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COLOURINGS OF VOLOSHIN FOR ATS (v)

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A mixed hypergraph is a triple H=(**S**,C,D), where **S** is the *vertex set* and each of C,D is a family of not-empty subsets of **S**, the C-*edges* and D-*edges* respectively. A strict k-*colouring* of H is a surjection f from the vertex set into a set of colours $\{1, 2, \ldots, k\}$ so that each C-edge contains at least two distinct vertices x, y such that $f(x) = f(y)$ and each D-edge contains at least two vertices x, y such that $f(x) \neq f(y)$. For each $k \in \{1, 2, \ldots, |\mathbf{S}|\}$, let r_k be the number of partitions of the vertex set into k not-empty parts (the *colour classes*) such that the colouring constraint is satisfied on each C-edge and D-edge. The vector $\mathbf{R}(H) = (r_1, \ldots, r_k)$ is called the *chromatic spectrum* of H. These concepts were introduced by V. Voloshin in 1993 [6].

In this paper we examine colourings of mixed hypergraphs in the case that H is an **ATS**(v).

1. Introduction.

A *mixed hypergraph* is a triple H=(**S**,C,D), where **S** is the *vertex set* and each of C,D is a family of subsets of **S**, the C-*edges* and D-*edges* respectively. A proper k-*colouring* of a mixed hypergraph is a mapping f from the vertex set into a set of colours $\{1, 2, \ldots, k\}$ so that each C-edge contains at least two distinct vertices x, y such that $f(x) = f(y)$ and each D-edge contains at least two vertices x, y such that $f(x) \neq f(y)$. If $C = D$, then H is called a *bi-hypergraph*.

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A mixed hypergraph is called *k-colourable* if it admits a proper colouring with at most k colours; it is called *uncolourable* if it admits no colouring. A *strict* k-colouring is a proper k-colouring using all k colours. The minimum number of colours in a colouring of H is called the *lower chromatic number* $\chi(H)$, the maximum number of colours in a strict colouring of H is called the *upper chromatic number* $\chi^*(H)$.

If $|\mathbf{S}| = n$, for each $k \in \{1, 2, \ldots, n\}$, let r_k be the number of partitions of the vertex set into k not-empty parts (called *colour classes*) such that the colouring constraint is satisfied on each C-edge and D-edge. In fact, r_k is the number of different strict k-colourings if we ignore permutations of colours. The vector $\mathbf{R}(H) = (r_1, \ldots, r_k)$ is called the *chromatic spectrum* of H.

These concepts were introduced by V. Voloshin in 1993 [6].

A **Steiner System** $\mathbf{S}_\lambda(t, k, v)$ $(t, k, v, \lambda \in \mathbf{N})$ is a pair $(\mathbf{S}, B)$ where $\mathbf{S}$ is a finite set of v vertices and B is a family of subsets of $\mathbf{S}$ called *blocks* such that:

1) each block contains k vertices;
2) for each t-subset $\mathbf{T}$ of $\mathbf{S}$, there exist *exactly* λ blocks containing $\mathbf{T}$.

If $\lambda = 1$, a system $\mathbf{S}_1(t, k, v)$ is denoted by $\mathbf{S}(t, k, v)$. A system $\mathbf{S}(2, 3, v)$ is called a **Steiner Triple System** and is denoted by $\mathbf{STS}(v)$. As it is well known, there exists an $\mathbf{STS}(v)$ if and only if $v \equiv 1 \pmod 6$ or $v \equiv 3 \pmod 6$.

An **Almost Triple System** of order v, briefly an $\mathbf{ATS}(v)$, is a pair $(\mathbf{S}, B)$ where $\mathbf{S}$ is a finite set of v vertices and B is a family of subsets of $\mathbf{S}$, called *blocks*, such that:

1) there exists *exactly* one block containing 5 vertices;
2) all the other blocks contain 3 vertices;
3) each pair of vertices of $\mathbf{S}$ is contained in *exactly* one block of B.

It is possible to prove that an $\mathbf{ATS}(v)$ exists if and only if $v \equiv 5 \pmod 6$.

In what follows, the block containing five vertices will be always denoted by b^*.

We illustrate now a technique for a recurrent construction of $\mathbf{ATS}(v)$. It is called $(v \to 2v+1)$-*construction* and allows to obtain an $\mathbf{ATS}(2v+1)$ from an $\mathbf{ATS}(v)$. We will refer to this construction as **construction A**.

Construction A

Let $(\mathbf{S}, B)$ be an $\mathbf{ATS}(v)$, where $\mathbf{S} = \{x_1, \ldots, x_v\}$, and let $\mathbf{T} = \{y_1, \ldots, y_{v+1}\}$ be a $(v+1)$-set of vertices disjoint from $\mathbf{S}$. As $v+1$ is an even number, it is possible to consider a 1-factorization $F = (F_1, F_2, \ldots, F_v)$ of the complete graph $\mathbf{K}_{v+1}$ defined on $\mathbf{T}$. Let be $\mathbf{S}' = \mathbf{S} \cup \mathbf{T}$, $B' = B \cup C$, where the set C is defined as follows:

$$\forall i \in \{1, \ldots, v\}\ \{x_i, y', y''\} \in C \leftrightarrow \{y', y''\} \in F_i\,.$$

It is easy to prove that $H' = (\mathbf{S}', B')$ is an $\mathbf{ATS}(2v+1)$.

In what follows, we will consider $\mathbf{ATS}(v)$ as mixed hypergraphs in which $C = D$: we will call them $\mathbf{BATS}(v)$.

2. Preliminary results.

In this section we prove some general properties for $\mathbf{BATS}(v)$.

Theorem 2.1. *Let H be a $\mathbf{BATS}(v)$ with $\chi^*(H) = k$ and let H' be a $\mathbf{BATS}(2v+1)$ obtained from H by a construction A. Then*

i) $\chi^*(H') \leq k+1$

ii) If H is h-colourable, then H' is $(h+1)$- colourable.

Proof. Following the symbolism of construction A, let be $H = (\mathbf{S}, B)$, $H' = (\mathbf{S}', B')$ respectively an $\mathbf{ATS}(v)$ and an $\mathbf{ATS}(2v+1)$, where $|\mathbf{S}| = v$, $|\mathbf{S}'| = 2v+1$, $\mathbf{T} = \mathbf{S}' - \mathbf{S} = \{y_1, y_2, \ldots, y_{v+1}\}$. Since $\chi^*(H) = k$, let f be a k-colouring of H. Suppose that g is an h-colouring of H', for $h \geq k+2$. Since $\chi^*(H) = k$, then there exist at least two vertices $y', y'' \in \mathbf{T}$ such that $g(y') \neq g(y'')$ and $\{g(y'), g(y'')\} \cap g(\mathbf{S}) = \emptyset$. If $\{y', y''\} \in F_j$, then $\{x_j, y', y''\} \in B'$ and the triple $\{x_j, y', y''\}$ doesn't contain two vertices with a common colour. Therefore, for every h-colouring of H', $h \leq k+1$. Further, there exists a $(k+1)$-colouring of H': it suffices to extend the k-colouring f of H to H', associating with all the vertices of $\mathbf{T}$ a same colour, different from the k colours used for the vertices of H. It follows $\chi^*(H') = k+1$.

The second statement follows considering that it is always possible to give a same colour to the vertices of $\mathbf{T}$, distinct from all the colours used for the vertices of $\mathbf{S}$. □

Theorem 2.2. *Let $H = (\mathbf{S}, B)$ be a $\mathbf{BATS}(v)$ with $\chi^*(H) = k$ and let H' be a $\mathbf{BATS}(2v+1)$ obtained from H by a construction $\mathbf{A}$. If there exists a k-colouring f of H', then*

$$\begin{cases} \sum_{i=1}^{k} \Big(x_i^2 + \big(2a_i - 1\big)x_i \Big) = v(v+1) \\ \sum_{i=1}^{k} x_i = v+1 \end{cases}$$

where, for each $i \in \{1, 2, \ldots, k\}$, a_i, x_i are respectively the number of vertices of $\mathbf{S}$ and $\mathbf{S}' - \mathbf{S}$ coloured by the colour i in f.

Proof. Since $\chi^*(H) = k$, then, for every k-colouring f of H', $f/\mathbf{S}$ is a k-colouring of $\mathbf{S}$. The second equality is immediate. Prove the first. Consider a colour i, $i \in \{1, 2, \ldots, k\}$. If F is the 1-factorization of $\mathbf{K}_{v+1}$ on $\mathbf{T} = \mathbf{S}' - \mathbf{S}$ used to define H', then there are a_i factors of F associated with a_i vertices of $\mathbf{S}$ coloured by i. So, in $\mathbf{T}$ there are $a_i x_i$ pairs having exactly one vertex coloured by the colour i and

$$\binom{x_i}{2}$$

pairs having both vertices coloured by i.

Therefore, the number of monochromatic pairs of $\mathbf{T}$ is:

$$\sum_{i=1}^{k} \binom{x_i}{2} = \sum_{i=1}^{k} a_i \left(\frac{v+1}{2} - x_i \right)$$

hence

$$\sum_{i=1}^{k} \left(x_i^2 - x_i \right) = \sum_{i=1}^{k} \left(a_i(v+1) - 2a_i x_i \right)$$

from which, by a simple calculation, we obtain the first equality and the statement follows. $\square$

Theorem 2.3. *Let H be a* **BATS**(v). *If $v > 5$, then H is not 2-colourable.*

Proof. Suppose $\chi(H) = 2$ and let **A, B** the colour classes of a 2-colouring of H, $|\mathbf{A}| = p$, $|\mathbf{B}| = v - p$. We say of type 1 the blocks b of H such that $|\mathbf{A} \cap b| = 1$, $|\mathbf{B} \cap b| = 2$ and of type 2 the blocks b of H such that $|\mathbf{A} \cap b| = 2$, $|\mathbf{B} \cap b| = 1$. Let b* be the block of size 5.

Suppose $|\mathbf{A} \cap b^*| = 2$, $|\mathbf{B} \cap b^*| = 3$. The number of blocks of H is:

$$\left[\binom{p}{2} - 1 \right] + \left[\binom{v-p}{2} - 3 \right] + 1 = \frac{v(v-1) - 14}{6}$$

hence

$$3p^2 - 3pv + v^2 - v - 2 = 0$$

and so $v = 5$, $p = 2, 3$.

Suppose $|\mathbf{A} \cap b^*| = 1$, $|\mathbf{B} \cap b^*| = 4$. If we add the number of blocks of type 1 to the number of blocks of type 2, we obtain:

$$\binom{p}{2} + \left[\binom{v-p}{2} - 6 \right] + 1 = \frac{v(v-1) - 14}{6}$$

hence

$$3p^2 - 3pv + v^2 - v - 8 = 0$$

and so $v = 5$, $p = 1, 4$.

The statement is proved. □

3. Colourings for BATS(11).

In what follows, we indicate by the sequence $\mathbf{A}^{n_1}\mathbf{B}^{n_2}, \ldots$ a colouring of a mixed hypergraph H which associates the colour **A** with n_1 vertices, the colour **B** with n_2 vertices,.... If H is a **BATS**(5), then it admits only the 2-colourings $\mathbf{A}^4\mathbf{B}$, $\mathbf{A}^3\mathbf{B}^2$, the 3-colourings $\mathbf{A}^3\mathbf{BC}$, $\mathbf{A}^2\mathbf{B}^2\mathbf{C}$ and the 4-colouring $\mathbf{A}^2\mathbf{BCD}$, so that $\chi(H) = 2$, $\chi^*(H) = 4$.

In what follows, H will be an **ATS**(11) defined on $\mathbf{S}' = \{1, 2, \ldots, 11\}$. Further: $\mathbf{S} = \{1, 2, 3, 4, 5\}$, $\mathbf{T} = \mathbf{S}' - \mathbf{S}$, f will be a colouring of H, $F = (F_1, F_2, \ldots, F_5)$ will be a 1-factorization of $\mathbf{K}_6$ defined on $\mathbf{T}$. The blocks of H which are not contained in $\mathbf{S}$ are the triples $\{i, x, y\}$, for every $i \in \{1, 2, .., 5\}$ and $\{x, y\} \in F_i$. To within to isomorphisms the triples of an **ATS**(11) are obtained from a 1-factorization of the type :

F_1	F_2	F_3	F_4	F_5
$6-7$	$6-10$	$6-9$	$6-11$	$6-8$
$8-10$	$7-11$	$7-10$	$7-8$	$7-9$
$9-11$	$8-9$	$8-11$	$9-10$	$10-11$

Theorem 3.1. *All possible* 3*-colourings for a* **BATS**(11) *are of type* $\mathbf{A}^6\mathbf{BC}^4$, $\mathbf{A}^3\mathbf{B}^2\mathbf{C}^6$, $\mathbf{A}^5\mathbf{B}^4\mathbf{C}^2$.

Proof. Let H be a **BATS**(11). From Theorem 2.1, ii), H is 3-colourable. Let f be a 3-colouring of H. Observe that $f/\mathbf{S}$ can be a 2-colouring or a 3-colouring on $\mathbf{S}$.

We denote by x_A, x_B, x_C the colour class cardinalities on $\mathbf{T}$ and a, b, c the colour class cardinalities on $\mathbf{S}$. By Theorem 2.2, we have $x_A^2 + x_B^2 + x_C^2 + (2a-1)x_A + (2b-1)x_B + (2c-1)x_C = 30$, $x_A + x_B + x_C = 6$.

If $f/\mathbf{S}$ is a 2-colouring $\mathbf{A}^4\mathbf{B}$ on $\mathbf{S}$, we have $a = 4$, $b = 1$, $c = 0$, and $x_A^2 + x_B^2 + x_C^2 + 7x_A + x_B - x_C = 30$, $x_A + x_B + x_C = 6$, $x_C > 0$. The possible solutions are: (0, 5, 1), (2, 3, 1), (2, 0, 4), (0, 0, 6). Since $x_A \le 3$, $x_B \le 3$, the first solution doesn't imply a colouring; further, in the second triple, $x_C = 1$ implies $x_A \ge 4$ and this is not possible. The triple (2, 0, 4) implies a 3-colouring $\mathbf{A}^6\mathbf{BC}^4$. The triple (0, 0, 6) implies the 3-colouring $\mathbf{A}^4\mathbf{BC}^6$.

If $f/\mathbf{S}$ is a 2-colouring $\mathbf{A}^3\mathbf{B}^2$ on $\mathbf{S}$, we have $a = 3$, $b = 2$, $c = 0$, and $x_A^2 + x_B^2 + x_C^2 + 5x_A + 3x_B - x_C = 30$, $x_A + x_B + x_C = 6$, $x_C > 0$. The possible solutions are: $(0, 4, 2)$, $(3, 1, 2)$, $(3, 0, 3)$, $(0, 0, 6)$. Since $x_B \le 3$, the first solution is not acceptable. The second and the third solutions imply the existence of 3-chromatic blocks. The solution $(0, 0, 6)$ implies a 3-colouring $\mathbf{A}^3\mathbf{B}^2\mathbf{C}^6$.

If $f/\mathbf{S}$ is a 3-colouring $\mathbf{A}^3\mathbf{BC}$ on $\mathbf{S}$, we have $a = 3$, $b = 1$, $c = 1$, and $x_A^2 + x_B^2 + x_C^2 + 5x_A + x_B + x_C = 30$, $x_A + x_B + x_C = 6$, $x_C > 0$. It is possible to prove that there are not natural solutions.

If $f/\mathbf{S}$ is a 3-colouring $\mathbf{A}^2\mathbf{B}^2\mathbf{C}$ on $\mathbf{S}$, we have $a = 2$, $b = 2$, $c = 1$, $x_A^2 + x_B^2 + x_C^2 + 3x_A + 3x_B + x_C = 30$, $x_A + x_B + x_C = 6$, $x_C > 0$. The possible solutions are: $(2, 3, 1)$, $(3, 2, 1)$, $(0, 3, 3)$, $(3, 0, 3)$, $(0, 2, 4)$, $(2, 0, 4)$. Since $x_C \le 3$, the last two solutions are not acceptable. The third and the fourth solutions imply the existence of 3-chromatic blocks. The triple $(2, 3, 1)$ implies the 3-colouring $\mathbf{A}^4\mathbf{B}^5\mathbf{C}^2$. The solution $(3, 2, 1)$ implies a 3-colouring $\mathbf{A}^5\mathbf{B}^4\mathbf{C}^2$.

The statement is proved. □

Theorem 3.2. *All possible* 4*-colourings for a* **BATS**(11) *are of type* $\mathbf{A}^3\mathbf{BCD}^6$, $\mathbf{A}^2\mathbf{B}^2\mathbf{CD}^6$.

Proof. Let H be a **BATS**(11) and let f be a 4-colouring of H. Observe that $f/\mathbf{S}$ can be a 3- or a 4-colouring on $\mathbf{S}$. Denote by x_A, x_B, x_C, x_D the colour class cardinalities on $\mathbf{T}$ and a, b, c, d the colour class cardinalities on $\mathbf{S}$.
By Theorem 2.2, $x_A^2 + x_B^2 + x_C^2 + x_D^2 + (2a - 1)x_A + (2b - 1)x_B + (2c - 1)x_C + (2d - 1)x_D = 30$, $x_A + x_B + x_C + x_D = 6$.

If $f/\mathbf{S}$ is a 3-colouring $\mathbf{A}^3\mathbf{BC}$ on $\mathbf{S}$, then $a = 3$, $b = 1$, $c = 1$, $d = 0$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + 5x_A + x_B + x_C - x_D = 30$, $x_A + x_B + x_C + x_D = 6$, $x_D > 0$. Further: *i)* $x_A \le 3$, $x_B \le 3$, $x_C \le 3$; and *ii)* if one among x_A, x_B, x_C is odd, then the other two must be positive. The only possible solution is $(0, 0, 0, 6)$, that implies the 4-colouring $\mathbf{A}^3\mathbf{BCD}^6$.

If $f/\mathbf{S}$ is a 3-colouring $\mathbf{A}^2\mathbf{B}^2\mathbf{C}$ on $\mathbf{S}$, then $a = 2$, $b = 2$, $c = 1$, $d = 0$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + 3x_A + 3x_B + x_C - x_D = 30$, $x_A + x_B + x_C + x_D = 6$, $x_D > 0$, with the condition *i)* and *ii)* shown above. Also in this case, the only possible solution is $(0, 0, 0, 6)$, that implies a 4-colouring of type $\mathbf{A}^2\mathbf{B}^2\mathbf{CD}^6$.

Finally, if $f/\mathbf{S}$ is a 4-colouring on $\mathbf{S}$, it is necessarily of type $\mathbf{A}^2\mathbf{BCD}$, so that we have $a = 2$, $b = 1$, $c = 1$, $d = 1$, $x_A^2 + x_B^2 + x_C^2 + x_D^2 + 3x_A + x_B + x_C + x_D = 30$, $x_A + x_B + x_C + x_D = 6$, with the condition *i)* and *ii)* shown above. There is no solution and then the assertion of theorem follows. □

Theorem 3.3. *All possible* 5*-colourings for a* **BATS**(11) *are of type* $\mathbf{A}^2\mathbf{BCDE}^6$.

Proof. Let H be a **BATS**(11) and let f be a 5-colouring of H. Necessarily, $f/\mathbf{S}$ is 4-colouring on $\mathbf{S}$ and it can be only of type $\mathbf{A}^2\mathbf{BCD}$. From Theorem 3.2, the only possible colouring for H is a 5-colouring, which can be only of type $\mathbf{A}^2\mathbf{BCDE}^6$. □

A consequence:

Corollary. *For each* **ATS**(11)*, there exist only* 3*-colourings,* 4*-colourings,* 5*-colourings.*

4. Colourings for BATS(23).

The terminology is the same of Section 3. In what follows, every **BATS**(23) is obtained from a **BATS**(11) by construction **A**; it will be $\mathbf{S} = \{1, 2, \ldots, 11\}$, $\mathbf{T} = \mathbf{S}' - \mathbf{S} = \{12, 13, \ldots, 23\}$.

By Theorem 2.3, $\chi(H) \geq 3$ for all colourable **BATS**(23). Further, if we denote by x_i the i-colour class cardinality on **T** and a_j the j-colour class cardinality on **S**, we can prove the following Lemma:

Lemma 4.1. *Let H be a* 3*-colourable* **BATS**(23) *obtained from a* **BATS**(11) *by construction* **A**. *Then*

i) $x_A \leq 6$, $x_B \leq 6$, $x_C \leq 6$
ii) if $x_i, x_j \in \{x_A, x_B, x_C\}$ *for* $i \neq j$, *then* $x_i \leq a_i + a_j$, $x_j \leq a_i + a_j$.

Proof. Observe that $i)$ is immediate, otherwise there exist a monochromatic triple. For $ii)$ consider that if $x_i > a_i + a_j$ for some pair i, j, then an item x of **T** coloured by j forms x_i pairs with items of **T** coloured by i. These pairs should form triples with an element of **S** coloured necessarily by i or j; it follows that $a_i + a_j > x_i$, and it is not possible. □

Theorem 4.2. *All possible* 3*-colourings for a* **BATS**(23) *are of type* $\mathbf{A}^{10}\mathbf{B}^4\mathbf{C}^9$, $\mathbf{A}^6\mathbf{B}^6\mathbf{C}^{11}$, $\mathbf{A}^{10}\mathbf{B}^8\mathbf{C}^5$.

Proof. Let H be a **BATS**(23) and let f be a 3-colouring of H. Observe that $f/\mathbf{S}$ must be a 3-colouring on **S**. We denote by x_A, x_B, x_C the colour class cardinalities on **T** and a, b, c the colour class cardinalities on **S**. By Theorem 2.2, we have $x_A^2+x_B^2+x_C^2+(2a-1)x_A+(2b-1)x_B+(2c-1)x_C = 132$, $x_A + x_B + x_C = 12$.

If $f/\mathbf{S}$ is a 3-colouring $\mathbf{A}^6\mathbf{BC}^4$ on **S**, then we have $a = 6, b = 1, c = 4$, so that $x_A^2 + x_B^2 + x_C^2 + 11x_A + x_B + 7x_C = 132$, $x_A + x_B + x_C = 12$, with the conditions *i)* and *ii)* of Lemma 4.1. There is only one possible solution: $(4, 3, 5)$. It gives a colouring $\mathbf{A}^{10}\mathbf{B}^4\mathbf{C}^9$. A possible colouring is: $\mathbf{A} = \{1, 2, 3, 4, 5, 6, 12, 13, 14, 15\}$, $\mathbf{B} = \{7, 16, 17, 18\}$, $\mathbf{C} =$

$\{8, 9, 10, 11, 19, 20, 21, 22, 23\}$, with the 1-factorization shown in Table 1 [*see Appendix*].

If $f/\mathbf{S}$ is a 3-colouring $\mathbf{A}^3\mathbf{B}^2\mathbf{C}^6$ on $\mathbf{S}$, then we have $a = 3$, $b = 2$, $c = 6$, so that $x_A^2 + x_B^2 + x_C^2 + 5x_A + 3x_B + 11x_C = 132$, $x_A + x_B + x_C = 12$, with the conditions *i)* and *ii)* of Lemma 4.1. There is only one possible solution: $(3, 4, 5)$. It gives a 3-colouring $\mathbf{A}^6\mathbf{B}^6\mathbf{C}^{11}$. A possible colouring is: $\mathbf{A} = \{1, 2, 3, 12, 13, 14\}$, $\mathbf{B} = \{4, 5, 15, 16, 17, 18\}$, $\mathbf{C} = \{6, 7, 8, 9, 10, 11, 19, 20, 21, 22, 23\}$, with the 1-factorization shown in Table 2 [*see Appendix*].

If $f/\mathbf{S}$ is a 3-colouring $\mathbf{A}^5\mathbf{B}^4\mathbf{C}^2$ on $\mathbf{S}$, then we have $a = 5$, $b = 4$, $c = 2$, so that $x_A^2 + x_B^2 + x_C^2 + 9x_A + 7x_B + 3x_C = 132$, $x_A + x_B + x_C = 12$, with the conditions *i)* and *ii)* of Lemma 4.1. The possible solutions are: $(0, 6, 6)$, $(3, 6, 3)$, $(5, 1, 6)$, $(5, 4, 3)$. The triple $(0, 6, 6)$ implies a 3-colouring $\mathbf{A}^5\mathbf{B}^{10}\mathbf{C}^8$. A possible colouring is: $\mathbf{A} = \{1, 2, 3, 4, 5\}$, $\mathbf{B} = \{6, 7, 8, 9, 18, 19, 20, 21, 22, 23\}$, $\mathbf{C} = \{10, 11, 12, 13, 14, 15, 16, 17\}$, with the 1-factorization shown in Table 3 [*see Appendix*].

The solution $(3, 6, 3)$ implies that a point $x \in \mathbf{S}$ coloured by $\mathbf{C}$ is associated with 3 pairs $\{y, z\} \subseteq \mathbf{T}$ coloured by $\mathbf{BC}$, one pair coloured by $\mathbf{AA}$, one pair coloured by $\mathbf{BB}$ and one pair coloured by $\mathbf{AB}$, and it is not acceptable. The solution $(5, 1, 6)$ implies that the pairs $\{y, z\} \subseteq \mathbf{T}$ coloured by $\mathbf{AA}$ cannot form a triple with a point $x \in \mathbf{S}$ coloured by $\mathbf{C}$; a point of $\mathbf{S}$ associated with a pair $\mathbf{AA}$ and it is not possible because $x_A = 5$ and $\mathbf{B}^4$. The only possible solution is the triple $(5, 4, 3)$ which gives a 3-colouring $\mathbf{A}^{10}\mathbf{B}^8\mathbf{C}^5$ similar to $\mathbf{A}^5\mathbf{B}^{10}\mathbf{C}^8$.

The assertion of theorem follows. □

Theorem 4.3. *All possible* 4*-colourings for a colourable* $\mathbf{BATS}(23)$ *are of type* $\mathbf{A}^6\mathbf{B}\mathbf{C}^4\mathbf{D}^{12}$, $\mathbf{A}^3\mathbf{B}^2\mathbf{C}^{12}\mathbf{D}^6$, $\mathbf{A}^5\mathbf{B}^4\mathbf{C}^2\mathbf{D}^{12}$.

Proof. Let H be a colourable $\mathbf{BATS}(23)$ and let f be a 4-colouring of H. Observe that $f/\mathbf{S}$ can be a 3-colouring on $\mathbf{S}$ of type $\mathbf{A}^6\mathbf{B}\mathbf{C}^4$, $\mathbf{A}^3\mathbf{B}^2\mathbf{C}^6$, $\mathbf{A}^5\mathbf{B}^4\mathbf{C}^2$ or a 4-colouring on $\mathbf{S}$ of type $\mathbf{A}^3\mathbf{BCD}^6$, $\mathbf{A}^2\mathbf{B}^2\mathbf{CD}^6$. If we denote by x_A, x_B, x_C, x_D the colour class cardinalities on $\mathbf{T}$ and a, b, c, d the colour class cardinalities on $\mathbf{S}$, then, from Theorem 2.2, we have $x_A^2 + x_B^2 + x_C^2 + x_D^2 + (2a-1)x_A + (2b-1)x_B + (2c-1)x_C + (2d-1)x_D = 132$, $x_A + x_B + x_C + x_D = 12$. Further: *i)* $x_A \leq 6$, $x_B \leq 6$, $x_C \leq 6$, $x_D > 0$; *ii)* if $x = 0$, $x \in \{x_A, x_B, x_C\}$, then all the others are even.

If $f/\mathbf{S}$ is a 3-colouring $\mathbf{A}^6\mathbf{B}\mathbf{C}^4$ on $\mathbf{S}$, then $a = 6$, $b = 1$, $c = 4$, $d = 0$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + 11x_A + x_B + 7x_C - x_D = 132$, $x_A + x_B + x_C + x_D = 12$, $x_D > 0$, with the conditions *i)* and *ii)* shown above. The possible solutions are: $(0, 0, 0, 12)$, $(6, 0, 0, 6)$, $(6, 0, 2, 4)$, $(6, 3, 2, 1)$.

The first solution gives a 4-colouring $\mathbf{A}^6\mathbf{B}\mathbf{C}^4\mathbf{D}^{12}$, for $\mathbf{A} = \{1, 2, 3, 4, 5, 6\}$, $\mathbf{B} = \{7\}$, $\mathbf{C} = \{8, 9, 10, 11\}$, $\mathbf{D} = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$.

The second and third solutions give 4-colourings of type $\mathbf{A}^{12}\mathbf{B}\mathbf{C}^4\mathbf{D}^6$, $\mathbf{A}^{12}\mathbf{B}\mathbf{C}^6\mathbf{D}^4$ respectively, which are similar to $\mathbf{A}^6\mathbf{B}\mathbf{C}^4\mathbf{D}^{12}$. The solution $(6, 3, 2, 1)$ is not acceptable because $x_B = 3$, $x_D = 1$ and only one 1-factor admits the existence of pairs coloured by **BD**.

If $f/\mathbf{S}$ is a 3-colouring of type $\mathbf{A}^3\mathbf{B}^2\mathbf{C}^6$ of $\mathbf{S}$, then $a = 3$, $b = 2$, $c = 6$, $d = 0$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + 5x_A + 3x_B + 11x_C - x_D = 132$, $x_A + x_B + x_C + x_D = 12$, $x_D > 0$, with the conditions *i)* and *ii)* shown above. The possible solutions are: $(0, 0, 0, 12)$, $(0, 0, 6, 6)$, $(0, 4, 6, 2)$, $(3, 1, 6, 2)$.

The first solution gives a 4-colouring $\mathbf{A}^3\mathbf{B}^2\mathbf{C}^6\mathbf{D}^{12}$, for $\mathbf{A} = \{1, 2, 3\}$, $\mathbf{B} = \{4, 5\}$, $\mathbf{C} = \{6, 7, 8, 9, 10, 11\}$, $\mathbf{D} = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The second solution gives a 4-colouring $\mathbf{A}^3\mathbf{B}^2\mathbf{C}^{12}\mathbf{D}^6$ similar to the previous one. The solution $(0, 4, 6, 2)$ is not acceptable because $x_D > 0$ and $x_B = 4$ imply the existence of at least 4 points of $\mathbf{S}$ coloured by $\mathbf{B}$, while it is $\mathbf{B}^2$. The solution $(3, 1, 6, 2)$ is not acceptable because $x_D = 2$ implies $x_B \geq 2$.

If $f/\mathbf{S}$ is a 3-colouring of type $\mathbf{A}^5\mathbf{B}^4\mathbf{C}^2$ of $\mathbf{S}$, then $a = 5$, $b = 4$, $c = 2$, $d = 0$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + 9x_A + 7x_B + 3x_C - x_D = 132$, $x_A + x_B + x_C + x_D = 12$, $x_D > 0$, with the conditions shown above. The possible solutions are: $(0, 0, 0, 12)$, $(4, 6, 0, 2)$. The first solution gives a 4-colouring $\mathbf{A}^5\mathbf{B}^4\mathbf{C}^2\mathbf{D}^{12}$, for $\mathbf{A} = \{1, 2, 3, 4, 5\}$, $\mathbf{B} = \{6, 7, 8, 9\}$, $\mathbf{C} = \{10, 11\}$, $\mathbf{D} = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The solution $(4, 6, 0, 2)$ is not acceptable because x_D=2 implies $x_C \geq$2.

Now we consider the cases in which $f/\mathbf{S}$ is a 4-colouring on $\mathbf{S}$. In these cases, *i)* $x_A \leq 6$, $x_B \leq 6$, $x_C \leq 6$, $x_D \leq 6$; *ii)* if $x = 0$, $x \in \{x_A, x_B, x_C, x_D\}$, then all the others are even.

If $f/\mathbf{S}$ is a 4-colouring of type $\mathbf{A}^3\mathbf{B}\mathbf{C}\mathbf{D}^6$ on $\mathbf{S}$, then $a = 3$, $b = 1$, $c = 1$, $d = 6$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + 5x_A + x_B + x_C + 11x_D = 132$, $x_A + x_B + x_C + x_D = 12$, with the conditions shown above. The possible solutions are: (6,0,2,4), (6,2,0,4). These solutions are not acceptable because $x_B = 2$ or $x_C = 2$ and $\mathbf{A}^3\mathbf{B}$ ($\mathbf{A}^3\mathbf{C}$) implies $x_A + x_B \leq 4$ ($x_A + x_C \leq 4$).

Finally, if $f/\mathbf{S}$ is a 4-colouring of type $\mathbf{A}^2\mathbf{B}^2\mathbf{C}\mathbf{D}^6$ on $\mathbf{S}$, then $a = 2$, $b = 2$, $c = 1$, $d = 6$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + 3x_A + 3x_B + x_C + 11x_D = 132$, $x_A + x_B + x_C + x_D = 12$, with the conditions shown above. The possible solutions are: $(0, 2, 4, 6)$, $(2, 0, 4, 6)$, $(2, 3, 1, 6)$, $(3, 2, 1, 6)$. The first two solutions are not acceptable because $x_D = 6$ implies $x \leq 3$ for every $x \in \{x_A, x_B, x_C\}$. The solutions $(2, 3, 1, 6)$, $(3, 2, 1, 6)$ imply 4-colourings $\mathbf{A}^4\mathbf{B}^5\mathbf{C}^2\mathbf{D}^{12}$ (respectively $\mathbf{A}^5\mathbf{B}^4\mathbf{C}^2\mathbf{D}^{12}$).

The assertion of theorem follows. □

Theorem 4.4. *All possible* 5*-colourings for a* **BATS**(23) *are of type* $\mathbf{A}^3\mathbf{BCD}^6\mathbf{E}^{12}$*,* $\mathbf{A}^2\mathbf{B}^2\mathbf{CD}^6\mathbf{E}^{12}$*,*

Proof. Let H be a colourable **BATS**(23) and let f be a 5-colouring of H. Observe that $f/\mathbf{S}$ can be a 4-colouring on **S** of type $\mathbf{A}^3\mathbf{BCD}^6$, $\mathbf{A}^2\mathbf{B}^2\mathbf{CD}^6$. If we denote by x_A, x_B, x_C, x_D, x_E the colour class cardinalities on **T** and a, b, c, d, e the colour class cardinalities on **S**, then, by Theorem 2.2, $x_A^2 + x_B^2 + x_C^2 + x_D^2 + x_E^2 + (2a-1)x_A + (2b-1)x_B + (2c-1)x_C + (2d-1)x_D + (2e-1)x_E = 132$, $x_A + x_B + x_C + x_D + x_E = 12$. Further: *i)* $x_A \leq 6$, $x_B \leq 6$, $x_C \leq 6$, $x_D \leq 6$; *ii)* if $x = 0$, $x \in \{x_A, x_B, x_C, x_D\}$, then all the others are even.

If $f/\mathbf{S}$ is a 4-colouring of type $\mathbf{A}^3\mathbf{BCD}^6$ on **S**, then $a = 3$, $b = 1$, $c = 1$, $d = 6$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + x_E^2 + 5x_A + x_B + x_C + 11x_D - x_E = 132$, $x_A + x_B + x_C + x_D + x_E = 12$, with the conditions *i)* and *ii)* shown above. The possible solutions are: $(0, 0, 0, 0, 12)$, $(0, 0, 0, 6, 6)$. The first solution implies a 5-colouring $\mathbf{A}^3\mathbf{BCD}^6\mathbf{E}^{12}$, for $\mathbf{A} = \{1, 2, 3\}$, $\mathbf{B} = \{4\}$, $\mathbf{C} = \{5\}$, $\mathbf{D} = \{6, 7, 8, 9, 10, 11\}$, $\mathbf{E} = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The second solution implies another 5-colouring of type $\mathbf{A}^3\mathbf{BCD}^{12}\mathbf{E}^6$, that is similar to the previous one.

If $f/\mathbf{S}$ is a 4-colouring of type $\mathbf{A}^2\mathbf{B}^2\mathbf{CD}^6$ on **S**, then $a = 2$, $b = 2$, $c = 1$, $d = 6$, so that $x_A^2 + x_B^2 + x_C^2 + x_D^2 + x_E^2 + 3x_A + 3x_B + x_C + 11x_D - x_E = 132$, $x_A + x_B + x_C + x_D + x_E = 12$, with the conditions i) and ii) shown above. The possible solutions are: $(0, 0, 0, 0, 12)$, $(0, 0, 0, 6, 6)$, $(0, 4, 0, 6, 2)$, $(4, 0, 0, 6, 2)$.

The first solution implies a 5-colouring $\mathbf{A}^2\mathbf{B}^2\mathbf{CD}^6\mathbf{E}^{12}$, for $\mathbf{A} = \{1, 2\}$, $\mathbf{B} = \{3, 4\}$, $\mathbf{C} = \{5\}$, $\mathbf{D} = \{6, 7, 8, 9, 10, 11\}$, $\mathbf{E} = \{12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\}$. The second solution implies a 5-colouring $\mathbf{A}^2\mathbf{B}^2\mathbf{CD}^{12}\mathbf{E}^6$. The solution $(0, 4, 0, 6, 2)$ (respectively $(4, 0, 0, 6, 2)$) implies that a point $x \in \mathbf{S}$ coloured by B (A) is associated with 8 pairs of **T** coloured by **BE** (**AE**), that is not possible.

The statement is proved. □

Theorem 4.5. *All possible* 6*-colourings for a* **BATS**(23) *are of type* $\mathbf{A}^2\mathbf{BCDE}^6\mathbf{F}^{12}$*. There are not* 7 *or more colourings.*

Proof. The statement is a consequence of the previous results and of Theorem 3.3. □

5. Appendix.

$$\mathbf{A} = \{1, 2, 3, 4, 5, 6\} \cup \{12, 13, 14, 15\},$$
$$\mathbf{B} = \{7\} \cup \{16, 17, 18\},$$
$$\mathbf{C} = \{8, 9, 10, 11\} \cup \{19, 20, 21, 22, 23\}$$

1	2	3	4	5	6	7	8	9	10	11
12-21	12-20	12-23	15-19	15-21	14-19	16-19	16-23	16-22	16-21	16-20
13-22	13-23	13-21	12-18	12-17	12-16	17-20	17-22	17-19	17-23	17-21
15-23	14-22	14-20	13-16	13-18	13-17	18-22	18-21	18-23	18-20	18-19
14-18	15-17	15-16	14-17	14-16	15-18	12-13	13-19	13-20	12-19	12-22
16-17	16-18	17-18	20-21	19-23	20-23	14-15	15-20	14-21	15-22	14-23
19-20	19-21	19-22	22-23	20-22	21-22	21-23	12-14	12-15	13-14	13-15

Table 1

$$\mathbf{A} = \{1, 2, 3\} \cup \{12, 13, 14\},$$
$$\mathbf{B} = \{4, 5\} \cup \{15, 16, 17, 18\},$$
$$\mathbf{C} = \{6, 7, 8, 9, 10, 11\} \cap \{19, 20, 21, 22, 23\}$$

1	2	3	4	5	6	7	8	9	10	11
12-17	12-15	13-16	12-16	12-18	12-13	12-14	13-14	15-17	16-18	17-18
13-18	14-18	14-17	13-17	13-15	14-19	13-19	12-23	12-21	12-19	12-22
15-16	16-17	15-18	14-15	14-16	15-20	16-20	17-20	14-22	13-20	14-20
14-23	13-22	12-20	18-21	17-19	16-21	17-21	15-21	13-23	14-21	13-21
19-20	19-21	19-22	19-23	20-21	17-22	18-22	16-22	16-19	15-22	15-19
21-22	20-23	21-23	20-22	22-23	18-23	15-23	18-19	18-20	17-23	16-23

Table 2

$$\mathbf{A} = \{1, 2, 3, 4, 5\},$$
$$\mathbf{B} = \{6, 7, 8, 9\} \cup \{18, 19, 20, 21, 22, 23\},$$
$$\mathbf{C} = \{10, 11\} \cup \{12, 13, 14, 15, 16, 17\}$$

1	2	3	4	5	6	7	8	9	10	11
12-13	12-14	12-15	12-16	12-17	12-18	12-19	12-20	12-21	12-22	12-23
14-15	13-16	13-17	13-15	13-14	13-19	13-20	13-21	13-22	13-23	13-18
16-17	15-17	14-16	14-17	15-16	14-20	14-21	14-22	14-23	14-18	14-19
18-19	18-20	18-21	18-22	18-23	15-21	15-22	15-23	15-18	15-19	15-20
20-21	19-22	19-23	19-21	19-20	16-22	16-23	16-18	16-19	16-20	16-21
22-23	21-23	20-22	20-23	21-22	17-23	17-18	17-19	17-20	17-21	17-22

Table 3

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