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RETRACTION OF CHAOTIC MANIFOLDS AND FRACTAL

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In this article we will describe some types of retractions of chaotic manifold. We also will introduce a definition for the fractional dimension of the chaotic manifold. Some theorems related to them will be obtained. Some applications will also be mentioned in this paper.

Introduction.

Modern discussions of chaos are almost always based on the work of Edward N. Lorenz. In his book The Essence of Chaos, he expands considerable effort in explaining deterministic chaos to the layman. He defines a deterministic sequence as one in which only one thing can happen next. He then defines randomness as being identical to the absence of determinism. Deterministic chaos is then something that looks random, but is really deterministic.

Lorenz devised a simplified mathematical model to describe this convection. His model involved a system of equations in three variables: x, y, and z. In the system, x was related to the speed of the convective motion, y was related to the temperature difference between the ascending and descending currents, and z was related to the vertical temperature profile. At any time t the corresponding point (x, y, z) described the phisical state of the system at that time.

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(Thus the triple (x, y, z) does not refer to a point in the water). What would happen when heat was applied to the system? With the help of a computer, it was possible to "follow" an initial state (x_0, y_0, z_0) over a period or time. If a small amount of heat was applied, then in time the motion of the water approached a steady state, meaning that the iterates of (x_0, y_0, z_0) approached an attracting fixed point. However, when enough heat was applied then as time progressed, the iterates of $x_0, y_0, z_0)$ became erratic, jumping around wildly in space. See Fig. (1) [15].

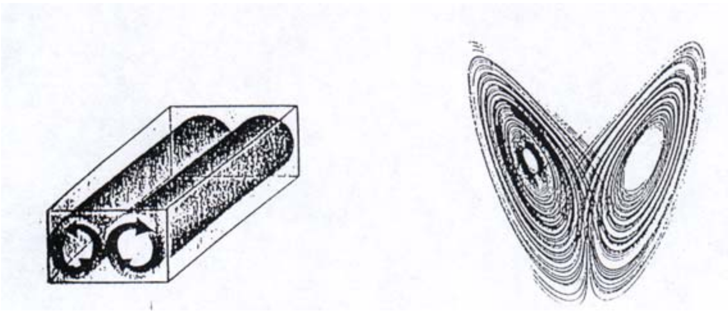

Figure 1.

In [1] Locally Recurrent Neural Networks are used to learn the chaotic trajectory of the Lorenz system starting from the measurements of an observable. The folding of a manifold is introduced by Robortson [16], the folding of a manifold into another manifold and or into itself are studied by El-kholy and M. El-Ghoul in [2], [3], [5], [12], [13], [14]. The retraction of a manifold is introduced in [10]. More studies and applications are discussed in [4], [6], [7], [8], [9], [10], [11].

Definitions. Let M be a non-empty (second-countable) Hausdorff topological space such that:

(i) M is the union of open subsets U_a and each U_a is equipped with a homeomorphism ϕ_a taking U_a to open set in R^n, i.e.;

$$\phi_a : U_a \to \phi_a(U_a) \subset R^n$$

(ii) If $U_a \bigcap U_\beta \neq \phi$, then the overlap map

$$\phi_\beta \phi_a^{-1} : \phi_a(U_a \cap U_\beta) \to \phi_\beta(U_a \cap U_\beta)$$

is a smooth map, see Fig. (2), where the map $\phi_a \phi_a^{-1}$ from an open set in R^n to R^n is smooth if all partial derivatives of all orders of each component of the map exist everywhere, where the map is defined.

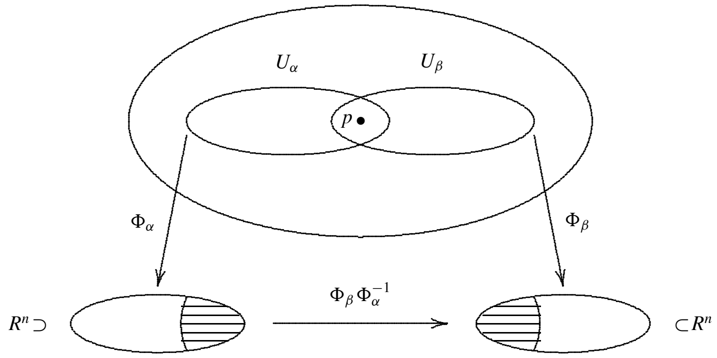

Fig. (2)

Each pair (U_a, Φ_a) is called a *chart* on M, and the collection $A = \{(U_a, \Phi_a)\}$ of charts is called (smooth) *atlas* on M. The space M taken toghether with the atlas A will be called *a smooth manifold of dimension n* or *smooth n-manifold or C^∞ n-manifold.*

For C^∞ - Riemannian manifolds M and N of dimensions m and n, respectively, a map $f : M \to N$ is said to be an isometric folding of M into N if for each piecewise geodesic path $\gamma : J \to M$, the induced path f o $\gamma : J \to N$ is also piecewise geodesic and has the same length as γ; if f does not preserve lengths it is just a topological folding [16].

A subset A of a topological space is called a *retract* of X if there exist a continuous map $r : X \to A$ called a retraction such that $r(a) = a$ for any $a \in A$.

A subset A of a topological space M is a deformation retract of M if there exist a retraction $R : M \to A$ and a homotopy $f : M \times I \to M$ such that

$$\left.\begin{array}{l} f(x,0) = x \\ f(x,1) = R(x) \end{array}\right\} \qquad x \in M$$

$f(a,t) = a, a \in A, t \in [0, 1]$.

The main results.

We will discuss some types of retraction of chaotic manifolds.

1. Retraction in volume

$r : M_{012...\infty h} - P_{012...\infty h} \rightarrow A_{012...\infty h}$, where $P_{012...\infty h}$ is a point $\in M_{012...\infty h}(P_{0h} \in M_{0h}, P_{1h} \in M_{1h}, \ldots, P_{\infty h} \in M_{\infty h})$, see Fig. (3). In case (a), the retraction may be $r_{11} : M_{012...\infty h} - P_{012...\infty h} \rightarrow M_{0h}$ or $r_{12} : M_{012...\infty h} - P_{012...\infty h} \rightarrow P$, where P is a common point. In case (b), r_2 every M_{ih} must retract to $A_{ih} \subset M_{ih}$.

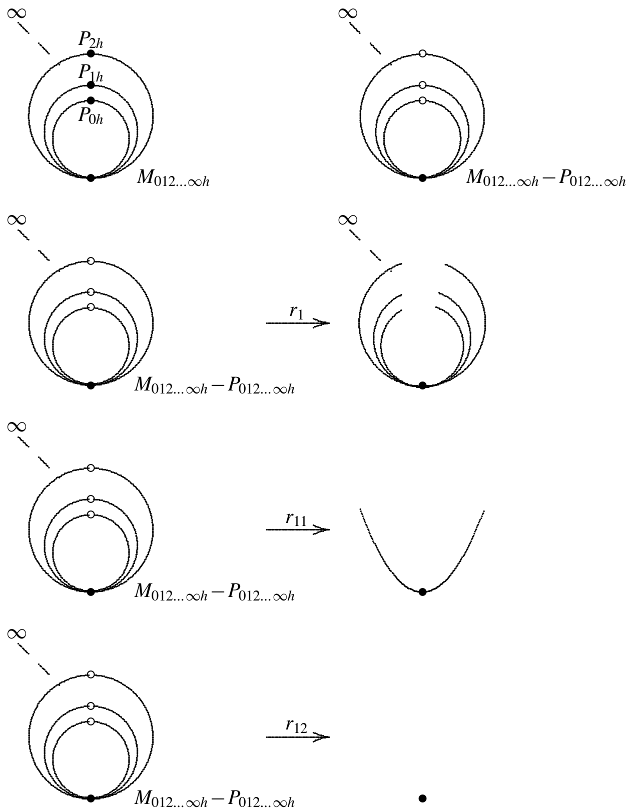

Fig. (3 a)

2. Retraction of density

Consider any chaotic manifolds $M_{012...\infty h}$, then, $M_{1h}, M_{2h}, \ldots, M_{\infty h}$ are all have the same phisical characters $d_{1h}, d_{2h}, \ldots, d_{\infty h}$ or $d_{1h}, I_{2h}, \ldots, t_{\infty h}$, where $d, I, \ldots, t$ are not all of the same phisical character.

(a) If all chaotic manifolds are all of the same physical character i.e. $d_{1h}, d_{2h}, \ldots, d_{\infty h}$, then, $r_i : d_{ih} - p_{ih} \rightarrow \overline{d}_{ih}$, $P_{ih} \in M_{ih}$. See fig. (4).

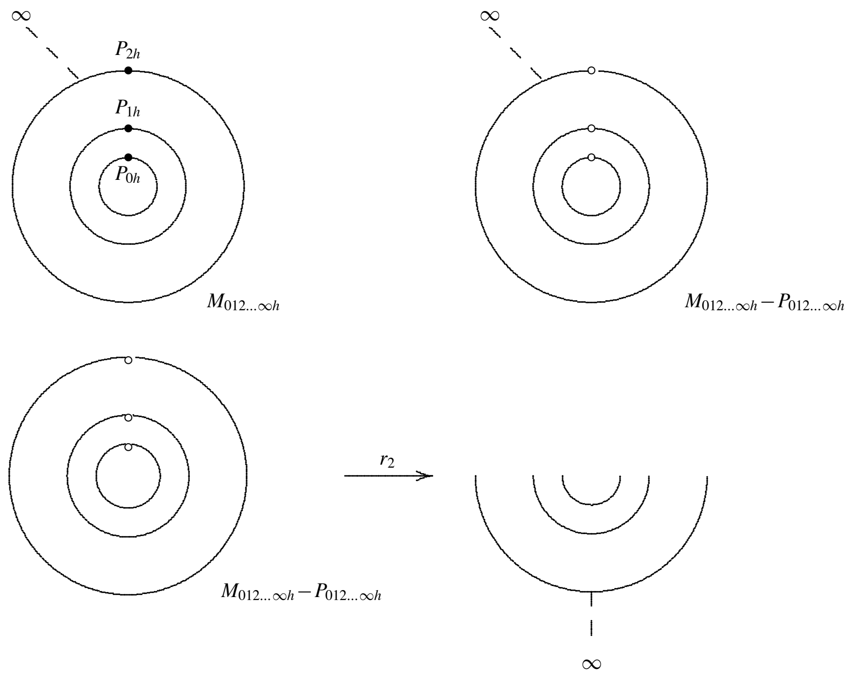

Fig. (3 b)

(b) If the chaotic manifolds are not all of the same physical character, then, there are an infinite number of retractions for every character.

$r_1 : d_{1h} \to \overline{d}_{1h}$
$r_2 : I_{2h} \to \overline{I}_{2h}$
|
|
|
$r_\infty : t_{\infty h} \to \overline{t}_{\infty h}$, where $\overline{d}_{1h} \prec d_{1h}, \overline{I}_{2h} \prec I_{2h}, \ldots, \overline{t}_{\infty h} \prec t_{\infty h}$. See Fig. (5).

(3) *Retraction in volume and density*:
$r_i : M_{012\ldots\infty h} - P_{012\ldots\infty h} \to \overline{M}_{012\ldots\infty h}$. See Fig. (6).

Theorem 1. *Any retraction of the geometric manifold M_{0h} induces retractions of every pure chaotic manifold.*

Proof. Let $t_g : M_{0h} \to \overline{M}_{0h}$, then, there are induced retractions in volume:
$r_1 : M_{1h} \to \overline{M}_{1h}$,
$r_2 : M_{2h} \to \overline{M}_{2h}$,

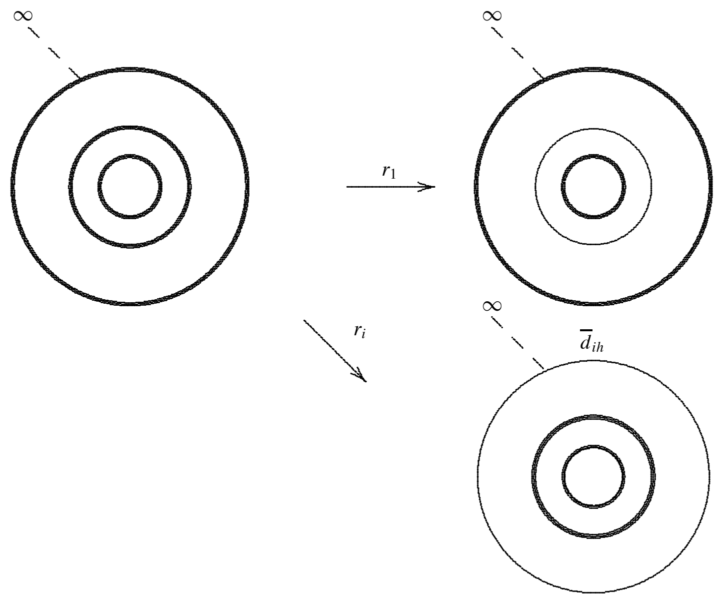

Fig. (4)

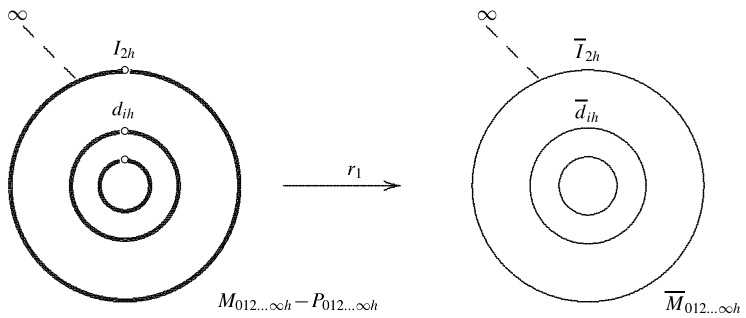

Fig. (5)

$$\vdots$$

$$r_\infty : M_{\infty h} \to \overline{M}_{\infty h}.$$

If $r_g : M_{0h} \to \overline{P}_g$, then,

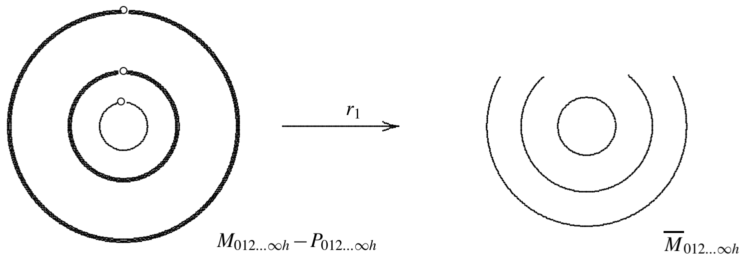

Fig. (6)

$r_1 : M_{1h} \to \overline{P}_{1h},$
$r_2 : M_{2h} \to \overline{P}_{2h},$
|
|
|
$r_\infty : M_{\infty h} \to \overline{P}_{\infty h}.$
is the minimum retraction of the manifold (i.e. minimum retraction of the geometric manifold ⇒ minimum retraction of the pure chaotic manifolds). The chaotic retraction is homeomorphic to the geometric retraction of the chaotic manifold but the chaotic manifold is not homeomorphic to its retraction. See Fig. (7).

Lemma 1. *The chaotic retraction of the manifold does not induce a geometric retraction.*

Proof. Let $r_{ih} : M_{012...\infty h} \to \overline{M}_{012...\infty h}$ be a pure chaotic retraction of $M_{012...\infty h}$ onto $\overline{M}_{012...\infty h}$, then there is an induced sequence of retractions

$M_{1h} - P_{1h} \xrightarrow{r_{1h}} \overline{M}_{1h} \xrightarrow{r_{2h}} \overline{M}_{1h} \longrightarrow \dots \xrightarrow{r_{\infty h}} \tilde{M}_{1h}$
$M_{2h} - P_{2h} \xrightarrow{r_{1h}} \overline{M}_{2h} \xrightarrow{r_{2h}} \overline{M}_{2h} \longrightarrow \dots \xrightarrow{r_{\infty h}} \tilde{M}_{2h},$
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|
$M_{\infty h} - P_{\infty h} \xrightarrow{r_{1h}} \overline{M}_{\infty h} \xrightarrow{r_{2h}} \overline{M} \longrightarrow \dots \xrightarrow{r_{\infty h}} \tilde{M}_{\infty h}$, and
$M_{0h} \xrightarrow{r_{1h}} M_{0h} \xrightarrow{r_{2h}} M_{0h} \longrightarrow \dots \xrightarrow{r_{\infty h}} M_{0h}$. See Fig. (8).
i.e. all r_{ih} on M_{0h} is the identity map.

Lemma 2. *The end of the limit of foldings is equal to the end of retractions.*

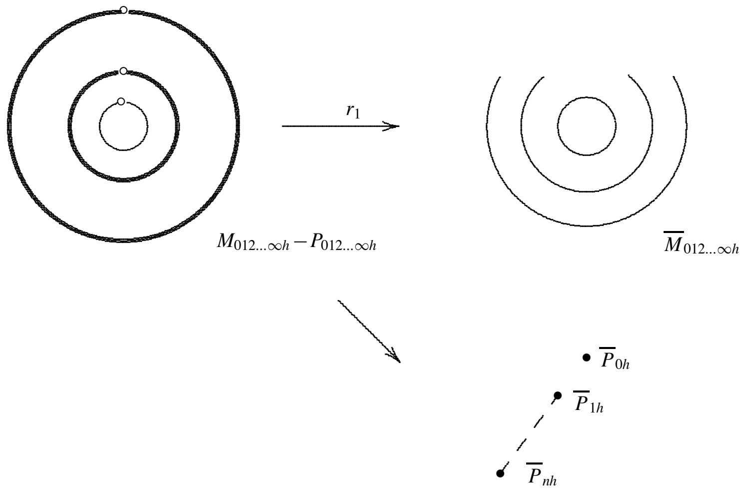
r1
M012...∞h − P012...∞h
M012...∞h
P0h
P1h
Pnh

Fig. (7)

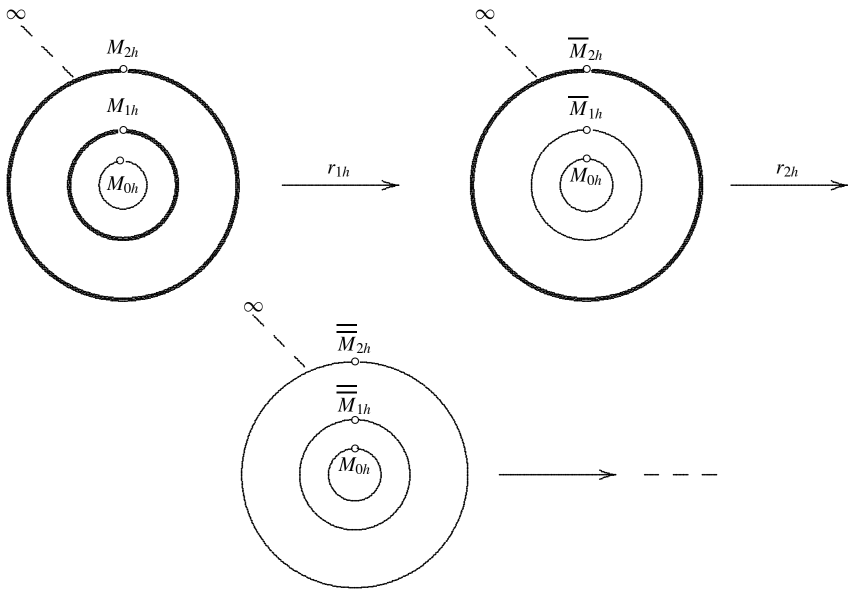
∞
M2h
M1h
M0h
r1h
∞
M2h
M1h
M0h
r2h
∞
M2h
M1h
M0h

Fig. (8)

Proof. Since the end of the limit of n-dimensional chaotic manifold is a 0-chaotic manifold (i.e. a point which carries ∞ physical character) and by Theorem 1 the result can be obtained. $M_{012\ldots\infty h} \xrightarrow{r_{1h}} M^1 \xrightarrow{r_2} M^2 \ldots \lim\limits_{n\to\infty} r_n(M^n) = M^{n-1}$. If r_i preserves the dimension there are induced equivalent sequence of foldings $M_{012\ldots\infty h} \xrightarrow{f_1} M^1 \xrightarrow{f_2} M^2 \ldots \lim\limits_{n\to\infty} f_n(M^n) = M^{n-1}$, where f_i is a folding.

$$\begin{array}{ccc} M_{012\ldots\infty h} - P & \xrightarrow{r_i} & M^i - P_i \\ {\scriptstyle P_{i-1}}\downarrow & & \downarrow{\scriptstyle P_i} \\ M_{012\ldots\infty h} - P & \xrightarrow{f_i} & M^i - P_i \end{array}$$

$P_i \circ r_i = P_{i-1} \circ f_i$ and $P_i \circ \lim\limits_{n\to\infty} r_{in} = P_{i-1} \circ \lim\limits_{n\to\infty} f_{in} = M^{n-1}_{012\ldots\infty h}$, $\lim\limits_{n\to\infty} f_{in}(M^{n-1}) = M^{n-2}$. The end of the limit of folding will be a point i.e. $\lim\limits_{n\to\infty} f_n(M^1) = P$ (0-manifold), which identical to the end of the limitof retractions.

Now we will discuss the retraction of the density and fractal chaotic manifolds and the effect of retraction on the fractal chaotic manifold.

Definition. The fractal dimension of the chaotic manifold:

Let $M_{012\ldots\infty h}$ be a chaotic manifold such that M_{0h} is the geometric manifold and let any point $P = (x_1, x_2, \ldots, x_{n+1}) \in M_{0h}$. If $\forall P_i \in M_{0h}$, $P_i = (x_1, x_2, \ldots, \varepsilon x_{n+1})$, there are chaotic points:

$P_1 = (x_1, x_2, \ldots, \varepsilon_1 x_{n+1})$,
$P_2 = (y_1, y_2, \ldots, \varepsilon_2 y_{n+1})$,
|
|
|
$P_n = (z_1, z_2, \ldots, \varepsilon_n z_{n+1})$, $\varepsilon_1 << I, \varepsilon_1 \to 0$,

then, the dimension of $M_{012\ldots\infty h} = n + 1/p$, p is positive integer $p > I$(i.e.$M^{n+i/p}_{012\ldots\infty h}$). Then, the dimension of M_{1h}, M_{2h}, ...and $M_{\infty h}$ is $n+1/p$, $p > 1$. If any point on M_{0h} is in the form $P_{0h} = (\varepsilon_1 x_1, \varepsilon_2 x_2, \ldots, \varepsilon_{n+1} x_{n+1})$ and $\varepsilon_j << 1$, where $\varepsilon_j = \max \varepsilon_1$, then $M^{1/p}_{Oh}$, $p > 1$, and any point on M_{1h} is $P_{1h} = (\varepsilon_{1h} x_{1h}, \varepsilon_{2h} x_{2h}, \ldots, \varepsilon_{(n+1)h} x_{(n+1)h})$, $\varepsilon_{jh} << 1$, where $\varepsilon_{jh} = \max \varepsilon_{ih}$ (i.e. the dimension of the pure chaotic manifold is equal to the dimension of the geometric manifold and is equal to $1/p$).

Fractal retraction of the chaotic manifold.

Let $r_i : (x_1, x_2, \dots, x_i, \dots, x_{n+1}) \to (x_1, x_2, \dots, \varepsilon_1 x_i, \dots x_{n+1})$ is the retraction of the coordinate x_i where $\varepsilon_1 << 1$. :par There are many retraction $r_{i(jh)}$ such that $r_{i(jh)} : (x_1^j, x_2^j, \dots, x_{n+1}^j) \to (x_1^j, x_2^j, \dots, \varepsilon_i x_i^j, x_{n+1}^j)$ and this retraction induces a fractal dimension $M_{ih}^{n+1/p}$ $\forall i = 1, 2, \dots, \infty$.

Fractal folding of the chaotic manifold.

We will define a type of folding:
$f : (x_1, x_2) \to (x_1, Ax_2), A \leq 1.$

The fractal folding of a chaotic manifold
$f_0 : M_{0h}^{n+1/p} \to M_{0h}^{n+1/p}$
$f_0(x_1, x_2, \dots, x_n, \varepsilon_1 x_{n+1}) \to (x_1, x_2, \dots, x_n, \varepsilon_1 |x_{n+1}|), |\varepsilon_1| \prec 1.$
$f_{01} : M_{0h}^{n+1/p} \to M_{0h}^{n+1/p}$
$f_{01}(x_1, x_2, \dots, x_n, \varepsilon_1 x_{n+1}) \to (x_1, x_2, \dots, x_n, \varepsilon_2 x_{n+1})$ where $\varepsilon_2 < \varepsilon_1$,
$f_{02}(x_1, x_2, \dots, x_n, \varepsilon_2 x_{n+1}) \to (x_1, x_2, \dots, x_n, \varepsilon_3 x_{n+1})$ where $\varepsilon_3 < \varepsilon_2$,
|
|
|
$f_{0n}(x_1, x_2, \dots, x_n, \varepsilon_n x_{n+1}) \to (x_1, x_2, \dots, x_n, \varepsilon_{n+1} x_{n+1})$ where $\varepsilon_{n+1} < \varepsilon_n, \varepsilon_{n+1} \to 0$.
$\lim_{n\to\infty} f_{0n}(M_{oh}^{n+1/p}) = M^n.$

There are a corresponding induced sequences of foldings
$f_{1h} : M_{1h}^{n+1/p} \to M_{1h}^{n+1/p}$
$f_{2h} : M_{2h}^{n+1/p} \to M_{2h}^{n+1/p}$
|
|
|
$f_{\infty h} : M_{\infty h}^{n+1/p} \to M_{\infty h}^{n+1/p}$,
$f_{1h1}, f_{1h2}, ---, f_{1hn}$,
$f_{2h1}, f_{2h2}, ---, f_{2hn}$,
|
|
|
$f_{nh1}, f_{nh2}, ---, f_{nhm}$
such that
$\lim_{n\to\infty} f_{1hn}(M_{1h}^{n+1/p}) = M_{1h}^n$,

$$\lim_{n\to\infty} f_{2hn}(M_{2h}^{n+1/p}) = M_{2h}^{n},$$
$$|$$
$$|$$
$$|$$
$$\lim_{n\to\infty} f_{mhn}(M_{nh}^{n+1/p}) = M_{mh}^{n}.$$

Theorem 4. *The fractal retraction of the geometric chaotic manifold induces a fractal dimensional of the pure chaotic manifold but the inverse is not true.*

Proof. Since the fractal retraction of the geometric chaotic manifold is the retraction of the coordinate, then by theorem (1) it induces a retractions of every pure chaotic manifold and this retractions will be also in dimension but the inverse by lemma (1) is not true.

Applications.

(1) The random motion of the electron during the disturbance of the effect of a strong field on the atom or during the distroy of the atoms the graph of the motion of the electron represents a chaotic manifold.
(2) The unstable periodic motion, this instability means that no two chaotic manifolds can be built that provides the same output. however, it has been shown that if two identical stable manifolds are driven by the same chaotic signal.

REFERENCES

[1] B. Cannas - S. Cincotti, *Neural reconstruction of Lorenz attractors by observable,* Chaos, Solitons and Fractals, 14 (2002), pp. 81–86.

[2] M. El-Ghoul, *Unfolding of graps and uncertain graph,* The Australian Senior Mathematics Journal Sandy Bay 7006 Tasmania Australia (accepted).

[3] M. El-Ghoul, *Folding of fuzzy graphs and fuzzy spheres,* Fuzzy Sets Syst., 58 (1993), pp. 355–363.

[4] M. El-Ghoul, *Folding of fuzzy torus and fuzzy graphs,* Fuzzy Sets Syst., 80 (1996), pp. 389–396.

[5] M. El-Ghoul, *The limit of folding of a manifold and its deformation retract,* J. Egypt Math. Soc., 5 – 2 (1997), pp. 133–140.

[6] M. El-Ghoul, *Unfolding of Riemannian manifolds,* Comm. Fac. Sci. Univ. Ankara Ser. A, 37 (1988), pp. 1–4.

[7] M. El-Ghoul, *Folding of manifoldd,* PhD Thesis, University of Tanta, Egypt, 1985.

[8] M. El-Ghoul - H. M. Shamara, *Folding of some types of fuzzy manifolds and their retractions,* Fuzzy Sets Syst., 97 (1988), pp. 387–391.

[9] M. El-Ghoul, *The deformation retract of the complex projective space and its topological folding,* J. Mater Sci., 30 – 4 (1995), pp. 45–48.

[10] M. El-Ghoul, *The deformation retract and topological folding of a manifold,* Comm. Fac. Sci. Univ. Ankara Ser. A, 37 (1998), pp. 1–4.

[11] M. El-Ghoul, *Fractional folding of manifold,* Chaos, Solitons and Fractals, 12 (2001), pp. 1019–1023.

[12] M. El-Ghoul, *Fractal dimension of a manifold,* Chaos, Solitons and Fractals, 14 (2002), pp. 77–80.

[13] E.M. El-Kholy - Al-Khusoni, *Folding of CW-complexes,* J. Inst. Math. Comp. Sci. (Math Ser), 4 – 1 (1991), pp. 41–48.

[14] E.M. El-Kholy - M. El-Ghoul, *Simpilicial folding,* J. Fac. Education, Ain Shams Un., 7 (part. II) (1984), pp. 127–139.

[15] Robertson Ellis - Denny Gulick, *Calculus with analytic geometry-Harcourt Brace Jovanovich,* New York, U.S.A.

[16] S.A. Robertson, *Isometric folding of Riemannian manifolds,* Proc. R. Soc. Edinburgh, 77 (1977) pp. 275–289.

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