda). \quad (15)$$

Proof. By using (5) with variable exponent $p(x, u_n(x))$, the definition of a_n and (13), the sequence $(u_n)_{n\in\mathbb{N}}$ verifies

$$\int_\Omega b(u_n)u_n dx + \frac{1}{C_2}\int_\Omega |\nabla u_n|^{p(x,u_n(x))} dx + \frac{1}{n}\int_\Omega |\nabla u_n|^{p_+} dx \leq \int_\Omega \mu u_n dx. \quad (16)$$

Applying Young's inequality on the right hand side of (16), we get

$$\begin{aligned}\int_\Omega \mu u_n dx &\leq \int_\Omega |\mu||u_n|dx \\ &\leq \|\mu\|_{W^{-1,(p_-)'}(\Omega)}\|u_n\|_{W_0^{1,p_-}(\Omega)} \\ &= \left(\frac{2C_2}{p_-}\right)^{\frac{1}{p_-}}\|\mu\|_{W^{-1,(p_-)'}(\Omega)}.\left(\frac{p_-}{2C_2}\right)^{\frac{1}{p_-}}\|u_n\|_{W_0^{1,p_-}(\Omega)} \\ &\leq \frac{1}{p'_-}\left(\frac{2C_2}{p_-}\right)^{\frac{p'_-}{p_-}}\|\mu\|^{(p_-)'}_{W^{-1,(p_-)'}(\Omega)} + \frac{1}{2C_2}\|u_n\|^{p_-}_{W_0^{1,p_-}(\Omega)}. \qquad (17)\end{aligned}$$

Moreover, as $p_- < \pi_n(.)$, we have

$$\int_\Omega |\nabla u_n|^{p_-}dx \leq meas(\Omega) + \int_\Omega |\nabla u_n|^{\pi_n(.)}dx. \qquad (18)$$

Furthermore, using (16), (17) and (18), we obtain

$$\begin{aligned}\int_\Omega b(u_n)u_n dx &+ \frac{1}{C_2}\int_\Omega |\nabla u_n|^{p(x,u_n(x))}dx + \frac{1}{n}\int_\Omega |\nabla u_n|^{p_+}dx \\ &\leq \frac{1}{p'_-}\left(\frac{2C_2}{p_-}\right)^{\frac{p'_-}{p_-}}\|\mu\|^{(p_-)'}_{W^{-1,(p_-)'}(\Omega)} \\ &+ \frac{meas(\Omega)}{2C_2} + \frac{1}{2C_2}\int_\Omega |\nabla u_n|^{\pi_n(.)}dx. \qquad (19)\end{aligned}$$

We infer from the above inequality that

$$\begin{aligned}\int_\Omega b(u_n)u_n dx + \frac{1}{2C_2}\int_\Omega |\nabla u_n|^{p(x,u_n(x))}dx + \frac{1}{n}\int_\Omega |\nabla u_n|^{p_+}dx \\ \leq Const\big(\mu, meas(\Omega), p_-\big), \qquad (20)\end{aligned}$$

where

$$Const\big(\mu, meas(\Omega), p_-\big) := \frac{1}{p'_-}\left(\frac{2C_2}{p_-}\right)^{\frac{p'_-}{p_-}}\|\mu\|^{(p_-)'}_{W^{-1,(p_-)'}(\Omega)} + \frac{meas(\Omega)}{2C_2}.$$

Thus, we deduce from the inequality (20) that

$$\int_\Omega |\nabla u_n|^{p(x,u_n(x))}dx \leq C_3, \qquad (21)$$

and $\frac{1}{n}|\nabla u_n|^{p_+-2}\nabla u_n$ converges to 0 in $L^1(\Omega)$, as n tends to ∞. Thus, using (18) and (21) the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,p_-}(\Omega)$. Then, up to a subsequence, u_n converges a.e. on Ω (and also weakly in $W_0^{1,p_-}(\Omega)$) to a limit u; using the representation of weakly convergent sequences in $L^1(\Omega)$ in terms of Young measures (see Theorem 2.3 and formula (9)), we can write

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda).$$

□

Assertion 2.
$|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N \times \Omega$. Moreover, $u \in \dot{E}^{\pi(.)}(\Omega)$.

Proof. We know that π_n converges in measure to π. Using Theorem 2.3 ii), iii), $(\pi_n, \nabla u_n)_{n\in\mathbb{N}}$ converges on $\mathbb{R}\times\mathbb{R}^N$ to the Young measure $\mu_x = \delta_{\pi(x)} \otimes \nu_x$. Thus, we can apply the weak convergence properties (8) to the Carathéodory function

$$F_m : (x, \lambda_0, \lambda) \in \Omega \times (\mathbb{R}\times\mathbb{R}^N) \mapsto |h_m(\lambda)|^{\lambda_0},$$

with $m \in \mathbb{N}$, where h_m is defined in the preliminary. We have

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^N} |h_m(\lambda)|^{\pi(.)} d\nu_x(\lambda)dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)} |h_m(\lambda)|^{\lambda_0} d\mu_x(\lambda_0,\lambda)dx \\
&= \int_\Omega \int_{\mathbb{R}\times\mathbb{R}^N} F_m(x,\lambda_0,\lambda) d\mu_x(\lambda_0,\lambda)dx \\
&= \lim_{n\to+\infty} \int_\Omega F_m(x,\pi_n(x),\nabla u_n(x))dx \\
&= \lim_{n\to+\infty} \int_\Omega |h_m(\nabla u_n)|^{\pi_n(.)}dx \\
&\le \lim_{n\to+\infty} \int_\Omega |\nabla u_n|^{\pi_n(.)}dx \\
&\le C_3.
\end{aligned}$$

Since $h_m(\lambda) \to \lambda$, as $m \to \infty$ and $m \mapsto h_m(\lambda)$ is increasing, then using Lebesgue convergence Theorem, we deduce that

$$\int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(.)} d\nu_x(\lambda)dx \le C_3.$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N \times \Omega$. Now, we prove that $\nabla u \in L^{\pi(.)}(\Omega)$. Using (15), Jensen inequality and the above inequality, we get

$$\int_\Omega |\nabla u|^{\pi(.)}dx = \int_\Omega \left| \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda) \right|^{\pi(.)} dx \le \int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(.)} d\nu_x(\lambda)dx < \infty.$$

Thus, $\nabla u \in L^{\pi(.)}(\Omega)$. Moreover, $u \in W_0^{1,p_-}(\Omega) \subset W_0^{1,1}(\Omega)$.
Hence, $u \in \dot{E}^{\pi(.)}(\Omega)$. □

Assertion 3.
The sequence Φ_n defined by $\Phi_n := a(x,u_n,\nabla u_n)$ is equi-integrable on Ω.

Proof. By using (4) with exponent $\pi_n(.)$, we obtain

$$|a(x,u_n,\nabla u_n)|^{\pi'_n(.)} \leq C_1(|\nabla u_n|^{\pi_n(.)} + \mathcal{M}(x)).$$

The above inequality give us

$$\begin{aligned} |a(x,u_n,\nabla u_n)| &\leq C\big((1+|\nabla u_n|^{\pi_n(.)}) + \mathcal{M}(x)\big)^{\frac{1}{\pi'_n(.)}} \\ &\leq C\big((1+\mathcal{M}(x))^{\frac{1}{\pi'_n(.)}} + |\nabla u_n|^{\frac{\pi_n(.)}{\pi'_n(.)}}\big) \\ &\leq C(1+\mathcal{M}(x) + |\nabla u_n|^{\pi_n(.)-1}). \end{aligned}$$

For all set $E \subset \Omega$,

$$\begin{aligned} \int_E |a(x,u_n,\nabla u_n)|dx &\leq C\int_E (1+\mathcal{M}(x))dx \\ &+ C_4\Big|\Big|\,|\nabla u_n|^{\pi_n(.)-1}\Big|\Big|_{L^{\pi'_n(.)}(\Omega)} ||\chi_E||_{L^{\pi_n(.)}(\Omega)}, \end{aligned}$$

where $C_4 = const(p_-)$. The first term of the right hand side of the last inequality is small for $meas(E)$ small enough, since $(1+\mathcal{M}) \in L^1(\Omega)$.
According to Proposition 2.2, we obtain

$$\begin{aligned} ||\chi_E||_{L^{\pi_n(.)}(\Omega)} &\leq \max\left\{\rho_{\pi_n(.)}(\chi_E)^{\frac{1}{p_+}}; \rho_{\pi_n(.)}(\chi_E)^{\frac{1}{p_-}}\right\} \\ &= \max\left\{(meas(E))^{\frac{1}{p_-}}, (meas(E))^{\frac{1}{p_+}}\right\}. \end{aligned}$$

Analogously,

$$\begin{aligned} \Big|\Big|\,|\nabla u_n|^{\pi_n(.)-1}\Big|\Big|_{L^{\pi'_n(.)}(\Omega)} &\leq \max\left\{\left(\rho_{\pi'_n(.)}(|\nabla u_n|^{\pi_n(.)-1})\right)^{\frac{1}{(p')_+}}, \left(\rho_{\pi'_n(.)}(|\nabla u_n|^{\pi_n(.)-1})\right)^{\frac{1}{(p')_-}}\right\} \\ &= \max\left\{\left(\int_\Omega |\nabla u_n|^{\pi_n(.)}dx\right)^{\frac{1}{(p')_+}}, \left(\int_\Omega |\nabla u_n|^{\pi_n(.)}dx\right)^{\frac{1}{(p')_-}}\right\}. \end{aligned}$$

Using (21), $\int_E |a(x,u_n,\nabla u_n)|dx$ is small for $meas(E)$ small enough.
Hence, $(\Phi_n)_{n\in\mathbb{N}}$ is equi-integrable. □

Assertion 4.
The weak limit Φ of Φ_n (or a subsequence) belongs to $L^{\pi'(.)}(\Omega)$ and we have

$$\Phi(x) = \int_{\mathbb{R}^N} a(x,u(x),\lambda)d\nu_x(\lambda). \tag{22}$$

Proof. Set $\tilde{\Phi}_n = a(x,u(x),\nabla v_n)$ with $\nabla v_n = \nabla u_n \chi_{S_n}$, where
$S_n = \{x \in \Omega, \quad |\pi(x) - \pi_n(x)| < \varepsilon\}$ and $0 < \varepsilon < 1$.

Firstly, we prove that $\tilde{\Phi}_n$ is equi-integrable on Ω. We applied (4) with variable exponent $\pi(.)$ on $\tilde{\Phi}_n(x)$. Let $E \subset \Omega$, we have

$$\begin{aligned}\int_E |a(x,u(x),\nabla v_n)|dx &\leq C\int_E (1+\mathcal{M}(x)+|\nabla v_n|^{\pi(.)-1})dx \\ &\leq C\bigg(\int_E (1+\mathcal{M}(x))dx + \int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1}dx\bigg).\end{aligned}$$

The first term of the right hand side of the last inequality is small for $meas(E)$ small enough.
For all $x \in S_n$, $\pi(x) < \pi_n(x) + \varepsilon$, thus

$$\int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1}dx \leq \int_E \left(1+|\nabla u_n|^{\pi_n(.)+\varepsilon-1}\right)dx$$

and

$$\int_\Omega \left(|\nabla u_n|^{\pi_n(.)+\varepsilon-1}\right)^{\left(\frac{\pi_n(.)}{1-\varepsilon}\right)'} dx = \int_\Omega |\nabla u_n|^{\pi_n(.)}dx < \infty, \tag{23}$$

which is equivalent to saying $|\nabla u_n|^{\pi_n(.)+\varepsilon-1} \in L^{\left(\frac{\pi_n(.)}{1-\varepsilon}\right)'}(\Omega)$. By using Hölder type inequality, we get

$$\begin{aligned}\int_{E\cap S_n} |\nabla u_n|^{\pi(.)-1}dx &\leq \int_E \left(1+|\nabla u_n|^{\pi_n(.)+\varepsilon-1}\right)dx \leq meas(E) \\ &+ 2\bigg|\bigg|\,|\nabla u_n|^{\pi_n(.)+\varepsilon-1}\bigg|\bigg|_{L^{\left(\frac{\pi_n(.)}{1-\varepsilon}\right)'}(\Omega)} ||\chi_E||_{L^{\left(\frac{\pi_n(.)}{1-\varepsilon}\right)}(\Omega)}. \end{aligned} \tag{24}$$

According to Proposition 2.2,

$$\begin{aligned}||\chi_E||_{L^{\left(\frac{\pi_n(.)}{1-\varepsilon}\right)}(\Omega)} &\leq \max\left\{\left(\rho_{\frac{\pi_n(.)}{1-\varepsilon}}(\chi_E)\right)^{\frac{1}{\frac{p_-}{1-\varepsilon}}}, \left(\rho_{\frac{\pi_n(.)}{1-\varepsilon}}(\chi_E)\right)^{\frac{1}{\frac{p_+}{1-\varepsilon}}}\right\} \\ &= \max\left\{(meas(E))^{\frac{1-\varepsilon}{p_-}}, (meas(E))^{\frac{1-\varepsilon}{p_+}}\right\}.\end{aligned}$$

Applying (23) and Proposition 2.2, it follows that $\left\|\left|\nabla u_n\right|^{\pi_n(.)+\varepsilon-1}\right\|_{L^{\left(\frac{\pi_n(.)}{1-\varepsilon}\right)'}(\Omega)}$ is bounded.

Therefore, the right hand side of (24) is uniformly small for $meas(E)$ small enough.

Hence, $\tilde{\Phi}_n$ is equi-integrable (up to a subsequence) and weakly converges in $L^1(\Omega)$ to $\tilde{\Phi}$, as $n \to \infty$.

Now, we prove that $\tilde{\Phi} = \Phi$; more precisely, we show that $\tilde{\Phi}_n - \Phi_n$ strongly converges in $L^1(\Omega)$ to 0.

From (21), $\int_\Omega |\nabla u_n|^{\pi_n(.)}dx$ is uniformly bounded, which implies that $\int_\Omega |\nabla u_n|dx$ is finite, since

$$\int_\Omega |\nabla u_n|dx \leq \int_\Omega (1+|\nabla u_n|^{\pi_n(x)})dx.$$

By Chebyschev inequality, we have

$$meas(\{|\nabla u_n| > L\}) \leq \frac{\int_\Omega |\nabla u_n|dx}{L}.$$

Therefore, $\sup_{n\in\mathbb{N}} meas(\{|\nabla u_n| > L\})$ tends to 0 for L large enough. Since $\tilde{\Phi}_n - \Phi_n$ is equi-integrable, then for all $\beta > 0$, there exists $\delta = \delta(\beta)$ such that for all $A \subset \Omega$, $meas(A) < \delta$ and $\int_A |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4}$.

Therefore, if we choose L large enough, we get $\frac{\int_\Omega |\nabla u_n|dx}{L} < \delta$, so $meas(\{|\nabla u_n| > L\}) < \delta$. Hence,

$$\int_{\{|\nabla u_n|>L\}} |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4}.$$

By lemma 2.7, we also have

$$meas\left(\left\{x \in \Omega;\ \sup_{\lambda\in K} |a(x,u_n(x),\lambda) - a(x,u(x),\lambda)| \geq \sigma\right\}\right) \longrightarrow 0,$$

as $n \to \infty$.

Thus, by the above equi-integrability, for all $\sigma > 0$, there exists $n_0 = n_0(\sigma, L) \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\int_{\left\{x\in\Omega;\ \sup_{|\lambda|\leq L} |a(x,u_n(x),\lambda)-a(x,u(x),\lambda)|\geq\sigma\right\}} |\tilde{\Phi}_n - \Phi_n|dx < \frac{\beta}{4}.$$

Using the definition of Φ_n and $\tilde{\Phi}_n$, we have

$$\tilde{\Phi}_n - \Phi_n = a(x,u_n(x),\nabla u_n) - a(x,u(x),\nabla u_n) \text{ on } S_n.$$

Now, we reason on

$$S_{n,L,\sigma} := \left\{ x \in \Omega;\ \sup_{|\lambda|\leq L} |a(x,u_n(x),\lambda) - a(x,u(x),\lambda)| < \sigma, |\nabla u_n| \leq L \right\}.$$

We get

$$\begin{aligned} \int_{S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n| dx &\leq \int_{S_{n,L,\sigma}} \sup_{|\lambda|\leq L} |a(x,u_n(x),\lambda) - a(x,u(x),\lambda)| dx \\ &\leq \sigma meas(\Omega). \end{aligned}$$

We observe that

$$\int_{S_n} |\tilde{\Phi}_n - \Phi_n| dx = \int_{S_n \cap S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n| dx + \int_{S_n \setminus S_{n,L,\sigma}} |\tilde{\Phi}_n - \Phi_n| dx$$

and

$$S_n \setminus S_{n,L,\sigma} \subset \left\{ x \in \Omega;\ \sup_{|\lambda|\leq L} |a(x,u_n(x),\lambda) - a(x,u(x),\lambda)| \geq \sigma \right\} \cup \left\{ |\nabla u_n| > L \right\}.$$

Consequently, by choosing $\sigma = \sigma(\beta) < \dfrac{\beta}{4meas(\Omega)}$, we get

$$\int_{S_n} |\tilde{\Phi}_n - \Phi_n| dx < \frac{\beta}{4} + \frac{\beta}{4} + \frac{\beta}{4} = \frac{3\beta}{4},$$

for all $n \geq n_0(\sigma, L)$.
By lemma 2.7, we also have $meas(\{x \in \Omega, |\pi(x) - \pi_n(x)| \geq \varepsilon\}) \to 0$ for n large enough; which means that $meas(\Omega \setminus S_n)$ converges to 0 for n large enough. Thus,

$$\int_{\Omega \setminus S_n} |\tilde{\Phi}_n - \Phi_n| dx = \int_{\Omega \setminus S_n} |\Phi_n| dx \leq \frac{\beta}{4}.$$

Therefore, for all $\beta > 0$, there exists $n_0 = n_0(\beta)$ such that for all $n \geq n_0$, $\int_\Omega |\tilde{\Phi}_n - \Phi_n| dx \leq \beta$. Hence, $\tilde{\Phi}_n - \Phi_n$ strongly converges to 0 in $L^1(\Omega)$. We prove that

$$\Phi(x) = \int_{\mathbb{R}^N} a(x,u(x),\lambda) d\nu_x(\lambda) \quad a.e.\ x \in \Omega \text{ and } \Phi \in L^{\pi'(\cdot)}(\Omega).$$

Notice that

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n| (1 - \chi_{S_n}) dx = \lim_{n\to\infty} \int_{\Omega \setminus S_n} |\nabla u_n| dx = 0,$$

since $(\nabla u_n)_{n\in\mathbb{N}}$ is equi-integrable and $meas(\Omega \setminus S_n)$ converges to 0 for n large enough.
Therefore, $(\nabla u_n)_{n\in\mathbb{N}}$ and $\nabla u_n \chi_{S_n}$ converge to the same Young measure $\nu_x(\lambda)$.
Moreover, by applying Theorem 2.3-i) to the Carathéodory function $F(x,(\lambda_0,\lambda)) := a(x,\lambda_0,\lambda)$, we infer that

$$\tilde{\Phi}(x) = \Phi(x) = \int_{\mathbb{R}^N} a(x,u(x),\lambda)d\nu_x(\lambda) \text{ a.e. } x \in \Omega.$$

Using (4), it follows that $|a(x,u(x),\lambda)|^{\pi'(.)} \leq C(\mathcal{M}(x) + |\lambda|^{\pi(.)})$. Thus, with Jensen Inequality, it follows that

$$\begin{aligned}
\int_\Omega |\Phi(x)|^{\pi'(.)}dx &= \int_\Omega \left| \int_{\mathbb{R}^N} a(x,u(x),\lambda)d\nu_x(\lambda) \right|^{\pi'(.)} dx \\
&\leq \int_{\Omega\times\mathbb{R}^N} |a(x,u(x),\lambda)|^{\pi'(.)}d\nu_x(\lambda)dx \\
&\leq C\int_{\Omega\times\mathbb{R}^N} \left(\mathcal{M}(x) + |\lambda|^{\pi(.)}\right) d\nu_x(\lambda)dx < \infty.
\end{aligned}$$

Hence, $\Phi \in L^{\pi'(.)}(\Omega)$. □

Assertion 5.

$$\int_\Omega \Phi.\nabla u dx \geq \int_{\Omega\times\mathbb{R}^N} a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx. \tag{25}$$

Proof. For all $\varphi \in \mathcal{D}(\Omega)$, we have

$$\int_\Omega \left[b(u_n)\varphi + a(x,u_n,\nabla u_n)\nabla\varphi + \frac{1}{n}|\nabla u_n|^{p_+-2}\nabla u_n \nabla\varphi\right]dx = \int_\Omega \mu\varphi dx. \tag{26}$$

Letting n goes to ∞ in the above equality, we obtain

$$\int_\Omega \left[b(u)\varphi + \Phi\nabla\varphi\right]dx = \int_\Omega \mu\varphi dx, \tag{27}$$

where Φ is a weak limit of the sequence $(a(x,u_n,\nabla u_n))_{n\in\mathbb{N}}$ given by (22). Moreover, $b(u_n)$ converges to $b(u)$ in $L^1(\Omega)$, thanks to Lebesgue dominated convergence Theorem. Since $b(u_n)$ converges to $b(u)$ a.e in Ω and $(b(u_n))_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$, due to the fact that $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$ $\big((u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,p_-}(\Omega)$ and $W_0^{1,p_-}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p_- > N\big)$. By density argument, we can replace φ with u_n in (26) to get

$$\int_\Omega \left[b(u_n)u_n + a(x,u_n,\nabla u_n)\nabla u_n + \frac{1}{n}|\nabla u_n|^{p_+}\right]dx = \int_\Omega \mu u_n dx. \tag{28}$$

Further, $u \in W_0^{1,p_-}(\Omega) \subset C^{0,\alpha}(\overline{\Omega})$ for $p_- > N$ and $p(.,.)$ is locally uniformly log-Hölder continuous, then $p(.,u(.)) := \pi(.)$ verifies (10). Therefore, $\mathcal{D}(\Omega)$ is dense in $\dot{E}^{\pi(.)}(\Omega)$, so, we change φ by u in (27) to get

$$\int_\Omega \left[b(u)u + \Phi\nabla u\right] dx = \int_\Omega \mu u dx. \tag{29}$$

The sequence $(b(u_n)u_n)_{n\in\mathbb{N}}$ is nonnegative and converges a.e. to $b(u)u$. By Fatou's Lemma, we obtain

$$\liminf_{n\to\infty} \int_\Omega b(u_n)u_n dx \geq \int_\Omega b(u)u dx. \tag{30}$$

Moreover, we have

$$\lim_{n\to\infty} \int_\Omega \mu u_n dx = \int_\Omega \mu u dx, \tag{31}$$

since $u_n \rightharpoonup u$ in $W_0^{1,p_-}(\Omega)$ and $\mu \in W^{-1,(p_-)'}(\Omega)$. Combining (30) and (31), we get

$$\liminf_{n\to\infty} \int_\Omega \mu u_n dx - \int_\Omega b(u)u dx \geq \liminf_{n\to\infty} \left(\int_\Omega \mu u_n dx - \int_\Omega b(u_n)u_n dx \right),$$

which is equivalent to say

$$\begin{aligned} \int_\Omega \mu u dx - \int_\Omega b(u)u dx &\geq \liminf_{n\to\infty} \left(\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n dx + \frac{1}{n}\int_\Omega |\nabla u_n|^{p_+} dx \right) \\ &\geq \liminf_{n\to\infty} \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n dx. \end{aligned}$$

Hence, by using (28) and (29), we obtain

$$\int_\Omega \Phi\nabla u dx \geq \liminf_{n\to\infty} \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n dx. \tag{32}$$

By ([1], Lemma 2.1), $m \mapsto a(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n)$ is increasing and converges to $a(x,u_n,\nabla u_n).\nabla u_n$ for m large enough; then,

$$a(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n) \leq a(x,u_n,\nabla u_n).\nabla u_n.$$

Therefore, by using (32) and Theorem 2.3, we have

$$\begin{aligned} \int_\Omega \Phi.\nabla u dx &\geq \liminf_{n\to\infty} \int_\Omega a(x,u_n,\nabla u_n).\nabla u_n dx \\ &\geq \lim_{n\to\infty} \int_\Omega a(x,u_n,h_m(\nabla u_n)).h_m(\nabla u_n) dx \\ &= \int_{\Omega\times\mathbb{R}^N} a(x,u,h_m(\lambda)).h_m(\lambda) d\nu_x(\lambda) dx. \end{aligned} \tag{33}$$

Using Lebesgue convergence Theorem in (33), as m goes to ∞, we get (25). □

Assertion 6. The "div-curl" inequality holds.

$$\int_{\Omega\times\mathbb{R}^N}\big(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x)\big).(\lambda-\nabla u(x))d\nu_x(\lambda)dx\leq 0.$$

Proof. We have

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^N}(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x))).(\lambda-\nabla u(x))d\nu_x(\lambda)dx\\
=&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx-\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\nabla u(x)d\nu_x(\lambda)dx\\
-&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\nabla u(x)).\lambda d\nu_x(\lambda)dx+\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\nabla u(x)).\nabla u(x)d\nu_x(\lambda)dx\\
=&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx-\int_\Omega\left(\int_{\mathbb{R}^N}a(x,u(x),\lambda)d\nu_x(\lambda)\right)\nabla u(x)dx\\
-&\int_\Omega a(x,u(x),\nabla u(x)).\left(\int_{\mathbb{R}^N}\lambda d\nu_x\right)dx+\int_\Omega a(x,u(x),\nabla u(x)).\nabla u(x)\left(\int_{\mathbb{R}^N}d\nu_x\right)dx\\
=&\int_{\Omega\times\mathbb{R}^N}a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx-\int_\Omega\Phi.\nabla u dx\leq 0.
\end{aligned}$$

We pass from the first equality to the second equality by using Fubini Theorem and from the second equality to the third one, by using (15) and the fact that ν_x is a probability measure on $\mathbb{R}^N$. Finally, (22) and (25) give us the desired inequality. □

Assertion 7.
$\Phi(x)=a(x,u(x),\nabla u(x))$ a.e. $x\in\Omega$ and ∇u_n converges to ∇u in measure on Ω.

Proof. From Assertion 6 and relation (3), we deduce that

$$\big(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x))\big).\big(\lambda-\nabla u(x)\big)=0\quad a.e.\quad x\in\Omega,\quad \lambda\in\mathbb{R}^N.$$

Thus, $\lambda=\nabla u(x)$ a.e. $x\in\Omega$ wrt ν_x on $\mathbb{R}^N$; therefore $\nu_x(\nabla u(x))=1$ and $\delta_{\nabla u}=\nu_x$. By using (22), we get

$$\Phi(x)=\int_{\mathbb{R}^N}a(x,u(x),\lambda)d\nu_x(\lambda)=a(x,u(x),\nabla u(x))\ \text{ a.e. }\ x\in\Omega.$$

Thus, Theorem 2.3-ii) shows that ∇u_n converges in measure to ∇u. □

Lemma 3.5. *u is a weak solution of the problem (1).*

Proof. Using Assertion 4, Assertion 7 and thanks to the density argument, $\mathcal{D}(\Omega)$ into the space $\dot{E}^{\pi(.)}(\Omega)$, we can take φ in $\dot{E}^{\pi(.)}(\Omega)$ as a test function in (27) to get

$$\int_\Omega\big[b(u)\varphi+a(x,u,\nabla u)\nabla\varphi\big]dx=\int_\Omega\mu\varphi dx.$$

Moreover, $b(u) \in L^1(\Omega)$, since $b(u_n)$ converges to $b(u)$ in $L^1(\Omega)$. Hence, u is a weak solution of the problem (1). $\square$

$\square$

In order to prove the uniqueness result, we make the following hypotheses on the function a, namely the local Lipschitz continuity with respect to z.
For all bounded subset K of $\mathbb{R} \times \mathbb{R}^N$, there exists a constant $C(K)$ such that

$$\begin{aligned} &\text{a.e. } x \in \Omega, \ \text{ for all } \ (z,\eta),(\tilde{z},\eta) \in K, \\ &|a(x,z,\eta) - a(x,\tilde{z},\eta)| \leq C(K)|z-\tilde{z}|. \end{aligned} \tag{34}$$

Remark 3.6. Let u be a weak solution of the problem (1), then $u \in C(\overline{\Omega})$, since $u \in W_0^{1,p_-}(\Omega)$ and $p_- > N$. Moreover, if u is a lipschitz continuous weak solution, then $u \in W^{1,\infty}(\Omega)$. Indeed, Ω is open bounded domain with smooth boundary $\partial\Omega$, so, the space of lipschitz functions $C^{0,1}(\overline{\Omega})$ and $W^{1,\infty}(\Omega)$ are homeomorphic and they can be identified.

Theorem 3.7. *(”weak-strong uniqueness”)*
Assume (2)-(5), with $\mathcal{M}$ in (4) constant, (6) and (34). *Moreover, assume that $\mu \in W^{-1,(p_-)'}(\Omega)$ is such that there exists a weak solution u with the $W^{1,\infty}$ regularity. Then any other weak solution of (1) coincides with u.*

Remark 3.8. In the proof of the Theorem 3.7, the relation (34) is used to obtain the inequality (38) below.

Proof. The existence has already been proved. Now, we show the uniqueness. For more details, see [2]-Proof of Theorem 2.8.
Let u be a Lipschitz continuous weak solution of the problem (1) and v be an other weak solution of (1).
Let $\phi := \dfrac{1}{k}T_k(u-v)$, then ϕ is an admissible test function in the weak formulations for both u and v. Indeed, since u is bounded

$$\frac{1}{k}T_k(u-v) = \frac{1}{k}T_k(u - T_{k+||u||_{L^\infty}}(v)),$$

then by using Lemma 2.5 and the fact that $u \in W^{1,\infty}(\Omega) \subset \dot{E}^{p(.,v(.))}(\Omega)$, $\phi \in \dot{E}^{p(.,v(.))}(\Omega) \cap L^\infty(\Omega)$.
Moreover, from assumption (4) of Theorem 3.7 and the fact that $|\nabla u| \in L^\infty(\Omega)$, it follows that

$$|a(x,u,\nabla u)| \leq C\big(1 + |\nabla u|^{p(.,u(.))}\big) \in L^\infty(\Omega).$$

Therefore, ϕ is admissible as a test function in the weak formulation for u. Obviously, ϕ is also an admissible test function in the weak formulation for v,

since $\phi \in \dot{E}^{p(.,v(.))}(\Omega) \cap L^\infty(\Omega)$.
Thus, with this test function, we have

$$\int_\Omega b(u)\frac{1}{k}T_k(u-v)dx + \frac{1}{k}\int_\Omega a(x,u,\nabla u).\nabla(u-v)\chi_{[0<|u-v|<k]}dx = \int_\Omega \mu \frac{1}{k}T_k(u-v)dx \tag{35}$$

and

$$\int_\Omega b(v)\frac{1}{k}T_k(u-v)dx + \frac{1}{k}\int_\Omega a(x,v,\nabla v).\nabla(u-v)\chi_{[0<|u-v|<k]}dx = \int_\Omega \mu \frac{1}{k}T_k(u-v)dx. \tag{36}$$

We substract (35) and (36) to get

$$\begin{aligned} \int_\Omega \big(b(u)-b(v)\big)\frac{1}{k}T_k(u-v)dx &+ \frac{1}{k}\int_\Omega \big(a(x,u,\nabla u)-a(x,v,\nabla v)\big).\nabla(u-v)\chi_{[0<|u-v|<k]}dx \\ &= 0. \end{aligned} \tag{37}$$

Let's denote by I the second term of left hand side of (37). We know that

$$\begin{aligned} (a(x,u,\nabla u)-a(x,v,\nabla v))\nabla(u-v) &= (a(x,u,\nabla u)-a(x,v,\nabla u))\nabla(u-v) \\ &+ \underbrace{(a(x,v,\nabla u)-a(x,v,\nabla v))\nabla(u-v)}_{\geq 0}. \end{aligned}$$

We have

$$I = I_k + \int_\Omega \big(a(x,v,\nabla u)-a(x,v,\nabla v)\big)\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx,$$

where

$$I_k = \int_\Omega \big(a(x,u,\nabla u)-a(x,v,\nabla u)\big)\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx.$$

Let's show that $I_k \to 0$ as $k \to 0$. Since u is bounded, then v is also bounded on the set $[0<|u-v|<k]$. Thus,

$$\begin{aligned} |I_k| &\leq \frac{1}{k}\int_{[0<|u-v|<k]} |a(x,u,\nabla u)-a(x,v,\nabla u)||\nabla u-\nabla v|dx \\ &\leq \frac{1}{k}\int_{[0<|u-v|<k]} C\big(||u||_{L^\infty(\Omega)}, ||\nabla u||_{L^\infty(\Omega)}\big)|u-v||\nabla u-\nabla v|dx \ \big(\text{by using } (34)\big) \\ &\leq C\big(||u||_{L^\infty(\Omega)}, ||\nabla u||_{L^\infty(\Omega)}\big)\int_{[0<|u-v|<k]} |\nabla u-\nabla v|dx \to 0, \quad \text{as } k\to 0. \end{aligned} \tag{38}$$

Note that $\lim_{k\to 0} meas([0<|u-v|<k])=0$ and $|\nabla u-\nabla v|\in L^1(\Omega)$.
For the first term of the left hand side of (37), one has

$$\begin{aligned}\lim_{k\to 0}\int_\Omega \big(b(u)-b(v)\big)\frac{1}{k}T_k(u-v)dx &= \int_\Omega (b(u)-b(v))sign_0(u-v)dx\\ &= \int_\Omega |b(u)-b(v)|dx. \qquad (39)\end{aligned}$$

Finally, one makes k goes to 0 in (37) and taking into account inequalities (38) and (39), we get

$$\int_\Omega |b(u)-b(v)|dx \;+\; \lim_{k\to 0}\int_\Omega \big(a(x,v,\nabla u)-a(x,v,\nabla v)\big)\frac{1}{k}\nabla(u-v)\chi_{[0<|u-v|<k]}dx=0. \qquad (40)$$

Since all the terms of equality (40) are nonnegative, we deduce that

$$\int_\Omega |b(u)-b(v)|dx=0.$$

Hence,

$$b(u)=b(v) \quad \text{a.e. on} \quad \Omega.$$

Thus, $u=v$ a.e. on Ω, since b is assume strictly increasing. □

4. Continuous dependence of the weak solution

Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of $W^{-1,(p_-)'}(\Omega)$ such that $\mu_n \rightharpoonup \mu$ in $W^{-1,(p_-)'}(\Omega)$.
Now, we are interested to the stability result of weak solutions to the problem

$$(P_{\mu_n})\begin{cases} b(u_n)-\mathrm{div} a_n(x,u_n,\nabla u_n)=\mu_n & \text{in} \quad \Omega\\ \\ u_n=0 & \text{on} \quad \partial\Omega.\end{cases}$$

$(a_n)_{n\in\mathbb{N}}$ is a sequence of diffusion flux function such that $a_n(x,z,\xi)$ satisfies (2)-(5) with variable exponent $p_n:\Omega\times\mathbb{R}\to[p_-,p_+]$ and $(\mathcal{M}_n)_{n\in\mathbb{N}}$ equi-integrable on Ω. In the sequel, we make the log-Hölder continuous and convergence hypothesis.

$$\left[\begin{array}{l} p_n:\overline{\Omega}\times\mathbb{R}\longrightarrow[p_-,p_+], \text{with} \quad p_->N \;\text{ and } \;\forall M>0,\\ p_n \;\text{ is log-Hölder continuous in }\; (x,z) \;\text{ uniformly on }\; \overline{\Omega}\times[-M,M].\end{array}\right. \qquad (41)$$

$$\left[\begin{array}{l} \text{For all bounded subset } K \text{ of } \mathbb{R}\times\mathbb{R}^N,\\ \sup_{(z,\xi)\in K}|a_n(.,z,\xi)-a(.,z,\xi)|\to 0 \;\text{ in measure on }\; \Omega.\end{array}\right. \qquad (42)$$

$$\left[\begin{array}{l} \text{For all bounded subset } K \text{ of } \mathbb{R},\\ \sup_{z\in K}|p_n(.,z)-p(.,z)| \;\text{ converges to zero in measure on }\; \Omega.\end{array}\right. \qquad (43)$$

The following structural stability result holds for weak solutions.

Theorem 4.1. *Let u_n be a weak solution of (P_{μ_n}). Assume that (2)-(6) hold. Assume also that $(a_n)_{n\in\mathbb{N}}$ is a sequence of diffusion flux functions of the form $a_n(x,z,\xi)$ such that (2)-(5) hold with $p_n(x,z)$, C independent of n with a sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ equi-integrable on Ω. Furthermore, let us consider the assumptions (41)-(43).*
Then, there exists a measurable function u on Ω such that u_n and ∇u_n (up to extraction of subsequences still denoted $(u_n)_{n\in\mathbb{N}}$ and $(\nabla u_n)_{n\in\mathbb{N}}$) converge to u and ∇u, respectively, a.e. in Ω, as $n\to\infty$. The function u is a weak solution of the problem (1) associated to the diffusion flux $a(.,.,.)$ and the measure μ.

The proof is organized in several steps and we reason up to an extracted subsequence of $(u_n)_{n\in\mathbb{N}}$ still denoted $(u_n)_{n\in\mathbb{N}}$.

Step 1. The sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges to a Young measure $\nu_x(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence and

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda). \tag{44}$$

Proof. As $(\mu_n)_{n\in\mathbb{N}}$ is weakly converges to μ in $W^{-1,(p_-)'}(\Omega)$, it is uniformly bounded in $W^{-1,(p_-)'}(\Omega)$.
Moreover, using (5) with variable exponent $p_n(.,u_n(.))$ on $a_n(x,u_n,\nabla u_n)$ and the reasoning that lead to (20) in the proof of Assertion 1, the sequence $(u_n)_{n\in\mathbb{N}}$ satisfies the following estimation.

$$\int_\Omega b(u_n)u_n dx + \frac{1}{2C_2}\int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx \leq C,$$

with C that depends on μ, $meas(\Omega)$ and p_- but not on n. From the above inequality, we deduce that

$$\int_\Omega |\nabla u_n|^{p_n(.,u_n(.))}dx \leq Const(\mu,\Omega,p_-). \tag{45}$$

Thus, using (18) with $\pi_n(.)$ replaced by $p_n(.,u_n(.))$ and (45), the sequence $(u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W_0^{1,p_-}(\Omega)$. Then, up to a subsequence, u_n converges a.e. in Ω (and also weakly in $W_0^{1,p_-}(\Omega)$) to a limit u; using the representation of weakly convergent sequences in $L^1(\Omega)$ in terms of Young measures (see Theorem 2.3 and formula (9)), we can write

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda)$$

□

Step 2.
$|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N\times\Omega$. Moreover, $u\in\dot{E}^{\pi(.)}(\Omega)$.

Proof. $|p_n(x,u_n(x))-p(x,u(x))|\leq|p_n(x,u_n(x))-p(x,u_n(x))|+|p(x,u_n(x))-p(x,u(x))|$. According to a.e. convergence of u_n to u, assumption (43) and the Lusin Theorem applied to the map

$$p:\Omega\mapsto p(x,.)\in C(\mathbb{R}),$$

we deduce that $p_n(.,u_n(.))\to p(.,u(.))=\pi(.)$ in measure on Ω. Now, using Theorem 2.3-ii), iii) $(p_n(.,u_n(.)),\nabla u_n)_{n\in\mathbb{N}}$ converges on $\mathbb{R}\times\mathbb{R}^N$ to Young measure $\mu_x=\delta_\pi(x)\otimes\nu_x$.
Thus, we can apply the weak convergence properties (8) to the Carathéodory function

$$F_m:(x,\lambda_0,\lambda)\in\Omega\times(\mathbb{R}\times\mathbb{R}^N)\mapsto|h_m(\lambda)|^{\lambda_0},$$

with $m\in\mathbb{N}$, where h_m is defined in the preliminary. We have

$$\begin{aligned}
\int_{\Omega\times\mathbb{R}^N}|h_m(\lambda)|^{\pi(.)}d\nu_x(\lambda)dx &= \int_{\Omega\times(\mathbb{R}\times\mathbb{R}^N)}|h_m(\lambda)|^{\lambda_0}d\mu_x(\lambda_0,\lambda)dx\\
&= \int_\Omega\int_{\mathbb{R}\times\mathbb{R}^N}F_m(x,\lambda_0,\lambda)d\mu_x(\lambda_0,\lambda)dx\\
&= \lim_{n\to+\infty}\int_\Omega F_m(x,p_n(x,u_n),\nabla u_n(x))dx\\
&= \lim_{n\to\infty}\int_\Omega|h_m(\nabla u_n)|^{p_n(x,u_n)}dx\\
&\leq \lim_{n\to\infty}\int_\Omega|\nabla u_n|^{p_n(x,u_n)}dx\\
&\leq Const(\mu,\Omega,p_-).
\end{aligned}$$

Since $h_m(\lambda)\to\lambda$, as $m\to\infty$ and $m\mapsto h_m(\lambda)$ is increasing, then using Lebesgue convergence Theorem, we deduce that

$$\int_{\Omega\times\mathbb{R}^N}|\lambda|^{\pi(.)}d\nu_x(\lambda)dx\leq Const(\mu,\Omega,p_-).$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N\times\Omega$.
Now, we prove that $\nabla u\in L^{\pi(.)}(\Omega)$. Using (44), Jensen inequality and the above inequality, we get

$$\int_\Omega|\nabla u|^{\pi(.)}dx=\int_\Omega\left|\int_{\mathbb{R}^N}\lambda d\nu_x(\lambda)\right|^{\pi(.)}dx\leq\int_{\Omega\times\mathbb{R}^N}|\lambda|^{\pi(.)}d\nu_x(\lambda)dx<\infty.$$

Thus, $\nabla u\in L^{\pi(.)}(\Omega)$. Moreover, $u\in W_0^{1,p_-}(\Omega)\subset W_0^{1,1}(\Omega)$.
Hence, $u\in\dot{E}^{\pi(.)}(\Omega)$. □

Step 3.
(a) The sequence $(A_n)_{n\in\mathbb{N}}$ defined by $A_n(x) := a_n(x,u_n(x),\nabla u_n(x))$ is equi-integrable on Ω.
(b) The weak limit A of (A_n) belongs to $L^{\pi'(.)}(\Omega)$ and one has

$$A(x) = \int_{\mathbb{R}^N} a(x,u(x),\lambda)d\nu_x(\lambda) \quad \text{a.e.} \quad x\in\Omega. \tag{46}$$

Proof. (a) Using (5), we have

$$|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))} \leq C\big(\mathcal{M}_n + |\nabla u_n|^{p_n(.,u_n(.))}\big).$$

The sequence $\big(|\nabla u_n|^{p_n(.,u_n(.))}\big)_{n\in\mathbb{N}}$ being bounded in $L^1(\Omega)$ and $\mathcal{M}_n$ is also equi-integrable on Ω, hence $\big(|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}\big)_{n\in\mathbb{N}}$ is equi-integrable.
Otherwise, as $p_n'(.,u_n(.)) > 1$, we have

$$|a_n(x,u_n,\nabla u_n)| \leq 1 + |a_n(x,u_n,\nabla u_n))|^{p_n'(.,u_n(.))}.$$

Thus, for all subset $E\subset\Omega$, we have

$$\int_E |a_n(x,u_n,\nabla u_n)|dx \leq meas(E) + \int_E |a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}dx.$$

Thus, for $meas(E)$ small enough, we deduce that $(A_n)_{n\in\mathbb{N}}$ is equi-integrable.

(b) Set $\nabla v_n := \nabla u_n \chi_{S_n'}$ and consider auxiliary functions $\tilde{A}_n := a(x,u,\nabla v_n)$, where

$$S_n' := \{x\in\Omega, \quad |p(x,u(x)) - p_n(x,u_n(x))| < \varepsilon\}, \text{ with } 0<\varepsilon<1.$$

Let us prove that for all $\sigma > 0$,

$$meas\big(\{x\in\Omega, \sup_{\lambda\in K}|a_n(x,u_n,\lambda) - a(x,u,\lambda)| \geq \sigma\}\big) \to 0, \text{ as } n\to\infty,$$

where K is a bounded subset of $\mathbb{R}^N$. We know that

$$|a_n(x,u_n,\lambda)-a(x,u,\lambda)| \leq |a_n(x,u_n,\lambda)-a(x,u_n,\lambda)| + |a(x,u_n,\lambda)-a(x,u,\lambda)|.$$

Thus, it follows from Lemma 2.7 and (42),

$$\sup_{\lambda\in K}|a_n(x,u_n,\lambda)-a(x,u,\lambda)| \to 0 \text{ in measure, as } n\to\infty.$$

Now, using the same argument as in the proof of Assertion 4 to A_n and $\tilde{A}_n$ instead of Φ_n and $\tilde{\Phi}_n$, we get the wished result. □

Step 4.
(a)

$$\int_\Omega A.\nabla u dx \geq \int_\Omega a(x,u(x),\lambda).\lambda d\nu_x(\lambda).$$

(b) The "div-curl" inequality holds.

$$\int_{\Omega\times\mathbb{R}^N}\big(a(x,u(x),\lambda)-a(x,u(x),\nabla u(x))\big)(\lambda-\nabla u(x))d\nu_x(\lambda)dx \leq 0. \quad (47)$$

(c)

$$A(x)=a(x,u(x),\nabla u(x)) \text{ for a.e. } x\in\Omega,$$

and ∇u_n converges to ∇u in measure on Ω, as $n\to\infty$.

Proof. We only give the proof of (a). The proofs of (b) and (c) are exactly the same as the proofs of Assertion 6 and Assertion 7.

(a) Let $\varphi\in\mathcal{D}(\Omega)$ (For more details, see the proof of Lemma 4.2). For n large enough, φ is an admissible test function in the weak formulation of u_n and we have

$$\int_\Omega b(u_n)\varphi dx+\int_\Omega a_n(x,u_n,\nabla u_n).\nabla\varphi dx=\int_\Omega \mu_n\varphi dx. \quad (48)$$

We can pass to the limit as n goes to ∞ to get

$$\int_\Omega b(u)\varphi dx+\int_\Omega A.\nabla\varphi dx=\int_\Omega \mu\varphi dx. \quad (49)$$

By density argument, we can replace φ with u_n in (48) to get

$$\int_\Omega b(u_n)u_n dx+\int_\Omega a_n(x,u_n,\nabla u_n).\nabla u_n dx=\int_\Omega \mu_n u_n dx. \quad (50)$$

Since $\mathcal{D}(\Omega)$ is dense in $\dot{E}^{\pi(.)}(\Omega)$, so we replace φ by u in (49) and we have

$$\int_\Omega b(u)u dx+\int_\Omega A.\nabla u dx=\int_\Omega \mu u dx. \quad (51)$$

By Fatou's Lemma, we deduce

$$\liminf_{n\to\infty}\int_\Omega b(u_n)u_n dx \geq \int_\Omega b(u)u dx. \quad (52)$$

Moreover,

$$\lim_{n\to\infty}\int_\Omega \mu_n u_n dx=\int_\Omega \mu u dx. \quad (53)$$

Indeed, the sequence $(\mu_n u_n)_{n\in\mathbb{N}}$ converges to μu a.e. in Ω. Now, we prove that $(\mu_n u_n)_{n\in\mathbb{N}}$ is equi-integrable. Since $\mu_n \rightharpoonup \mu$ in $W^{-1,(p_-)'}(\Omega)$, then $(\mu_n u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $W^{-1,(p_-)'}(\Omega)$. Let E be a subset of Ω, we have

$$\begin{aligned}\int_E |\mu_n u_n| dx &= \int_\Omega |\mu_n||u_n\chi_E| dx \\ &\leq \|\mu_n\|_{W^{-1,(p_-)'}(\Omega)} \|u_n\chi_E\|_{W_0^{1,p_-}(\Omega)} \\ &\leq C\left(\int_E |\nabla u_n|^{p_-} dx\right)^{\frac{1}{p_-}}. \end{aligned} \tag{54}$$

As $(\nabla u_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^{p_-}(\Omega)$, then $(|\nabla u_n|^{p_-})_{n\in\mathbb{N}}$ is equi-integrable in Ω. Therefore, it follows from (54) that

$$\lim_{meas(E)\to 0} \int_E |\mu_n u_n| dx = 0.$$

Then, applying Vitali's Theorem, we obtain (53).

Combining (52) and (53), we have

$$\liminf_{n\to\infty} \int_\Omega \mu_n u_n dx - \int_\Omega b(u) u dx \geq \liminf_{n\to\infty} \left(\int_\Omega \mu_n u_n dx - \int_\Omega b(u_n) u_n dx\right)$$

which is equivalent to say

$$\int_\Omega \mu u dx - \int_\Omega b(u) u dx \quad \geq \quad \liminf_{n\to\infty} \int_\Omega a_n(x, u_n, \nabla u_n) \nabla u_n dx.$$

Hence, by using (50) and (51) we obtain

$$\int_\Omega A.\nabla u dx \geq \liminf_{n\to\infty} \int_\Omega a_n(x, u_n, \nabla u_n) \nabla u_n dx. \tag{55}$$

By [1]-Lemma 2.1, $m \mapsto a_n(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n)$ is increasing and converges to $a_n(x, u_n, \nabla u_n).\nabla u_n$ for m large enough. Then,

$$a_n(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n) \leq a_n(x, u_n, \nabla u_n).\nabla u_n.$$

Therefore, by using (55) and Theorem 2.3, we have

$$\begin{aligned}\int_\Omega A.\nabla u dx &\geq \liminf_{n\to\infty} \int_\Omega a_n(x, u_n, \nabla u_n) \nabla u_n.\nabla u_n dx \\ &\geq \lim_{n\to\infty} \int_\Omega a_n(x, u_n, h_m(\nabla u_n)).h_m(\nabla u_n) dx \\ &= \int_{\Omega\times\mathbb{R}^N} a(x, u, h_m(\lambda)).h_m(\lambda) d\nu_x(\lambda) dx. \end{aligned} \tag{56}$$

Using Lebesgue convergence Theorem in (56), as m goes to ∞, we get (a). □

Lemma 4.2. *u is a weak solution of the problem (1).*

Proof. Let u_n be a weak solution of the problem (P_{μ_n}). Due to (45) $u_n \in W_0^{1,p_n(.,u_n(.))}(\Omega) \hookrightarrow W_0^{1,p_-}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ for $p_- > N$ and using (41), $\mathcal{D}(\Omega)$ is dense in $\dot{E}^{p_n(.,u_n(.))}(\Omega)$. Therefore, for n large enough , we can choose φ in $\mathcal{D}(\Omega)$ as a test function in the weak formulation of the problem (P_{μ_n}). Thus, we have

$$\int_\Omega b(u_n)\varphi dx + \int_\Omega a_n(x,u_n,\nabla u_n)\nabla\varphi dx = \int_\Omega \mu_n \varphi dx. \tag{57}$$

$(u_n)_{n\in\mathbb{N}^*}$ is uniformly bounded in the space $W_0^{1,p_-}(\Omega)$, so, uniformly bounded in $L^\infty(\Omega)$, thanks to embedding result.
Thus, as $b(.)$ is continuous, then $(b(u_n))_{n\in\mathbb{N}^*}$ is bounded in $L^\infty(\Omega)$ and $b(u_n)$ converges a.e. in Ω to $b(u)$. Therefore, $(b(u_n))_{n\in\mathbb{N}}$ converges to $b(u)$ in $L^1(\Omega)$, as n tends to ∞, thanks to Lebesgue dominated convergence Theorem. Hence

$$\lim_{n\to\infty}\int_\Omega b(u_n)\varphi dx = \int_\Omega b(u)\varphi dx. \tag{58}$$

We also have

$$\lim_{n\to\infty}\int_\Omega \mu_n\varphi dx = \int_\Omega \mu\varphi dx, \tag{59}$$

since $\mu_n \rightharpoonup \mu$ in $W^{-1,(p_-)'}(\Omega)$, as n tends to ∞ and $\varphi \in L^\infty(\Omega)$. Now, we are interested to the second term of the left-hand side of (57). We prove that $a_n(x,u_n,\nabla u_n).\nabla\varphi$ is equi-integrable. Let E be a subset of Ω, by using Young's inequality, we have

$$\begin{aligned}
\int_E a_n(x,u_n,\nabla u_n).\nabla\varphi dx \;&\leq\; \int_E |a_n(x,u_n,\nabla u_n)||\nabla\varphi| dx \\
&\leq\; \int_E \frac{1}{p_n'(.,u_n(.))}|a_n(x,u_n,\nabla u_n)|^{p_n'(.,u_n(.))}dx \\
&+\; \int_E \frac{1}{p_n(.,u_n(.))}|\nabla\varphi|^{p_n(.,u_n(.))}dx \\
&\leq\; C\int_E \left(\mathcal{M}_n(x) + |\nabla u_n|^{p_n(.,u_n(.))}\right)dx + \int_E |\nabla\varphi|^{p_n(.,u_n(.))}dx \\
&\leq\; C\int_E \left(\mathcal{M}_n(x) + |\nabla u_n|^{p_n(,u_n(.))}\right)dx \\
&+\; \int_{E\cap\{|\nabla\varphi|\leq 1\}} |\nabla\varphi|^{\pi_n(.)}dx + \int_{E\cap\{|\nabla\varphi|>1\}} |\nabla\varphi|^{p_n(.,u_n(.))}dx \\
&\leq\; C\int_E \left(\mathcal{M}_n(x) + |\nabla u_n|^{p_n(,u_n(.))}\right)dx \\
&+\; meas(E) + \int_E |\nabla\varphi|^{p_+}dx.
\end{aligned} \tag{60}$$

Due to (45) and the fact that $(\mathcal{M}_n)_{n\in\mathbb{N}}$ is equi-integrable, $\left(\mathcal{M}_n + |\nabla u_n|^{p_n(\cdot,u_n(\cdot))}\right)_{n\in\mathbb{N}}$ is equi-integrable. Moreover, $|\nabla\varphi|^{p_+} \in L^1(\Omega)$, since $\nabla\varphi$ is bounded. Thus, we deduce from (60) that

$$\lim_{meas(E)\to 0} \int_E a_n(x,u_n,\nabla u_n).\nabla\varphi dx = 0.$$

Furthermore, due to Step 3-b) and Step 4-c),

$$a_n(x,u_n,\nabla u_n).\nabla\varphi \to a(x,u,\nabla u).\nabla\varphi \quad \text{a.e.} \quad \text{in } \Omega.$$

By applying Vitali's Theorem, we obtain

$$\lim_{n\to+\infty} \int_\Omega a_n(x,u_n,\nabla u_n).\nabla\varphi dx = \int_\Omega a(x,u,\nabla u).\nabla\varphi dx. \tag{61}$$

Finally, using (58), (59) and (61), we get the weak formulation (14) for u with all test function in $\mathcal{D}(\Omega)$. Moreover, $u \in \dot{E}^{\pi(\cdot)}(\Omega)$, thanks to Step 2. Further, $b(u) \in L^1(\Omega)$, since $(b(u_n))_{n\in\mathbb{N}}$ strongly converges to $b(u)$ in $L^1(\Omega)$. □

Remark 4.3. Under the assumptions (34) and (6), the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to u a.e. on Ω and the whole sequence $(\nabla u_n)_{n\in\mathbb{N}}$ converges to ∇u a.e. on Ω, as $n\to\infty$.
Indeed, by Step 1, we deduce that $(u_n)_{n\in\mathbb{N}}$ converges to u a.e. on Ω, up to a subsequence. From Step 4-c), $(\nabla u_n)_{n\in\mathbb{N}}$ converges to ∇u a.e. on Ω, up to a subsequence.
Now, by Lemma 4.2 and the uniqueness of the weak solution of the problem(1) according to in Theorem 3.7, we conclude that all convergent subsequences of $(u_n)_{n\in\mathbb{N}}$ and $(\nabla u_n)_{n\in\mathbb{N}}$ converge to the same limits u and ∇u respectively.

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REFERENCES

[1] B. Andreianov, M. Bendahmane, S. Ouaro; *Structural stability for variable exponent elliptic problems. I. The $p(x)$-Laplacian kind problems*, Nonlinear Analysis, **73** (2010), 2 – 24.

[2] B. Andreianov, M. Bendahmane, S. Ouaro; *Structural stability for variable exponent elliptic problems, II: The $p(u)$-Laplacian and coupled problems*, Nonlinear Analysis, **72** (2010), 4649 – 4660.

[3] B. Andreianov, F. Bouhsiss; *Uniqueness for an elliptic parabolic problem with Neumann boundary condition*, J. Evol. Equ. 4 (2) (2004), 273-295.

[4] S.N. Antontsev, J.F. Rodrigues; *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (1) (2006), 19 – 36.

[5] J.M. Ball; *A version of the fundamental theorem for Young measures. PDEs and continum models of phase transitions* (Nice, 1988), 207 – 215, Lecture Notes in Phys., 344, Springer, 1989.

[6] L. Boccardo, T. Gallouët, L. Orsina; *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*, vol. 13, n^o5, (1996), 539 – 551.

[7] Y. Chen, S. Levine, M. Rao; *Variable exponent, linear growth functionals in image restoration*. SIAM. J.Appl. Math., 66 (2006), 1383 – 1406.

[8] M. Chipot, H. B. de Oliveira, *Some results on the $p(u)$-Laplacian problem*, Math. Ann. 375 (2019), no. 1-2, pp.283 – 306.

[9] R. Eymard, T. Gallouët, R. Herbin; *Finite volume methods. Handbook of numerical analysis*, Vol. VII, 713 – 1020, North-Holland, 2000.

[10] N. Hungerbühler; *A refinement of Ball's theorem on Young measures*. New York J. Math. **3** (1997), 48 – 53.

[11] I. Nyanquini, S. Ouaro, S. Safimba; *Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data*, Ann. Univ. Craiova Ser. Mat. Inform. **40**(2013), no. 2, 174 – 198.

[12] P. Pedregal; Parametrized measures and variational principles. Progress in Nonlinear Differential Equations and their Applications, 30. Birkhäuser, Basel, 1997.

[13] M. Ruzicka; *Electrorheological fluids: modelling and mathematical theory*. Lecture Notes in Mathematics 1748, Springer-Verlag, Berlin, 2002.

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