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AN IDENTIFICATION PROBLEM FOR SOME DEGENERATE DIFFERENTIAL EQUATIONS

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We are concerned with a degenerate first order identification problem in a Banach space. Suitable hypotheses on the involved operators are made in order to reduce the given problem to a non-degenerate case. Some applications to partial differential equations are indicated extending well known results in the regular case.

1. Introduction.

In this paper, we are concerned with an identification problem for a first order degenerate linear system, more precisely, our goal is to find $u \in C^1([0, \tau]; X)$, X being a Banach space and $f \in C^1([0, \tau]; R)$, $\tau > 0$, such that

$$\frac{d}{dt}(Mu) + Lu = f(t)\,z, \qquad 0 \le t \le \tau, \tag{1}$$

$$(Mu)(0) = Mu_0, \tag{2}$$

$$\Phi[Mu(t)] = g(t), \qquad 0 \le t \le \tau, \tag{3}$$

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where L, M are two closed linear operators with $D(L) \subseteq D(M)$, L is invertible, z, $u_0 \in X$, $\Phi \in X^*$, X^* being the dual space to X, $g \in C^1([0, \tau]; R)$, and M may have no bounded inverse.

If $M = I$, the problem was discussed in A. Lorenzi [2] where a fixed point argument furnishes the desired solution. Here, we parallel his main results in two cases, the so-called non singular case and the singular case. Correspondingly, we obtain two main results. The key assumption here is $\lambda = 0$ to be a simple pole for the resolvent operator $(\lambda L + M)^{-1}$ and this enables us to extend the technique in A. Favini and A. Yagi [1]. It is shown that our problem is reduced to the identification problem in $R(T)$:

$$\begin{cases} \dfrac{d}{dt} w(t) + \tilde{T}^{-1} w(t) = f(t)(I - P)z, \qquad 0 \le t \le \tau, p \\ w(0) = w_0, \\ \Phi[w(t)] = g(t), \qquad 0 \le t \le \tau, \end{cases}$$

where $\tilde{T}$ is the restriction of T to $R(T)$ $\big(T = ML^{-1} \in \mathcal{L}(X)\big)$, $\mathcal{L}(X)$ being the linear space of all continuous linear operators from X into X and P is the projection onto $N(T)$ along $R(T)$. Then we are allowed to apply the methods described in A. Lorenzi [2] to find an explicit solution $u \in C^1([0, \tau]; R(T))$ and $f \in C^1([0, \tau]; R)$ to such a problem.

As a possible application of the abstract theorems, some examples from partial differential equations are given.

2. The Non Singular Case.

Consider now problem (1)-(3) with the compatibility relation

$$\Phi[(Mu)(0)] = \Phi[Mu_0] = g(0), \qquad 0 \le t \le \tau. \tag{4}$$

Let us remark that the compatibility relation $g(0) = \Phi[Mu_0]$ must hold, as one easily observe from (2), (3). Let $Lu = v$, $ML^{-1} = T \in \mathcal{L}(X)$; then (1)-(3) can be rewritten as

$$\frac{d}{dt}(Tv) + v = f(t)z, \qquad 0 \le t \le \tau, \tag{5}$$

$$(Tv)(0) = Tv_0 = ML^{-1}v_0, \tag{6}$$

$$\Phi[Tv(t)] = g(t), \qquad 0 \le t \le \tau, \tag{7}$$

where $v_0 = Lu_0$. Suppose $\lambda = 0$ to be a simple pole for $(\lambda + T)^{-1}$, i.e. there exists ϵ such that $\lambda + T$ has a bounded inverse operator for $0 < |\lambda| \leq \epsilon$ and $\|(\lambda + T)^{-1}\|_{\mathcal{L}(X)} \leq c|\lambda|^{-1}, \quad 0 < |\lambda| \leq \epsilon$. Then we can represent X in the following form (see 3):

$$X = N(T) \oplus R(T)$$

where $N(T)$ is the null space of T and $R(T)$ is the (closed) range of T. Let $\tilde{T}$ be the restriction of T to $R(T)$; then $\tilde{T} = T_{R(T)} : R(T) \to R(T)$. Clearly $\tilde{T}$ is a one to one from $R(T)$ onto $R(T)$. Since $\tilde{T}$ is closed on $R(T)$, $\tilde{T}^{-1}$ is also continuous on $R(T)$. Let P be the corresponding projection onto $N(T)$ along $R(T)$; then (5)-(7) are written as follows :

$$\frac{d}{dt}T(Pv + (I - P)v) + Pv + (I - P)v = f(t)z, \qquad 0 \leq t \leq \tau, \tag{8}$$

$$Tv(0) = Tv_0, \tag{9}$$

$$\Phi[Tv(t)] = g(t), \qquad 0 \leq t \leq \tau. \tag{10}$$

Therefore, (1)-(3) are equivalent to the couple of problems

$$\frac{d}{dt}\tilde{T}(I - P)v + (I - P)v = f(t)(I - P)z, \tag{11}$$

$$\tilde{T}(I - P)v(0) = \tilde{T}(I - P)v_0, \tag{12}$$

$$\Phi[\tilde{T}(I - P)v(t)] = g(t), \tag{13}$$

and

$$Pv(t) = f(t)Pz. \tag{14}$$

Of course, problem (11)-(13) is a problem in the space $R(T)$.

Let $w = \tilde{T}(I-P)v$, then $(I-P)v = \tilde{T}^{-1}w$, and hence, (11)-(13) becomes

$$\frac{d}{dt}w(t) + \tilde{T}^{-1}w(t) = f(t)(I - P)z, \qquad 0 \leq t \leq \tau, \tag{15}$$

(16) $$w(0) = w_0 = \tilde{T}(I - P)v_0 = Tv_0,$$

(17) $$\Phi[w(t)] = g(t), \qquad 0 \le t \le \tau.$$

Now, we can solve problem (15)-(17) directly and here we recall the main steps. Notice that from (4) we have $g(0) = \Phi[w_0]$. For any given f, the solution w of (15)-(16) is assigned by the formula

(18) $$w(t) = e^{-t\tilde{T}^{-1}} w_0 + \int_0^t e^{-(t-s)\tilde{T}^{-1}} f(s)(I - P)z\, ds$$

where by definition

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$$

for any $A \in \mathcal{L}(R(T))$. Apply Φ to (15) and take equation (17) into account; we obtain the following equation for the unknown $f(t)$:

(19) $$g'(t) + \Phi[\tilde{T}^{-1} w(t)] = f(t)\Phi[(I - P)z]$$

Suppose the condition

(20) $$\chi(z)^{-1} = \Phi[(I - P)z] \neq 0$$

to hold. Then we can write (19) under the form :

(21) $$f(t) = \chi(z)g'(t) + \chi(z)\Phi[\tilde{T}^{-1} w(t)]$$

and by using the previous representation (18), we arrive at the following equation for w :

(22) $$w(t) = \big[e^{-t\tilde{T}^{-1}} w_0 + \chi(z) \int_0^t e^{-(t-s)\tilde{T}^{-1}} g'(s)(I - P)z\, ds\big]$$
$$+ \big[\chi(z) \int_0^t e^{-(t-s)\tilde{T}^{-1}} \Phi[\tilde{T}^{-1} w(s)](I - P)z\, ds\big].$$

This last equation has the form :

(23) $$w(t) = Aw(t) + r(t),$$

where

$$Aw(t) = \chi(z) \int_0^t e^{-(t-s)\tilde{T}^{-1}} \Phi[\tilde{T}^{-1} w(s)](I-P)z\, ds, \tag{24}$$

and

$$r(t) = e^{-t\tilde{T}^{-1}} + \chi(z) \int_0^t e^{-(t-s)\tilde{T}^{-1}} g'(s)(I-P)z\, ds. \tag{25}$$

In order to solve equation (23) by a fixed point argument, let us introduce the Banach space

$$Y = C\big([0,\tau];\, R(T)\big), \tag{26}$$

with norm $\|u\| = \sup_{t\in[0,\tau]} \|u(t)\|_{R(T)}$ ($R(T)$ is a Banach space with the norm induced by X). Then equation (23) is written in the operator form :

$$(I-A)w = r. \tag{27}$$

Clearly, if we prove that

$$\sum_{n=0}^{+\infty} \|A^n\|_{\mathcal{L}(Y)} < +\infty, \tag{28}$$

then $(I-A)^{-1}$ exists and belong to $\mathcal{L}(Y)$, and hence the solution $w(t)$ of the equation (27) is given by

$$w = (I-A)^{-1} r = \sum_{n=0}^{+\infty} A^n r.$$

Now it is easy to conclude from (24) that (for sake of brevity, we shall use the same notation $\|.\|$ for different norms)

$$\begin{aligned} \|Aw(t)\| &\le |\chi(z)|\, \|\Phi\|_{X^*} \|I-P\|\|z\|\|\tilde{T}^{-1}\| \int_0^t e^{(t-s)\|\tilde{T}^{-1}\|} \|w(s)\|\, ds \\ &= K(z) \int_0^t e^{(t-s)\|\tilde{T}^{-1}\|} \|w(s)\|\, ds. \end{aligned} \tag{29}$$

Proceeding by induction, we can find the estimates

$$\|A^n w(t)\| \leq \frac{K(z)^n}{(n-1)!}\int_0^t (t-s)^{n-1} e^{(t-s)\|\tilde{T}^{-1}\|}\|w(s)\|\,ds \quad \forall t\in[0,\tau]. \tag{30}$$

In particular, we deduce that

$$\begin{aligned}
\|A^n w(t)\| &\leq \frac{K(z)^n}{(n-1)!}\|w\|_Y \int_0^t (t-s)^{n-1} e^{(t-s)\|\tilde{T}^{-1}\|}\,ds \\
&\leq \frac{K(z)^n}{(n-1)!}\|w\|_Y e^{t\|\tilde{T}^{-1}\|}\int_0^t (t-s)^{n-1}\,ds \\
&\leq \frac{K(z)^n}{n!}\|w\|_Y t^n e^{t\|\tilde{T}^{-1}\|} \\
&\leq \frac{(K(z)\tau)^n}{n!}\|w\|_Y e^{\tau\|\tilde{T}^{-1}\|} \qquad \forall n\in\mathbb{N}.
\end{aligned}$$

for every $t\in[0,\tau]$ and every $w\in Y$. Hence, A satisfies the condition (28) and, more precisely,

$$\sum_{n=0}^{+\infty}\|A^n\| \leq \exp[K(z)+\|\tilde{T}^{-1}\|]\tau.$$

Now, since $w(t)=\sum_{n=0}^{+\infty} A^n r(t)$, we have

$$\tilde{T}^{-1}w(t) = \tilde{T}^{-1}\sum_{n=0}^{+\infty} A^n r(t) = \sum_{n=0}^{+\infty}\tilde{T}^{-1}A^n r(t), \tag{31}$$

where $r(t)$ is given by (25). Using (21) we have

$$\begin{aligned}
f(t) &= \chi(z)g'(t) + \chi(z)\Phi\Big[\sum_{n=0}^{+\infty}\tilde{T}^{-1}A^n r(t)\Big] \\
&= \chi(z)g'(t) + \chi(z)\sum_{n=0}^{+\infty}\Phi\big[\tilde{T}^{-1}A^n r(t)\big].
\end{aligned} \tag{32}$$

Since $f(t)$ is now known, (14) becomes

$$Pv(t) = f(t)Pz \tag{33}$$

$$= \chi(z)g'(t)Pz + \chi(z)\sum_{n=0}^{+\infty}\Phi\big[\tilde{T}^{-1}A^n r(t)\big]Pz.$$

Now, we know that

$$(I-P)v(t) = \tilde{T}^{-1}w(t) = \sum_{n=0}^{+\infty}\tilde{T}^{-1}A^n r(t),$$

and so,

$$v(t) = Pv(t) + (I-P)v(t)$$

$$= \chi(z)g'(t)Pz + \chi(z)\sum_{n=0}^{+\infty}\Phi\big[\tilde{T}^{-1}A^n r(t)\big]Pz + \sum_{n=0}^{+\infty}\tilde{T}^{-1}A^n r(t).$$

At last, we have the explicit solution $u(t)$ from

$$u(t) = L^{-1}v(t) \tag{34}$$

$$= \chi(z)g'(t)L^{-1}Pz + \chi(z)\sum_{n=0}^{+\infty}\Phi\big[\tilde{T}^{-1}A^n r(t)\big]L^{-1}Pz + \sum_{n=0}^{+\infty}L^{-1}\tilde{T}^{-1}A^n r(t).$$

Therefore, the problem is completely solved .

We have established the following result:

Theorem 2.1. *Let* $\lambda = 0$ *be a simple pole for* $(\lambda + T)^{-1}$*, where* $T = ML^{-1}$*,* L*,* M *are two closed linear operators with* $D(L) \subseteq D(M)$*,* L *invertible. Let* $z, u_0 \in X$*,* $\Phi \in X^*$*,* $g \in C^1\big([0,\tau]; R\big)$*, with (4), too. If, in addition,* $\Phi[(I-P)z] \neq 0$*, where* P *is the projection onto* $N(T)$ *along* $R(T)$*, then problem (1)-(3) admits a unique solution* $(u, f) \in C^1\big([0,\tau]; X\big) \times C^1\big([0,\tau]; R\big)$ *such that* $(Mu)' \in C\big([0,\tau]; X\big)$*.*

3. The Singular Case.

We consider the case when

$$\Phi[(I-P)z] = 0. \tag{35}$$

To solve the problem in (15)-(17) under the condition (35), we introduce $\zeta(t) = \tilde{T}^{-1}w(t)$ so that by applying $\tilde{T}^{-1}$ to (15) we have

$$\zeta'(t) + \tilde{T}^{-1}\zeta(t) = f(t)\tilde{T}^{-1}(I - P)z, \tag{36}$$

$$\zeta(0) = \tilde{T}^{-1}w_0, \tag{37}$$

$$\Phi[\zeta(t)] = -g'(t), \tag{38}$$

$$g(0) = \Phi[w_0]. \tag{39}$$

with the compatibility condition

$$g'(0) = -\Phi[\tilde{T}^{-1}w_0] = -\Phi[(I - P)Lu_0] \tag{40}$$

Recall that the compatibility relations (39) and (40) must be assumed in any case. Condition (38) is due to

$$\Phi[\zeta(t)] = \Phi[\tilde{T}^{-1}w(t)] = -\Phi[w'(t)] = -g'(t).$$

Now, we need the solvability condition

$$\Phi[\tilde{T}^{-1}(I - P)z] \neq 0, \tag{41}$$

(recall that $\Phi[(I - P)z] = 0$). Correspondingly, the solution to (36)-(39) is given by (32) and (35) with $\big(g, w_0, (I - P)z\big)$ being replaced with $\big(-g', \tilde{T}^{-1}w_0, \tilde{T}^{-1}(I - P)z\big)$. We find out the following representations (42) and (43) for f and u:

$$\begin{aligned} f(t) &= -\chi(z)g''(t) + \chi(z)\Phi\Big[\sum_{n=0}^{+\infty} \tilde{T}^{-2}A^n r(t)\Big] \\ &= -\chi(z)g''(t) + \chi(z)\sum_{n=0}^{+\infty} \Phi[\tilde{T}^{-2}A^n r(t)]. \end{aligned} \tag{42}$$

$$\begin{aligned} u(t) &= L^{-1}v(t) \\ &= -\chi(z)g''(t)L^{-1}Pz + \chi(z)\sum_{n=0}^{+\infty} \Phi[\tilde{T}^{-2}A^n r(t)]L^{-1}Pz \\ &\quad + \sum_{n=0}^{+\infty} L^{-1}\tilde{T}^{-1}A^n r(t). \end{aligned} \tag{43}$$

We have established the result as follows:

Theorem 3.1. *Let $\lambda = 0$ be a simple pole for $(l+T)^{-1}$, where $T = ML^{-1}$, L, M are two closed linear operators with $D(L) \subseteq D(M)$, L invertible, $z, u_0 \in X$, $\Phi \in X^*$, and $g \in C^2([0,\tau]; R)$. Assume that (4) and (40) hold and*

$$\Phi[(I-P)z] = 0 \ \ with \ \ \Phi[\tilde{T}^{-1}(I-P)z] \neq 0,$$

where P is the corresponding projection onto $N(T)$ along $R(T)$ and $\tilde{T}$ is the restriction of T to $R(T)$. Then problem (36)-(39) admits a unique solution $(u, f) \in C^1([0,\tau]; X) \times C^1([0,\tau]; R)$ such that $(Mu)' \in C([0,\tau]; X)$.

Remark 3.1. In the case when

$$\Phi[(I-P)z] = \Phi[\tilde{T}^{-1}(I-P)z] = 0, \tag{44}$$

we can go on as previously applying $\tilde{T}^{-1}$ another time and find the same type of equation provided that

$$\Phi[\tilde{T}^{-2}(I-P)z] \neq 0. \tag{45}$$

Of course, if g is sufficiently regular, the only case which is really difficult reduces to the so called supersingular case:

$$\Phi[\tilde{T}^{-j}(I-P)z] = 0 \qquad for \ any \ j \in N, \tag{46}$$

and it will be considered elsewhere.

4. Examples.

We devote this section to listing some initial value problems to which the previous abstract results apply.

Example 4.1. Consider the following initial value problem:

$$\frac{d}{dt}[(I+K)v] = Kv + f(t)z, \qquad 0 < t < \tau,$$

$$(I+K)v(0) = (I+K)v_0,$$

where K is a densely defined closed linear operator in a Banach space X with -1 an eigenvalue of K of multiplicity one, $f(t)$ is a given function, $v_0 \in D(K)$, and $v = v(t)$ is the unknown function. By the change of the unknown function to $\tilde{v}(t) = e^{-(\nu+1)t}v(t)$, our problem is transformed into

$$\frac{d}{dt}[(I+K)\tilde{v}] = -\nu(I+K)\tilde{v} - \tilde{v} + f_\nu(t)z, \qquad 0 < t < \tau,$$

$$(I+K)\tilde{v}(0) = (I+K)v_0,$$

where $f_\nu(t) = e^{-(\nu+1)t}f(t)$. So that this equation is viewed as an equation of the form (1)-(3) with $M = I + K$ and $L = -\nu(I+K) - I$. Since $\lambda M - L = (\lambda+\nu)(\frac{1}{\lambda+\nu} + M)$, the M resolvent $(\lambda M - L)^{-1}$ exists if $\left|\frac{1}{\lambda+\nu}\right| \le \epsilon$, i.e. if $|\lambda+\nu| \ge \frac{1}{\epsilon}$. Therefore, $(\lambda M - L)^{-1}$ exists as a bounded operator on X for all $|\lambda| \ge \epsilon_0$, ($|\lambda| \ge \epsilon_0 > |\nu + \frac{1}{\epsilon}|$). Hence for all μ, $0 < |\mu| \le \frac{1}{\epsilon_0}$ with a suitable $\epsilon_0 > 0$,

$$(\mu L - M)^{-1} = \mu^{-1}(L - \mu^{-1}M)^{-1},$$

and it is readily seen that $\mu = 0$ is a simple pole for $M(\mu L - M)^{-1}$. All the previous results work.

Example 4.2.

$$\frac{\partial}{\partial t}(1 + \frac{\partial^2}{\partial x^2})v = \frac{\partial^2 v}{\partial x^2} + f(x,t) \qquad 0 \le x \le l\pi, \qquad 0 < t \le \tau,$$

$$v(0,t) = v(l\pi,t) = 0, \qquad 0 < t \le \tau,$$

$$(1 + \frac{\partial^2}{\partial x^2})v(x,0) = (1 + \frac{\partial^2}{\partial x^2})v_0(x), \qquad 0 \le x \le l\pi,$$

where $X = \{f \in C([0,l\pi];\mathbb{C}); \quad f(0) = f(l\pi) = 0\}$. K is then given by

$$\begin{cases} D(K) = \left\{v \in C^2\big([0,l\pi];\mathbb{C}\big); \quad v(0) = v(l\pi) = v''(0) = v''(l\pi) = 0\right\} \\ Kv = \frac{d^2 v}{dx^2}, \end{cases}$$

where l is a positive integer. It is a simple matter to check that all abstract assumptions are satisfied, it only needs to take $K = \frac{\partial^2}{\partial x^2}$ in Example 4.1. For more details, we refer to A. Favini and A. Yagi [1].

Moreover, one can take in Example 4.1. $K = \Delta$, the Laplace operator, and all previous results work.

A more generally example is as follows:

Example 4.3.

$$\frac{d}{dt}\{(l_0 + K)v\} = Kv + f(t)z, \qquad 0 < t \leq T,$$

$$(l_0 + K)v(0) = (l_0 + K)v_0,$$

where K is a second-order elliptic differential operator with λ_0 being the first positive eigenvalue of K. All hypotheses hold, and for more details, see (A. Favini and A. Yagi [1]).

Example 4.4. Let L, M be two $n \times n$ real matrices such that $z = 0$ is a simple pole for $(zL + M)^{-1}$, i.e., $z = 0$ is a simple pole for $(z + T)^{-1}$, in other words, det(zM+L) has a simple zero, i.e., $det(zL + M) = zp(z), \quad p(0) \neq 0$. All the previous abstract results hold.

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