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SOLVABILITY OF THE DIRICHLET PROBLEM IN $W^{2,p}$ FOR ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS IN UNBOUNDED DOMAINS

LOREDANA CASO - PAOLA CAVALIERE - MARIA TRANSIRICO

In this paper some $W^{2,p}$-estimates for the solutions of the Dirichlet problem for a class of elliptic equations with discontinuous coefficients in unbounded domains are obtained. As a consequence, an existence and uniqueness theorem for such a problem is proved.

1. Introduction.

The aim of this paper is to study the Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ Lu = f, \; f \in L^p(\Omega), \end{cases} \tag{1.1}$$

where Ω is an unbounded open subset of $\mathbb{R}^n$, $p \in]1, +\infty[$, L is the uniformly elliptic differential operator defined by the position

$$L = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} + a(x) \quad \text{a.e. in } \Omega \tag{1.2}$$

and the coefficients a_{ij}, a_i, a are discontinuous functions. If Ω is bounded, the above problem has been widely investigated by several authors under various

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hypotheses on the leading coefficients. In particular, if the coefficients a_{ij} belong to the space $C^\circ(\bar{\Omega})$ and the a_i's and a satisfy some suitable assumptions, then $W^{2,p}$-bounds for the solutions of the problem (1.1) and related existence and uniqueness results have been obtained (see [16], [17], [12], [15]). On the other hand, when the coefficients a_{ij} are required to be discontinuous, it must be mentioned the classical contribution by C. Miranda [19], where the author assumed that the a_{ij}'s belong to $W^{1,n}(\Omega)$ (and considered the case $p = 2$); among the other results on this subject, we quote here those proved in [20], [11] (where the Cordes hypothesis is assumed to be true for the a_{ij}'s, and again $p = 2$), and in [13], [2], [14] (where the coefficients lie in certain classes wider than $W^{1,n}(\Omega)$). More recently, a relevant contribution has been given in [9], [10], [25], [26] where the coefficients a_{ij} are assumed to be in the class VMO and $p \in\ ]1, +\infty[$; observe here that VMO contains both classes $C^\circ(\bar{\Omega})$ and $W^{1,n}(\Omega)$.

If the open set Ω is unbounded, the problem (1.1) has for instance been studied in [21], [22], [4], [5], [6] under assumptions similar to those required in [19] with $p = 2$. In this paper we extend this investigation to the case $p \in\ ]1, +\infty[$. More precisely, under suitable hypotheses on the coefficients a_{ij} (see condition (h_2) in Section 4), we obtain the following a priori bound:

$$||u||_{W^{2,p}(\Omega)} \leq c\big(||Lu||_{L^p(\Omega)} + ||u||_{L^p(\Omega_\circ)}\big), \qquad \forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \tag{1.3}$$

where $c \in \mathbb{R}_+$ is independent of u, and $\Omega_\circ$ is a bounded open subset of Ω. The existence and uniqueness of the solution of (1.1) can be deduced from this result.

In order to prove the estimate (1.3), some preliminaries are needed (see Section 3); in fact, using these lemmas, we will previously obtain a bound similar to (1.3) for more regular functions u (see Lemma 4.2). Then a suitable density result will allow to complete the proof.

2. Some notation.

In this paper we will use the following notation: E, a generic Lebesgue measurable subset of $\mathbb{R}^n$; $\Sigma(E)$, the Lebesgue σ-algebra on E; $|A|$, the Lebesgue measure of $A \in \Sigma(E)$; χ_A, the characteristic function of A; $\mathfrak{D}(A)$ (respectively, $\mathfrak{D}^0(A)$), the class of restrictions to A of functions $\zeta \in C_\circ^\infty(\mathbb{R}^n)$ (respectively $\zeta \in C_\circ^0(\mathbb{R}^n)$) with $\bar{A} \cap \operatorname{supp}\zeta \subseteq A$; $L^p_{loc}(A)$, the class of functions g, defined on A, such that $\zeta g \in L^p(A)$ for all $\zeta \in \mathfrak{D}(A)$; $B(x, r)$, the open ball of radius r centered at x and $B_r = B(0, r)$; Ω, an unbounded open subset of $\mathbb{R}^n$ and $D(x, r) = D \cap B(x, r)$ for every $D \in \Sigma(\Omega)$.

We now recall the definitions of the function spaces in which the coefficients of the operator will be chosen. For $p \in [1, +\infty[$, $\lambda \in [0, n[$ and $t \in \mathbb{R}_+$, we denote by $M^{p,\lambda}(\Omega, t)$ the set of all functions g in $L^p_{loc}(\bar{\Omega})$ such that

$$||g||_{M^{p,\lambda}(\Omega,t)} = \sup_{\substack{r\in]0,t] \\ x\in\Omega}} r^{-\lambda/p} ||g||_{L^p(\Omega(x,r))} < +\infty, \tag{2.1}$$

endowed with the norm defined by (2.1). It is easy to show that for any $t_1, t_2 \in \mathbb{R}_+$ a function g belongs to $M^{p,\lambda}(\Omega, t_1)$ if and only if it is in $M^{p,\lambda}(\Omega, t_2)$, and the norms of g in the two spaces are equivalent. This allows to restrict the attention to the space $M^{p,\lambda}(\Omega) = M^{p,\lambda}(\Omega, 1)$. Then we define $M^{p,\lambda}_\circ(\Omega)$ as the closure of $C^\infty_\circ(\Omega)$ in $M^{p,\lambda}(\Omega)$. In particular, we put $M^p(\Omega) = M^{p,0}(\Omega)$, and $M^p_\circ(\Omega) = M^{p,0}_\circ(\Omega)$. In order to define the modulus of continuity of a function g in $M^{p,\lambda}_\circ(\Omega)$, recall first that for a function $g \in M^{p,\lambda}(\Omega)$ the following characterization holds:

$$g \in M^{p,\lambda}_\circ(\Omega) \iff \lim_{t\to 0^+} \big(p_g(t) + ||(1 - \zeta_{1/t})g||_{M^{p,\lambda}(\Omega)}\big) = 0, \tag{2.2}$$

where

$$p_g(t) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_{x\in\Omega} |E(x,1)|\le t}} ||\chi_E g||_{M^{p,\lambda}(\Omega)}, \quad t \in \mathbb{R}_+ ,$$

and ζ_r, $r \in \mathbb{R}_+$, is a function in $C^\infty_\circ(\mathbb{R}^n)$ such that

$$0 \le \zeta_r \le 1, \quad \zeta_{r|B_r} = 1, \quad \text{supp } \zeta_r \subset B_{2r}.$$

Thus the *modulus of continuity* of $g \in M^{p,\lambda}_\circ(\Omega)$ is a function

$$\sigma_\circ :]0, 1] \longrightarrow \mathbb{R}_+$$

such that

$$p_g(t) + ||(1 - \zeta_{1/t})g||_{M^{p,\lambda}(\Omega)} \le \sigma_\circ(t) \quad \forall t \in]0, 1], \quad \lim_{t\to 0^+} \sigma_\circ(t) = 0.$$

A more detailed account of properties of the above defined function spaces can be found in [23].

3. Some preliminaries.

In our results certain regularity properties of open subsets of $\mathbb{R}^n$ will often occur; for the corresponding definitions we will refer to [1].

Lemma 3.1. *Let Ω be an unbounded open subset of $\mathbb{R}^n$ with the uniform C^1-regularity property. Then for every $v \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$ there exists a sequence $(\Phi_h)_{h\in\mathbb{N}}$ of functions such that*

$$\Phi_h \in C^\infty_\circ(\Omega), \quad \Phi_h \to v \ in\ W^{1,2}(\Omega), \quad \sup_{h\in\mathbb{N}} ||\Phi_h||_{L^\infty(\Omega)} \le ||v||_{L^\infty(\Omega)}. \tag{3.1}$$

Proof. Given $g \in C^\infty([0, +\infty[)$ such that $g(t) = 1$ if $t \le 1$, $g(t) = 0$ if $t \ge 2$, $0 \le g \le 1$, we put

$$\delta_h\ : x \in \mathbb{R}^n \longrightarrow g(|x|/h), \quad h \in \mathbb{N}.$$

Clearly δ_h belongs to $C^\infty_\circ(\mathbb{R}^n)$ and

$$0 \le \delta_h \le 1, \quad \sup_{\mathbb{R}^n} \sup_{h\in\mathbb{N}} (\delta_h)_x \ < \ +\infty\ ,$$
$$\lim_{h\to+\infty} (1 - \delta_h(x)) = \lim_{h\to+\infty} (\delta_h)_x(x) = 0, \quad x \in \mathbb{R}^n.$$

Moreover, it is easy to show that

$$\delta_h\, v \ \longrightarrow \ v \quad \text{in} \quad W^{1,2}(\Omega) \tag{3.2}$$

for all $v \in W^{1,2}(\Omega)$.

Denote by $(\zeta_i)_{i\in\mathbb{N}}$ a sequence of functions in $C^\infty_\circ(\mathbb{R}^n)$ such that

$$\operatorname{supp} \zeta_i \ \subset \ \Omega, \quad 0 \le \zeta_i \le 1, \quad d_\circ = \sup_{\Omega} \sup_{i\in\mathbb{N}} d\ (\zeta_i)_x \ < \ +\infty,$$
$$\lim_{i\to+\infty} (1 - \zeta_i(x)) = \lim_{i\to+\infty} (\zeta_i)_x(x) = 0, \quad x \in \Omega,$$

where

$$d : x \in \Omega \longrightarrow dist(x, \partial\Omega)$$

(see [24], Corollary 4.1).

We prove now that

$$\zeta_i \delta_h v \longrightarrow \delta_h v \quad \text{in} \ \ W^{1,2}(\Omega) \tag{3.3}$$

for all $h \in \mathbb{N}$ and $v \in \overset{\circ}{W}{}^{1,2}(\Omega)$.

In fact, we have

$$\zeta_i\, \delta_h\, v \longrightarrow \delta_h\, v \quad \text{in } L^2(\Omega), \tag{3.4}$$

$$\big((\zeta_i - 1)\delta_h\, v\big)_{x_j} \longrightarrow 0 \quad \text{a.e. in } \Omega,\ j = 1, \dots, n, \tag{3.5}$$

$$\begin{aligned} \big|\big((\zeta_i - 1)\, \delta_h\, v\big)_{x_j}\big| &= \big|(\zeta_i - 1)\,(\delta_h\, v)_{x_j} + (\zeta_i)_{x_j}\, \delta_h\, v\big| \\ &\le (\delta_h\, v)_x + d_\circ\, |(\delta_h\, v)/d|, \end{aligned} \tag{3.6}$$

so that, in order to deduce (3.3) from (3.4) - (3.6), it is enough to show that

$$(\delta_h\, v)/d \in L^2(\Omega). \tag{3.7}$$

To this end, denote by Ω_h a bounded open subset of Ω with C^1-boundary such that

$$\bar{\Omega}_h \subset \bar{\Omega}\,, \qquad \operatorname{supp} \delta_h \cap \bar{\Omega} \subset \Omega_h \cup \partial\Omega\,,$$

and observe that $\delta_h\, v$ belongs to $\overset{\circ}{W}{}^{1,2}(\Omega_h)$. An application of the Hardy inequality (see for instance [3]) then yields that (3.7) holds.

For any $v \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$, denote by $v_\circ$ the extension of v to $\mathbb{R}^n$ with zero values out of Ω and put

$$v_{hik} = \big(J_k * (\zeta_i\, \delta_h\, v_\circ)\big)_{|\Omega}\,,$$

where $(J_k)_{k\in\mathbb{N}}$ is a given sequence of mollifiers. It is well known that

$$v_{hik} \in C_\circ^\infty(\Omega)\,,\ ||v_{hik}||_{L^\infty(\Omega)} \le ||v||_{L^\infty(\Omega)}\,,\ k \in \mathbb{N}\,, \tag{3.8}$$

$$v_{hik} \longrightarrow \zeta_i\, \delta_h\, v \quad \text{in} \quad W^{1,2}(\Omega) \tag{3.9}$$

for all $h,\ i \in \mathbb{N}$. On the other hand, we obtain from (3.3) and (3.9) that for every $h \in \mathbb{N}$ there exist $i_h,\ k_h \in \mathbb{N}$ such that

$$||\zeta_{i_h}\, \delta_h\, v - \delta_h\, v||_{W^{1,2}(\Omega)} \le 1/h\,, \quad ||v_{hi_hk_h} - \zeta_{i_h}\, \delta_h\, v||_{W^{1,2}(\Omega)} \le 1/h. \tag{3.10}$$

Therefore it follows from (3.2), (3.8) and (3.10) that the functions $\Phi_h = v_{hi_hk_h}$ ($h \in \mathbb{N}$) satisfy the statement of the lemma. □

The above lemma can be used to prove the following result, which will be essential in the proof of Lemma 4.2.

Lemma 3.2. *Let Ω be an unbounded open subset of $\mathbb{R}^n$ with the uniform C^1-regularity property. If $v \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$, then $|v|^{q-2}v$ belongs to $\overset{\circ}{W}{}^{1,2}(\Omega)$ for every $q \in [2, +\infty[$.*

Proof. It is easy to show that $|v|^{q-2}v \in W^{1,2}(\Omega)$ for each $q \in [2, +\infty[$. In order to prove that $|v|^{q-2}v$ even belongs to $\overset{\circ}{W}{}^{1,2}(\Omega)$, we need different arguments corresponding to the cases $q < 3$ and $q \geq 3$. Suppose first $q < 3$ and denote by $(\psi_h)_{h\in\mathbb{N}}$ a sequence of functions of class $C^\infty_\circ(\Omega)$ which converges to v in $W^{1,2}(\Omega)$. Let $\varphi \in C^\infty_\circ(\mathbb{R}^n)$ and $i \in \{1, \dots, n\}$. Then

$$\Big| \int_\Omega |v|^{q-2} v\, \varphi_{x_i}\, dx - \int_\Omega |\psi_h|^{q-2}\psi_h\, \varphi_{x_i}\, dx \Big| \tag{3.11}$$

$$\leq \int_\Omega |v|^{q-2}|v - \psi_h|\, \varphi_x dx + \int_\Omega \Big| |v|^{q-2} - |\psi_h|^{q-2} \Big| |\psi_h|\, \varphi_x dx$$

$$\leq ||v||^{q-2}_{L^\infty(\Omega)}\, ||v - \psi_h||_{L^2(\Omega)}\, ||\varphi_x||_{L^2(\Omega)} + ||v - \psi_h||^{q-2}_{L^2(\Omega)}\, ||\psi_h||_{L^2(\Omega)}\, ||\varphi_x||_{L^{2/(3-q)}(\Omega)}$$

$$\leq c_1\big(||v - \psi_h||_{L^2(\Omega)} + ||v - \psi_h||^{q-2}_{L^2(\Omega)}\big),$$

where $c_1 \in \mathbb{R}_+$ depends on q, v and φ_x. From (3.11) it follows that

$$\int_\Omega |v|^{q-2} v\, \varphi_{x_i}\, dx = \lim_{h\to+\infty} \int_\Omega |\psi_h|^{q-2}\psi_h\, \varphi_{x_i}\, dx. \tag{3.12}$$

The same argument also shows that

$$\int_\Omega |v|^{q-2} v_{x_i}\, \varphi\, dx = \lim_{h\to+\infty} \int_\Omega |\psi_h|^{q-2}(\psi_h)_{x_i}\, \varphi dx\, . \tag{3.13}$$

Using (3.12) and (3.13) we obtain that

$$\Big| \int_\Omega |v|^{q-2} v\, \varphi_{x_i}\, dx \Big| = (q-1) \Big| \int_\Omega |v|^{q-2} v_{x_i}\, \varphi\, dx \Big|$$

$$\leq (q-1)||v||^{q-2}_{L^\infty(\Omega)}||v_x||_{L^2(\Omega)}||\varphi||_{L^2(\Omega)},$$

and hence $|v|^{q-2}v$ belongs to $\overset{\circ}{W}{}^{1,2}(\Omega)$ by a characterization of the elements of the space $\overset{\circ}{W}{}^{1,2}(\Omega)$ (see for instance [3]).

Suppose now $q \geq 3$, and consider a sequence $(\Phi_h)_{h\in\mathbb{N}}$ of functions satisfying (3.1). If we put

$$v_h = |v|^{q-2}(\Phi_h - v), \quad h \in \mathbb{N},$$

an easy computation yields that

$$(3.14)\qquad ||v_h||^2_{W^{1,2}(\Omega)} \le c_2\Big(||v||^{2(q-2)}_{L^\infty(\Omega)}\,||\Phi_h - v||^2_{W^{1,2}(\Omega)}+ +||v||^{2(q-3)}_{L^\infty(\Omega)}\int_\Omega (\Phi_h - v)^2 v_x^2 dx\Big),$$

where $c_2 \in \mathbb{R}_+$ depends only on q. On the other hand, it follows from the last condition of (3.1) that

$$(3.15)\qquad (\Phi_h - v)^2 v_x^2 \le 4||v||^2_{L^\infty(\Omega)} v_x^2, \quad h \in \mathbb{N}.$$

Applying now (3.14) and (3.15), the sequence $(v_h)_{h\in\mathbb{N}}$ can be replaced by a suitable subsequence which goes to 0 in $W^{1,2}(\Omega)$, i.e. we may assume that

$$(3.16)\qquad |v|^{q-2}\Phi_h \longrightarrow |v|^{q-2}v \quad \text{in} \quad W^{1,2}(\Omega).$$

Since $|v|^{q-2}\Phi_h \in \overset{\circ}{W}{}^{1,2}(\Omega)$ for every $h \in \mathbb{N}$, there exists a sequence $(\psi_{hm})_{m\in\mathbb{N}}$ of functions of class $C^\infty_\circ(\Omega)$ such that

$$\psi_{hm} \longrightarrow |v|^{q-2}\Phi_h \quad \text{in} \quad W^{1,2}(\Omega),$$

and hence we can find $m_h \in \mathbb{N}$ with

$$(3.17)\qquad ||\psi_{hm_h} - |v|^{q-2}\Phi_h||_{W^{1,2}(\Omega)} \le 1/h.$$

It follows from (3.16) and (3.17) that

$$\psi_{hm_h} \longrightarrow |v|^{q-2}v \quad \text{in} \quad W^{1,2}(\Omega),$$

and so $|v|^{q-2}v$ belongs to $\overset{\circ}{W}{}^{1,2}(\Omega)$. The lemma is proved. □

4. Key lemmas.

In the following we will suppose that $n \ge 3$. Consider the conditions:

$$(h_1)\qquad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i,j = 1,\ldots,n, \\ \exists\, \nu > 0 \ : \ \sum_{i,j=1}^n a_{ij}\,\xi_i\,\xi_j \ge \nu|\xi|^2 \quad \text{a.e. in } \Omega, \ \forall\, \xi \in \mathbb{R}^n, \\ b \in L^\infty(\Omega), \ \operatorname{ess\,inf}_\Omega b = b_\circ > 0, \end{cases}$$

(h_2) $\quad \exists\, s \in\,]2, n] \;:\; (a_{ij})_{x_h} \in M_\circ^{s,n-s}(\Omega)\,, \quad i, j, h = 1, \ldots, n.$

Remark 4.1. *If* (h_1) *holds, for the bilinear form*

$$a(v, w) = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij} v_{x_i} w_{x_j} + bvw \Big) dx\,, \qquad v, w \in W^{1,2}(\Omega)\,, \tag{4.1}$$

we have

$$a(v, |v|^{q-2}v) \;\geq\; (q-1)\nu \int_\Omega |v|^{q-2} v_x^2 dx + b_\circ \int_\Omega |v|^q dx\,, \tag{4.2}$$

$$\forall\, v \in W^{1,2}(\Omega) \,\cap\, L^\infty(\Omega)\,,\ \forall\, q \in [2, +\infty[\,.$$

We can now prove the main result of this section; in its statement we will consider the operator

$$L_\circ = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} \qquad \text{a.e. in } \Omega.$$

Lemma 4.2. *If* Ω *has the uniform* $C^{1,1}$*-regularity property and conditions* (h_1) *and* (h_2) *hold, then for any* $p \in\,]1, +\infty[$ *there exist a constant* $c \in \mathbb{R}_+$ *and a bounded open subset* $\Omega_\circ \subset\subset \Omega$*, with the cone property, such that*

$$||u||_{W^{2,p}(\Omega)} \;\leq\; c\,\big(||L_\circ u + bu||_{L^p(\Omega)} + ||u||_{L^p(\Omega_\circ)}\big)\,, \tag{4.3}$$

$$\forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega) \cap \mathfrak{D}^0(\bar{\Omega})\,,$$

where c *and* $\Omega_\circ$ *depend on* $n, p, \nu, b_\circ, \Omega, s, ||b||_{L^\infty(\Omega)}, ||a_{ij}||_{L^\infty(\Omega)}$ *and on the continuity moduli of* $(a_{ij})_{x_h}$ *in* $M_\circ^{s,n-s}(\Omega)$.

Proof. Consider a function $u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega) \cap \mathfrak{D}^0(\bar{\Omega})$. It follows from Theorem 5.1 of [7] and from Lemmas 4.1 and 4.2 of [5] (see also Section 2 of [9]) that

$$||u||_{W^{2,p}(\Omega)} \;\leq\; \tilde{c}\,\big(||L_\circ u + bu||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}\big)\,, \tag{4.4}$$

where $\tilde{c} \in \mathbb{R}_+$ depends on $n, p, \nu, \Omega, s, ||b||_{L^\infty(\Omega)}, ||a_{ij}||_{L^\infty(\Omega)}$ and on the continuity moduli of $(a_{ij})_{x_h}$ in $M_\circ^{s,n-s}(\Omega)$.

We will now provide a bound for $||u||_{L^p(\Omega)}$, studying separately the cases $p \geq 2$ and $p < 2$. Suppose first $p \geq 2$. Then by Remark 4.1 and Lemma 3.2 we obtain

$$(4.5) \qquad (p-1)\nu \int_\Omega |u|^{p-2} u_x^2 dx + b_\circ \int_\Omega |u|^p dx \leq a(u, |u|^{p-2}u)$$

$$= \int_\Omega (L_\circ u + bu)|u|^{p-2} u dx - \int_\Omega \sum_{i,j=1}^n (a_{ij})_{x_j} \, u_{x_i} \, |u|^{p-2} u dx$$

$$\leq ||L_\circ u + bu||_{L^p(\Omega)} ||u||^{p-1}_{L^p(\Omega)} + \sum_{i,j=1}^n \int_\Omega |(a_{ij})_{x_j}| \, |u_{x_i}| \, |u|^{p-1} dx \, .$$

On the other hand,

$$(4.6) \qquad \int_\Omega |(a_{ij})_{x_j}| \, |u_{x_i}| \, |u|^{p-1} dx$$

$$\leq \varepsilon_1/2 \int_\Omega |u|^{p-2} u_x^2 dx + 1/(2\varepsilon_1) \int_\Omega (a_{ij})^2_{x_j} |u|^p dx$$

for each $\varepsilon_1 \in R_+$; moreover, for every $\varepsilon_2 \in \mathbb{R}_+$ there exist a constant $c(\varepsilon_2) \in \mathbb{R}_+$ and a bounded open subset $\Omega_{\varepsilon_2} \subset\subset \Omega$, with the cone property, such that

$$(4.7) \qquad \int_\Omega (a_{ij})^2_{x_j} |u|^p dx \; \leq \; \varepsilon_2 || \, |u|^{p/2} ||^2_{W^{1,2}(\Omega)} + c(\varepsilon_2) \int_{\Omega_{\varepsilon_2}} |u|^p dx$$

$$\leq \; \varepsilon_2 \int_\Omega |u|^p dx + \varepsilon_2 p^2/4 \int_\Omega |u|^{p-2} u_x^2 dx + c(\varepsilon_2) \int_{\Omega_{\varepsilon_2}} |u|^p dx \, ,$$

where $c(\varepsilon_2)$ and Ω_{ε_2} depend on n, Ω, s and on the continuity moduli of $(a_{ij})_{x_j}$ in $M^{s,n-s}_\circ(\Omega)$ (see [23], Corollary 3.5). Therefore it follows from (4.5), (4.6) and (4.7) that

$$(4.8) \qquad \big((p-1)\nu - n^2(\varepsilon_1/2 + \varepsilon_2 p^2/(8\varepsilon_1))\big) \int_\Omega |u|^{p-2} u_x^2 dx +$$

$$+ (b_\circ - n^2 \varepsilon_2/(2\varepsilon_1)) \int_\Omega |u|^p dx$$

$$\leq \; ||L_\circ u + bu||_{L^p(\Omega)} ||u||^{p-1}_{L^p(\Omega)} + n^2 c(\varepsilon_2)/(2\varepsilon_1) \int_{\Omega_{\varepsilon_2}} |u|^p dx .$$

For a suitable choice of ε_1 and ε_2, the relation (4.8) gives

$$(4.9)\qquad b_\circ/2\int_\Omega |u|^p dx \;\le\; ||L_\circ u + bu||_{L^p(\Omega)}||u||^{p-1}_{L^p(\Omega)} + c_1\int_{\Omega_1}|u|^p dx\,,$$

where c_1 and Ω_1 depend on $n, p, \nu, b_\circ, \Omega, s$ and on the continuity moduli of $(a_{ij})_{x_h}$ in $M^{s,n-s}_\circ(\Omega)$. From (4.9) we obtain

$$(4.10)\qquad ||u||_{L^p(\Omega)} \;\le\; 2/b_\circ\big(||L_\circ u + bu||_{L^p(\Omega)} + c_1||u||_{L^p(\Omega_1)}\big).$$

Applying now (4.4) and (4.10) we complete the proof in the first case.

Suppose now that $p < 2$. In this case our argument is suggested by a trick already used in the proof of Lemma 1 in [12]. If $f = |u|^{p-1} sign\, u$, it follows from the Theorem in [18] that there exists a unique function $w \in \overset{\circ}{W}{}^{1,2}(\Omega)\cap L^\infty(\Omega)$ such that

$$(4.11)\qquad a(w,v) = \int_\Omega f v dx \qquad \forall v\in \overset{\circ}{W}{}^{1,2}(\Omega).$$

Then by Remark 4.1 and Lemma 3.2 we have that

$$(4.12)\qquad \int_\Omega |w|^{p'}dx \;\le\; 1/b_\circ\; a(w,|w|^{p'-2}w) = 1/b_\circ\int_\Omega f|w|^{p'-2}w dx$$

$$\le\; 1/b_\circ\int_\Omega |u|^{p-1}|w|^{p'-1}dx \;\le\; 1/b_\circ\,||u||^{p-1}_{L^p(\Omega)}||w||^{p'-1}_{L^{p'}(\Omega)},$$

where $1/p + 1/p' = 1$, and hence

$$(4.13)\qquad ||w||_{L^{p'}(\Omega)} \;\le\; 1/b_\circ\,||u||^{p-1}_{L^p(\Omega)}.$$

An application of (4.13) yields that

$$(4.14)\qquad \int_\Omega |u|^p dx = \int_\Omega f u dx$$

$$= \int_\Omega (L_\circ u + bu)w dx - \int_\Omega \sum_{i,j=1}^n (a_{ij})_{x_j} u_{x_i} w dx$$

$$\le\; \Big(||L_\circ u + bu||_{L^p(\Omega)} + \sum_{i,j=1}^n ||(a_{ij})_{x_j} u_{x_i}||_{L^p(\Omega)}\Big)||w||_{L^{p'}(\Omega)}$$

$$\leq 1/b_\circ\Big(||L_\circ u + bu||_{L^p(\Omega)} + \sum_{i,j=1}^{n} ||(a_{ij})_{x_j} u_{x_i}||_{L^p(\Omega)}\Big)||u||^{p-1}_{L^p(\Omega)},$$

so that by (4.14)

$$||u||_{L^p(\Omega)} \leq 1/b_\circ\Big(||L_\circ u + bu||_{L^p(\Omega)} + \sum_{i,j=1}^{n} ||(a_{ij})_{x_j} u_{x_i}||_{L^p(\Omega)}\Big). \tag{4.15}$$

On the other hand, it follows from Corollary 3.5 of [23] that for every $\varepsilon \in \mathbb{R}_+$ there exist a constant $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset\subset \Omega$, with the cone property, such that

$$||(a_{ij})_{x_j} u_{x_i}||_{L^p(\Omega)} \leq \varepsilon||u||_{W^{2,p}(\Omega)} + c(\varepsilon)||u_x||_{L^p(\Omega_\varepsilon)}, \tag{4.16}$$

where $c(\varepsilon)$ and Ω_ε depend on n, p, Ω, s and on the continuity moduli of $(a_{ij})_{x_j}$ in $M^{s,n-s}_\circ(\Omega)$. A final application of (4.4), (4.15) and (4.16) completes the proof of the lemma. □

Lemma 4.3. *If Ω has the uniform $C^{1,1}$-regularity property and if $p \in \,]1, +\infty[$, then the problem*

$$u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \quad -\Delta u + u = f, \quad f \in L^p(\Omega)\,, \tag{4.17}$$

is uniquely solvable and the solution u satisfies the bound

$$||u||_{W^{2,p}(\Omega)} \leq c\,||f||_{L^p(\Omega)}\,, \tag{4.18}$$

where the constant $c \in \mathbb{R}_+$ depends only on n, p and Ω.

Proof. It has already been proved that the problem (4.17) is uniquely solvable if $p = 2$ (see, e.g., [6], Lemma 4.4); in this case we will denote by Af the solution. Let now f be a function in $C^\infty_\circ(\Omega)$. Then for every $q \in [1, +\infty]$, Af belongs to $L^q(\Omega)$ and

$$||Af||_{L^q(\Omega)} \leq c_1||f||_{L^q(\Omega)}\,, \tag{4.19}$$

where $c_1 \in \mathbb{R}_+$ depends only on n (see Theorem in [18]).

On the other hand, a suitable application of Theorem 5.1 in [7] yields that $Af \in W^{2,p}(\Omega)$ and there exists a constant $c_2 = c_2(n, p, \Omega) \in \mathbb{R}_+$ such that

$$||Af||_{W^{2,p}(\Omega)} \leq c_2\big(||f||_{L^p(\Omega)} + ||Af||_{L^p(\Omega)}\big)\,. \tag{4.20}$$

Since $W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega) \subseteq \overset{\circ}{W}{}^{1,p}(\Omega)$, Af is a solution of problem (4.17) and, by (4.19) and (4.20), it satisfies the estimate (4.18). The result follows now in the general case from the density of $C^\infty_\circ(\Omega)$ in $L^p(\Omega)$. □

Using the above lemma, the following density result can be proved.

Lemma 4.4. *If Ω has the uniform $C^{1,1}$-regularity property and if $p \in]1,+\infty[$, then for every $u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$ there exists a sequence of functions $(u_h)_{h \in \mathbb{N}}$ such that*

$$u_h \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega) \cap \mathfrak{D}^0(\bar{\Omega}),\ h \in \mathbb{N}, \quad u_h \to u \ \ in \ \ W^{2,p}(\Omega). \tag{4.21}$$

Proof. Let $p \in]1,+\infty[$, $u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$ and consider a sequence $(v_h)_{h \in \mathbb{N}}$ of functions such that

$$v_h \in \mathfrak{D}(\bar{\Omega}),\ h \in \mathbb{N}, \quad v_h \to u \quad \text{in } W^{2,p}(\Omega). \tag{4.22}$$

It follows from Lemma 4.3 that the problem

$$w_h \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \quad -\Delta w_h + w_h = -\Delta v_h + v_h \tag{4.23}$$

is uniquely solvable for all $h \in \mathbb{N}$ and the solution w_h satisfies the bound

$$||w_h||_{W^{2,p}(\Omega)} \le c\,|| - \Delta v_h + v_h||_{L^p(\Omega)}, \tag{4.24}$$

with $c \in \mathbb{R}_+$ dependent on n, p and Ω. Observe that $w_h \in \overset{\circ}{W}{}^{1,2}(\Omega)$ (see the proof of Lemma 4.3). Clearly w_h belongs to $C^0(\bar{\Omega})$ when $p > n/2$; if $p \le n/2$, we have that $w_h \in W^{2,n/2+\varepsilon}(\Omega)$ for $\varepsilon > 0$, (see [7], Theorem 5.1) and so $w_h \in C^0(\bar{\Omega})$ also in this case. Moreover, we deduce from (4.22), (4.23) and (4.24) that

$$w_h \to u \quad \text{in } W^{2,p}(\Omega).$$

Denote now by $(\delta_h)_{h \in \mathbb{N}}$ the sequence of functions defined in the proof of Lemma 3.1, and note that

$$\sup_{\mathbb{R}^n} \sup_{h \in \mathbb{N}} (\delta_h)_{xx} < +\infty, \quad \lim_{h \to +\infty} (\delta_h)_{xx}(x) = 0, \quad x \in \mathbb{R}^n,$$

$$\delta_h u \to u \quad \text{in } W^{2,p}(\Omega).$$

Thus it follows from the properties of δ_h and w_h that the functions $u_h = \delta_h w_h$, $h \in \mathbb{N}$, satisfy the conditions of the statement. □

5. Main results.

In this section we will suppose that the coefficient a of the operator L has the form $a = a' + b$, where the function b satisfies the condition (h_1), and we will consider the following additional condition:

$$(h_3) \qquad a_i \in M^r_\circ(\Omega),\ i = 1, \dots, n\,, \quad a' \in M^t_\circ(\Omega)\,,$$

where

$$r > n \quad \text{if } p \le n\,, \quad r = p \quad \text{if } p > n\,,$$

$$t > n/2 \quad \text{if } p \le n/2\,, \quad t = p \quad \text{if } p > n/2\,.$$

We can now prove the main result of the paper.

Theorem 5.1. *If Ω has the uniform $C^{1,1}$-regularity property and conditions (h_1), (h_2) and (h_3) hold, then for any $p \in \,]1, +\infty[$ there exist a constant $c \in \mathbb{R}_+$ and a bounded open subset $\Omega_\circ \subset\subset \Omega$, with the cone property, such that*

$$(5.1) \qquad ||u||_{W^{2,p}(\Omega)} \le c\left(||Lu||_{L^p(\Omega)} + ||u||_{L^p(\Omega_\circ)}\right),$$

$$\forall\, u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega),$$

where c and $\Omega_\circ$ depend on n, p, ν, $b_\circ$, Ω, s, r, t, $||b||_{L^\infty(\Omega)}$, $||a_{ij}||_{L^\infty(\Omega)}$ and on the continuity moduli of $(a_{ij})_{x_h}$, a_i and a' in $M^{s,n-s}_\circ(\Omega)$, $M^r_\circ(\Omega)$ and $M^t_\circ(\Omega)$, respectively.

Proof. Let $u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$. By Lemma 4.4 there exists a sequence $(u_h)_{h\in\mathbb{N}}$ of functions satisfying (4.21), and hence it follows from Lemma 4.2 that

$$(5.2) \qquad ||u_h||_{W^{2,p}(\Omega)} \le c\left(||L_\circ u_h + bu_h||_{L^p(\Omega)} + ||u_h||_{L^p(\Omega_\circ)}\right), \quad h \in \mathbb{N}\,,$$

where c and $\Omega_\circ$ are those in (4.3). Moreover,

$$(5.3) \quad ||L_\circ u_h + bu_h||_{L^p(\Omega)} \le c_1||u_h - u||_{W^{2,p}(\Omega)} + ||L_\circ u + bu||_{L^p(\Omega)},\ h \in \mathbb{N},$$

where $c_1 \in \mathbb{R}_+$ depends on n, $||b||_{L^\infty(\Omega)}$ and $||a_{ij}||_{L^\infty(\Omega)}$. An application of (5.2), (5.3) and (4.21) yields now that

$$(5.4) \qquad ||u||_{W^{2,p}(\Omega)} \le c\left(||L_\circ u + bu||_{L^p(\Omega)} + ||u||_{L^p(\Omega_\circ)}\right).$$

On the other hand, using the argument of the proof of Corollary 3.5 of [23], it follows from Theorem 3.2 of [7] that for any $\varepsilon \in \mathbb{R}_+$ there exist a constant

$c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \subset\subset \Omega$, with the cone property, such that

$$||\sum_{i=1}^{n} a_i u_{x_i} + a' u||_{L^p(\Omega)} \tag{5.5}$$

$$\leq \varepsilon \, ||u||_{W^{2,p}(\Omega)} + c(\varepsilon) \left(||u_x||_{L^p(\Omega_\varepsilon)} + ||u||_{L^p(\Omega_\varepsilon)} \right),$$

where $c(\varepsilon)$ and Ω_ε depend on n, p, Ω, r, t and on the continuity moduli of a_i and a' in $M^r_\circ(\Omega)$ and $M^t_\circ(\Omega)$, respectively. Relations (5.4) and (5.5) complete the proof of the theorem. □

Theorem 5.2. *If Ω has the uniform $C^{1,1}$-regularity property, conditions (h_1), (h_2) and (h_3) hold, and $a \geq 0$ a.e. in Ω, then the problem*

$$u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)\,, \;\; Lu = f\,, \quad f \in L^p(\Omega)\,, \tag{5.6}$$

is uniquely solvable for every $p \in \,]1, +\infty[$.

Proof. Let f be a function in $C^\infty_\circ(\Omega)$. Then there exists a unique $u \in W^{2,2}(\Omega) \cap \overset{\circ}{W}{}^{1,2}(\Omega)$ such that $L_\circ u + bu = f$ (see for instance [6], Lemma 4.4). On the other hand, it follows from Theorem 5.1 of [7] and Lemmas 4.1 and 4.2 of [5] that u belongs to $W^{2,p}(\Omega)$. Therefore $u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega)$, and it is a solution of the equation $L_\circ u + bu = f$, so that $C^\infty_\circ(\Omega) \subseteq R(L_\circ + b)$. Since, by Theorem 5.1, $R(L_\circ + b)$ is a closed subspace of $L^p(\Omega)$, we obtain that $R(L_\circ + b) = L^p(\Omega)$. Thus Corollary in [8] gives that the problem

$$u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \quad L_\circ u + bu = f\,, \quad f \in L^p(\Omega)\,, \tag{5.7}$$

is uniquely solvable. Moreover, the operator

$$u \in W^{2,p}(\Omega) \longrightarrow \sum_{i=1}^{n} a_i u_{x_i} + a' u \in L^p(\Omega)$$

is compact by (5.5), and hence (5.6) is a zero index problem. Since for such problem a uniqueness result holds (see Corollary of [8]), the statement follows from well known results. □

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L. Caso - M. Transirico
Dipartimento di Matematica e Informatica
Università di Salerno
via S. Allende
84081 Baronissi (SA) (ITALY)

P. Cavaliere
Dipartimento di Ingegneria dell'Informazione e Matematica Applicata
Facoltà di Scienze MM. FF. NN.
Università di Salerno
via S. Allende
84081 Baronissi (SA) (ITALY)