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THE FOUCAULT'S CURRENTS PROBLEM WITH TEMPERATURE DEPENDENT RESISTIVITY

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A result of existence and uniqueness of solutions is presented for the system of P.D.E. modelling the eddy currents and the heating in a cylindrical conductor.

An external varying magnetic field induces electric currents, known as Foucault's currents, in massive conductors which in turn heat the body by Joule effects. We consider the special situation of an indefinite cylindrical conductor of cross section Ω, an open and bounded domain of $\mathbb{R}^2$ with a regular boundary Γ immersed in an insulating medium where a magnetic field $\mathbf{H}$, parallel to the axis of the cylinder, acts. The geometrical situation justifies the hypothesis

$$\mathbf{H} = w(x, y, t)\mathbf{k} \tag{1.1}$$

where $\mathbf{k}$ is the unit vector of the axis of the cylinder. In the medium surrounding the cylinder the current density $\mathbf{J}$ is zero. From (1.1) it follows $\nabla \times \mathbf{H} = \nabla w \times \mathbf{k}$. Assuming to be in a quasi-stationary situation, we have $\nabla \times \mathbf{H} = \mathbf{J}$; thus the magnetic field is constant in the medium surrounding the cylinder and we have

$$\mathbf{H} = \tilde{h}(t)\mathbf{k},$$

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where we assume $\tilde{h}(t)$ to be a given function such that $\tilde{h}(0) = 0$. Inside the cylinder we have, by Ohm's law,

$$\mathbf{E} = \tilde{\rho}(u)\mathbf{J},$$

where $\tilde{\rho}(u)$, a given function of the temperature u, is the electrical resistivity and $\mathbf{E}$ the electric field. Hence

$$\mathbf{E} = \tilde{\rho}(u)\nabla w \times \mathbf{k}.$$

Recalling (1.1) and using the Maxwell equation $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and the constitutive relation $\mathbf{B} = \mu\mathbf{H}$, where μ is the constant permeability, we obtain $\nabla \cdot (\tilde{\rho}(u)\nabla w) = \mu w_t$ or, setting $\rho(u) = \frac{\tilde{\rho}(u)}{\mu}$,

$$w_t = \nabla \cdot (\rho(u)\nabla w). \tag{1.2}$$

The tangential component of $\mathbf{H}$ is continuous across boundaries, thus

$$w = \tilde{h}(t) \; on \; \Gamma \times (0, T). \tag{1.3}$$

Inside the cylinder the temperature obeys the energy equation $cu_t - k\Delta = \rho(u)|\nabla w|^2$ or, after a rescaling,

$$u_t - \Delta u = \rho(u)|\nabla w|^2. \tag{1.4}$$

We assume for the temperature a boundary condition of the form

$$u = 0 \; on \; \Gamma \times (0, T). \tag{1.5}$$

The set of equations (1.2)-(1.5) is completed with the initial conditions

$$w(x, 0) = 0, \; u(x, 0) = u_0, \; x = (x_1, x_2) \in \Omega. \tag{1.6}$$

This system, closely related to the thermistor problem, was proposed in [3]. In this note we present a proof of existence of solutions assuming

$$u_0(x) \in L^2(\Omega). \tag{1.7}$$

Defining $h(x, t) = w(x, t) - \tilde{h}(t)$ we can rewrite the problem (1.2)-(1.6) as follows

$$h_t = \nabla \cdot (\rho(u)\nabla h) + f(t) \; in \; Q \tag{1.8}$$

$$u_t - \Delta u = \rho(u)|\nabla h|^2 \text{ in } Q \tag{1.9}$$

$$h = 0, \ u = 0 \text{ on } \Gamma \times (0, T) \tag{1.10}$$

$$h(x, 0) = 0, \ u(x, 0) = u_0(x) \text{ in } \Omega, \tag{1.11}$$

where $Q = \Omega \times (0, T)$ and $f(t) = \tilde{h}'(t)$. We suppose

$$\rho(u) \in C(\mathbb{R}^1), \ 0 < \rho_0 \le \rho(u) \le \rho_1 \ , \ |\rho(u_1) - \rho(u_2)| \le L|u_1 - u_2| \tag{1.12}$$

for all $u_1, u_2 \in \mathbb{R}$. Moreover, for greater generality (although purely mathematical), we assume, in the right hand side of (1.8), f to depend on x and t and to satisfy

$$f(x, t) \in L^4(0, T; L^2(\Omega)). \tag{1.13}$$

We denote (,) the scalar product in both $L^2(\Omega)$ and $\mathbf{L}^2(\Omega) = (L^2(\Omega))^2$ and write $\| \ \|$ for the corresponding norms. Subscripts are used for other norms.

An Integral Inequality 2. The proof of existence and uniqueness for problem (1.8)-(1.11) relies crucially on an "a priori" L^p estimate for the gradient of solutions of parabolic equations obtained by G. Pulvirenti in [7] and [8], which we quote below for the case at hand.

Theorem 2.1. Let $a(x, t) \in L^\infty(Q)$ satisfy

$$0 < a_0 \le a(x, t) \le a_1 \tag{2.1}$$

and $\mathbf{f} = (f_1, f_2, f_3) \in (L^p(Q))^3, \ p \ge 2$. Then there exists a constant $r > 2$ such that if $p \in [2, r)$, the weak solution of the initial-boundary value problem

$$u_t - \nabla \cdot (a(x, t)\nabla u) = -\nabla \cdot \mathbf{f} \text{ in } Q \tag{2.2}$$

$$u(x, 0) = 0, \ x \in \Omega, \tag{2.3}$$

$$u = 0 \text{ on } \Gamma \times (0, T) \tag{2.4}$$

belongs to $L^p(0, T; H^{1,p}(\Omega))$ and the estimate

$$\|\nabla u\|_{L^p(Q)} \le C\|\mathbf{f}\|_{L^p(Q)} \tag{2.5}$$

holds.

We present here a way to estimate the constant r in terms of a_0 and a_1. In the following lemma we quote a property, crucial in the sequel, of the solution of the problem

$$v_t - \Delta v = -\nabla \cdot \mathbf{f} \tag{2.6}$$

$$v(x,0) = 0 \; in \; \Omega \;\; v = 0 \; on \; \Gamma \times (0,T). \tag{2.7}$$

We refer to [1] (page 275) for the proof which is based on the Riesz-Thorin theorem.

Lemma 2.1. Let $p \in [2, \infty)$ and $\mathbf{f}(x,t) \in \mathbf{L}^p(Q)$. Then there exists a unique weak solution to problem (2.6), (2.7) and this solution satisfies the "a priori" estimate

$$\|\nabla v\|_{\mathbf{L}^p(Q)} \leq \varphi(p) \|\nabla \mathbf{f}\|_{\mathbf{L}^p(Q)}, \tag{2.8}$$

where $\varphi(p)$ is a continuous function for $\infty > p \geq 2$, depending only on Ω, such that $\varphi(2) = 1$.

Theorem 2.2. Let $\varphi(p)$ be the function entering in Lemma 2.1 and u the solution of (2.2)-(2.4). If

$$1 - \frac{a_0}{a_1} < \frac{1}{\varphi(p)}, \tag{2.9}$$

then we have

$$\|\nabla u\|_{\mathbf{L}^p(Q)} \leq C \|\nabla \mathbf{f}\|_{\mathbf{L}^p(Q)}. \tag{2.10}$$

Proof. We may assume $a_0 < a_1$, otherwise we are in the case of the laplacian and the result follows from Lemma 2.1. We make a time-rescaling of equation (2.2) defining

$$v(x,t) = u\Big(x, \frac{t}{a_1}\Big), \; m(x,t) = a\Big(x, \frac{t}{a_1}\Big)\frac{1}{a_1}, \; \mathbf{g}(x,t) = \mathbf{f}\Big(x, \frac{t}{a_1}\Big)\frac{1}{a_1}.$$

In this way (2.2)-(2.4) become

$$v_t - \nabla \cdot (m \nabla v) = -\nabla \cdot \mathbf{g}, \tag{2.11}$$

$$v(x,0)=0 \ in\ \Omega,\ \ v=0 \ on\ \Gamma\times(0,T) \tag{2.12}$$

and $m(x,t)\in L^\infty(Q)$ satisfies

$$0<\mu_0\le m(x,t)\le 1 \tag{2.13}$$

with $\mu_0=\frac{a_0}{a_1}<1$. The hypothesis (2.9) becomes

$$1-\mu_0<\frac{1}{\varphi(p)}. \tag{2.14}$$

Let $\lambda\in[0,1]$ and consider the following family of problems depending on the parameter λ

$$v_t-\nabla\cdot\{[1-\lambda(1-m)]\nabla v\}=-\nabla\cdot\ g \tag{2.15}$$

$$v(x,0)=0,\ \ v=0 \ on\ \Gamma\times(0,T). \tag{2.16}$$

Let $\mathcal{A}$ be the set of the values $(\lambda,p)\in[0,1]\times[2,\infty)$ for which (2.15), (2.16) has a solution in $L^p(0,T;H_0^{1,p}(\Omega))$ and the estimate

$$\|\nabla v\|_{\mathbf{L}^p(Q)}\le C\|\nabla \mathbf{g}\|_{\mathbf{L}^p(Q)}. \tag{2.17}$$

holds. $\mathcal{A}$ is not empty since $(0,p)\in\mathcal{A}$ for all $p\ge 2$ by Lemma 2.1 and it is easy to verify that $(\lambda,2)\in\mathcal{A}$ if $\lambda\in[0,1]$. When $(\lambda,p)\in\mathcal{A}$ we define the best constant for which (2.17) holds, i.e.

$$C(\lambda,p)=sup\left\{\frac{\|\nabla v\|_{\mathbf{L}^p(Q)}}{\|\mathbf{g}\|_{\mathbf{L}^p(Q)}},\ \|\mathbf{g}\|_{\mathbf{L}^p(Q)}\ne 0\right\}.$$

For all $p\ge 2$ we have $C(0,p)<\infty$ and

$$C(0,2)=1. \tag{2.18}$$

Moreover, if $\lambda\in[0,1]$ and $p=2$ again we have

$$C(\lambda,2)<\infty. \tag{2.19}$$

Define $k=1-\mu_0<1$. We claim that

$$C(\lambda,p)\le\frac{C(0,p)}{1-k\lambda C(0,p)}. \tag{2.20}$$

Let $(\lambda, p) \in \mathcal{A}$. The corresponding solution v of (2.15), (2.16) is differentiable with respect to the parameter λ and $v_\lambda = \frac{\partial v}{\partial \lambda}$ satisfies the problem

$$\frac{\partial v_\lambda}{\partial t} - \nabla \cdot \{[1 - \lambda(1 - m)]\nabla v_\lambda\} = -\nabla \cdot [(1 - m)\nabla v]$$

(2.21)

$$v_\lambda(x, 0) = 0,\ x \in \Omega,\ v_\lambda = 0 \ on\ \Gamma \times (0, T).$$

Therefore we have the estimate

$$\|\nabla v_\lambda\|_{\mathbf{L}^p(Q)} \leq C(\lambda, p)\|(1 - m)\nabla v\|_{\mathbf{L}^p(Q)} \leq C(\lambda, p)k\|\nabla v\|_{\mathbf{L}^p(Q)}. \tag{2.22}$$

Using the Hoelder inequality we obtain, with elementary calculations,

$$\frac{d}{d\lambda}\|\nabla v\|^p_{\mathbf{L}^p(Q)} \leq p\|\nabla v\|^{p-1}_{\mathbf{L}^p(Q)}\|\nabla v_\lambda\|_{\mathbf{L}^p(Q)}. \tag{2.23}$$

Hence, by (2.22), we have

$$\frac{d}{d\lambda}\|\nabla v\|^p_{\mathbf{L}^p(Q)} \leq pkC(\lambda, p)\|\nabla v\|^p_{\mathbf{L}^p(Q)}. \tag{2.24}$$

Integrating (2.24) between 0 and λ we obtain, recalling the definition of $C(\lambda, p)$,

$$\|\nabla v(\lambda)\|^p_{\mathbf{L}^p(Q)} \leq \|\nabla v(0)\|^p_{\mathbf{L}^p(Q)} + pk\int_0^\lambda C(\xi, p)\|\nabla v(\xi)\|^p_{\mathbf{L}^p(Q)} d\xi \leq$$

$$C^p(0, p)\|\mathbf{g}\|^p_{\mathbf{L}^p(Q)} + pk\|\mathbf{g}\|^p_{\mathbf{L}^p(Q)}\int_0^\lambda C^{p+1}(\xi, p)d\xi.$$

Dividing the above inequality by $\|\mathbf{g}\|^p_{\mathbf{L}^p(Q)}$ and using again the definition of $C(\lambda, p)$ we have

$$C^p(\lambda, p) \leq C^p(0, p) + pk\int_0^\lambda C^{p+1}(\xi, p)d\xi. \tag{2.25}$$

Let $y(\lambda) = C^p(\lambda, p)$, we obtain, by (2.25),

$$y(\lambda) \leq y(0) + pk\int_0^\lambda y^{1+1/p}(\xi)d\xi. \tag{2.26}$$

The solution of this integral inequality can easily be computed (see [5], page 38) and it is given by

$$y(\lambda) \leq \frac{y(0)}{\left[1 - k\lambda y^{1/p}(0)\right]^p}.$$

This implies (2.20). Setting $\lambda = 1$ in (2.20) we have

$$C(1, p) \leq \frac{C(0, p)}{1 - kC(0, p)}. \tag{2.27}$$

Since $C(0, p) \leq \varphi(p)$, we have also

$$C(1, p) \leq \frac{\varphi(p)}{1 - k\varphi(p)}. \tag{2.28}$$

By assumption $1 - k\varphi(p) > 0$, thus the right hand side of (2.28) remains bounded and (2.10) follows. $\square$

Remark 2.1. From (2.9) we obtain, in particular, that $a_1 \to a_0$ if $p \to \infty$ and $a_0 \to 0$ when $p \to 2$.

We return now to the nonlinear problem (1.8)-(1.11). For the proof of existence presented in the next section an estimate of the form

$$\|\nabla u\|_{\mathbf{L}^4(Q)} \leq C\|\mathbf{f}\|_{\mathbf{L}^4(Q)} \tag{2.29}$$

is needed. Therefore we assume, in addition to (2.12)

$$\frac{\rho_1 - \rho_0}{\rho_1} < \frac{1}{\varphi(4)}. \tag{2.30}$$

Existence and Uniqueness for the Non-linear Problem 3.

We rewrite (1.8) in the form

$$h_t - \nabla \cdot (\rho(u)\nabla h) = \nabla \cdot \mathbf{f} \tag{3.1}$$

which is more convenient for the application of Theorem 2.2. To this end let $\phi(x, t)$ be weak solution of the Dirichlet problem

$$\phi \in H_0^1(\Omega),\ \Delta\phi = f(x, t)$$

a.e. for $t \in [0, T]$ where $f(x, t)$ satisfies (1.13) and define $\mathbf{f} = \nabla\phi$. By standard results of regularity we have $\mathbf{f} \in L^4(0, T; H_2^1(\Omega))$ and the estimate

$$\|\mathbf{f}(t)\|_{\mathbf{L}^4(\Omega)} \leq C_1\|\mathbf{f}(t)\|_{\mathbf{H}^1(\Omega)} \leq C_2\|\phi(t)\|_{H^2(\Omega)} \leq C_3\|f(t)\|_{L^2(\Omega)}. \tag{3.2}$$

We say that (h, u) is a weak solution of problem (3.1), (1.8)-(1.11) if

$$h \in L^2(0, T; H_2^1(\Omega)), h_t \in L^2(0, T; H^1(\Omega)) \tag{3.3}$$

$$u \in L^2(0, T; H_0^1(\Omega))u_t \in L^2(0, T; H^1(\Omega)) \tag{3.4}$$

$$\rho(u)|\nabla h|^2 \in L^2(0, T; L^2(\Omega)) \tag{3.5}$$

$$< h', v > + \int_\Omega \rho(u)\nabla h \cdot \nabla v dx = -(\mathbf{f}, \nabla v) \tag{3.6}$$

for all $v \in H_0^1(\Omega)$ and for a.e. $t \in [0, T]$

$$h(0) = 0 \tag{3.7}$$

$$< u', \eta > +(\nabla u, \nabla \eta) = \int_\Omega \rho(u)|\nabla h|^2 \eta dx \tag{3.8}$$

for all $\eta \in H_0^1(\Omega)$ and for a.e. $t \in [0, T]$

$$u(0) = u_0. \tag{3.9}$$

Since h and $u \in C([0, T]; L^2(\Omega))$, the conditions (3.7) and (3.9) make sense. To prove existence we apply the Galerkin method. Let $w_k(x)$, $k = 1, ...$ be smooth functions such that $\{w_k\}_{k=1}^\infty$ is an orthogonal basis of $H_0^1(\Omega)$ and an orthonormal basis of $L^2(\Omega)$. We fix $m \in \mathbf{N}$ and consider the function

$$u_m = \sum_{k=1}^m d_k(t) w_k(x)$$

where $d_k(t) \in C^1([0, T])$. We solve the problem

$$h'_m = \nabla \cdot (\rho(u_m)\nabla h_m) + \nabla \cdot \mathbf{f} \tag{3.10}$$

$$h_m = 0 \; on \; \Gamma \times (0, T), \; h_m(x, 0) = 0 \tag{3.11}$$

and then we compute the functions $d_k(t)$ solving the following system of nonlinear O.D.E.

$$(u'_m, w_k) + (\nabla u_m, \nabla w_k) = \int_\Omega \rho(u_m)|\nabla h_m|^2 w_k dx \tag{3.12}$$

with the initial conditions

$$d_k(0) = (u_0, w_k), \quad k = 1, .., m.$$

By Theorem 2.1 and recalling (2.30) and (3.2), we have

$$\nabla h_m \text{ is bounded in } \mathbf{L}^4(Q). \tag{3.13}$$

Multiplying (3.10) by h_m we obtain, by (1.12),

$$\frac{1}{2}\frac{d}{dt}\|h_m(t)\|^2 + \rho_0\|\nabla h_m(t)\|^2 \le \|\,\mathbf{f}(t)\|\|\nabla h_m\| \le \frac{1}{2\rho_0}\|\mathbf{f}(t)\|^2 + \frac{\rho_0}{2}\|\nabla h_m\|^2.$$

Therefore we have

$$\frac{d}{dt}\|h_m\|^2 + \rho_0\|\nabla h_m\|^2 \le \frac{1}{2\rho_0}\|\mathbf{f}\|^2.$$

Thus

$$\{h_m\} \text{ is bounded in } L^2(0,T;H_0^1(\Omega)) \text{ and in } L^\infty(0,T;L^2(\Omega)). \tag{3.14}$$

Multiplying (3.12) by $d_k(t)$ and summing over k we obtain, recalling (3.13) and denoting C_4 the constant entering in the Poincaré inequality,

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2 + \|\nabla u_m\|^2 = \int_\Omega \rho(u)|\nabla h_m|^2 u_m dx$$

$$\le \rho_1 \int_\Omega |\nabla h_m|^2 |u_m| dx \le \frac{\rho_1}{2\varepsilon}\|\nabla h_m\|^4 + \frac{\rho_1\varepsilon}{2}C_4\|\nabla u_m\|^2.$$

Choosing $\varepsilon = \frac{1}{C_4\rho_1}$ we conclude that

$$\{u_m\} \text{ is bounded in } L^2(0,T;H_0^1(\Omega)) \text{ and in } L^\infty(0,T;L^2(\Omega)). \tag{3.15}$$

It remains to find an *a priori* estimate for u'. Let v be any function of $H_0^1(\Omega)$ such that $\|v\|_{H_0^1(\Omega)} \le 1$ and write $v_1 + v_2$ with $v_1 \in \ span\{w_k;\ k = 1, .., m\}$. Since $\|v_1\|_{H_0^1(\Omega)} \le 1$, we have

$$| < u_m', v > | = |(u_m', v_1)| = |(\nabla u_m, \nabla v_1)| + \int_\Omega \rho(u_m)|\nabla h_m|^2 v_1 dx$$

$$\le \|\nabla u_m\| + \rho_1 C \|\nabla h_m\|^2_{L^4(\Omega)}.$$

By (3.13) and (3.15)

$$\{u_m'\} \ is \ bounded \ in \ L^2(0, T; H^{-1}(\Omega)). \tag{3.16}$$

In as similar way we obtain:

$$\{h_m'\} \ is \ bounded \ in \ L^2(0, T; H^{-1}(\Omega)). \tag{3.17}$$

Recalling (see [6]) that the space

$$\left\{u \in L^2(0, T; H_0^1(\Omega)),\ u' \in L^2(0, T; H^{-1}(\Omega))\right\}$$

is compactly imbedded in $L^2(Q)$, we can extract from $\{u_m\}$ and $\{h_m\}$ two sequences, not relabelled, such that

$$\lim_{m\to\infty} u_m = u,\ \lim_{m\to\infty} h_m = h \ \ in \ L^2(Q) \ and \ a.e. \ in \ Q \tag{3.18}$$

$$\lim_{m\to\infty} u_m = u,\ \lim_{m\to\infty} h_m = h \ \ weakly \ \ in \ L^2(0, T; H_0^1(\Omega)) \tag{3.19}$$

$$\lim_{m\to\infty} u_m = u,\ \lim_{m\to\infty} h_m = h \ \ in \ L^2(0, T; H^{-1}(\Omega)). \tag{3.20}$$

From (3.18) we have

$$\|h_m(t)\| \to \|h(t)\| \ \ in \ L^2(0, T).$$

Hence we can extract from h_m a subsequence, still denoted h_m, such that

$$\|h_m(t)\| \to \|h(t)\| \ \ for \ \ a.e. \ t \in [0, T]. \tag{3.21}$$

Let $v \in L^2(0, T; H_0^1(\Omega))$. From (3.10) we have

$$\int_0^T < h'_m, v > dt + \int_0^T \int_\Omega \rho(u_m)\nabla h_m \cdot \nabla v dx dt = -\int_0^T (\mathbf{f}, \nabla v) dt. \tag{3.22}$$

To pass to the limit for $m \to \infty$ in (3.22) we add and subtract $\rho(u)\nabla h_m \cdot \nabla v$ in the second integral. In this way we need to estimate

$$\Big| \int_0^T \int_\Omega [\rho(u_m) - \rho(u)]\nabla h_m \cdot v dx dt \Big| \tag{3.23}$$

since the limits of all other terms for $m \to \infty$ are easily found. We can majorize (3.23) using (3.13) with

$$\|\rho(u_m) - \rho(u)\|_{L^4(Q)} \|\nabla h_m\|_{\mathbf{L}^4(Q)} \|\nabla v\|_{\mathbf{L}^2(Q)}$$

$$\leq C\|\rho(u_m) - \rho(u)\|_{L^4(Q)} \leq 4\rho_1^2 L^2 \|u_m - u\|_{L^2(Q)} \to 0,$$

the last inequality follows from (1.12) and the limit is zero in view of (3.18). Hence, letting $m \to \infty$ in (3.22), we have

$$\int_0^T < h', v > dt + \int_0^T \int_\Omega \rho(u)\nabla h \cdot \nabla v dx dt = -\int_0^T (\mathbf{f}, \nabla v) dt$$

for all functions $v \in L^2(0, T; H_0^1(\Omega))$ and, in particular,

$$< h', v > + \int_\Omega \rho(u)\nabla h \cdot \nabla v dx = -(\mathbf{f}, \nabla v) \tag{3.6}$$

for all $v \in H_0^1(\Omega)$ and for a.e. $t \in [0, T]$.

It remains to obtain (3.8). To this end we use the following

Lemma 3.1. We have

$$\lim_{m\to\infty} \|\rho(u_m)|\nabla h_m|^2 - \rho(u)|\nabla h|^2\|_{L^1(Q)} = 0. \tag{3.24}$$

Proof. Let us multiply (3.10) by h_m and integrate by parts recalling (3.11). By (3.21) we have, for almost every $T > 0$,

$$\lim_{m\to\infty}\int_0^T\int_\Omega \rho(u_m)|\nabla h_m|^2 dxdt = \lim_{m\to\infty}\Big[-\frac{1}{2}\int_\Omega h_m^2(T)dx + \int_0^T (\mathbf{f}, \nabla h_m)dt\Big]$$

(3.25)

$$= -\frac{1}{2}\int_\Omega h^2(T)dx + \int_0^T (\mathbf{f}, \nabla h)dt = \int_0^T\int_\Omega \rho(u)|\nabla h|^2 dxdt.$$

Let $\boldsymbol{\xi} = (\xi_1, \xi_2) \in (L^2(0, T; H_0^1(\Omega)))^2$. Proceeding as for estimating (3.22) we have, in view of (3.18) and (3.19),

$$\lim_{m\to\infty}\int_Q \sqrt{\rho(u_m)}\nabla h_m \cdot \boldsymbol{\xi} dxdt = \lim_{m\to\infty}\Big[\int_Q \sqrt{\rho(u)}\nabla h_m \cdot \boldsymbol{\xi} dxdt$$

(3.26)

$$+\int_Q \Big(\sqrt{\rho(u_m)} - \sqrt{\rho(u)}\Big)\nabla h_m \cdot \boldsymbol{\xi} dxdt\Big] = \int_Q \sqrt{\rho(u)}\nabla h \cdot \boldsymbol{\xi} dxdt.$$

On the other hand

$$\int_Q \Big|\rho(u_m)|\nabla h_m|^2 - \rho(u)|\nabla h|^2\Big| dxdt$$

$$= \int_Q \Big|\Big[\sqrt{\rho(u_m)}\nabla h_m - \sqrt{\rho(u)}\nabla h\Big]\cdot\Big[\sqrt{\rho(u_m)}\nabla h_m + \sqrt{\rho(u)}\nabla h\Big]\Big| dxdt$$

$$\leq C\|\sqrt{\rho(u_m)}\nabla h_m - \sqrt{\rho(u)}\nabla h\|_{L^2(Q)}.$$

Hence (3.24) follows from (3.25) and (3.26). □

We are now in a position to obtain (3.8). To this goal we fix an integer N and choose a function $\eta(x, t)$ having the form

$$\eta = \sum_{k=1}^N d_k(t) w_k(x), \tag{3.27}$$

where $\{d_k(t)\}_{k=1}^N$ are given smooth functions. We choose $m \geq N$, multiply (3.12) by $d_k(t)$, sum for $k = 1, .., N$ and then integrate with respect to t to find

$$\int_0^T (u'_m, \eta)dt + \int_0^T (\nabla u_m, \nabla \eta)dt = \int_0^T \int_\Omega \rho(u_m)|\nabla h_m|^2 \eta dx dt.$$

Letting $m \to \infty$ we have, recalling (3.19), (3.20) and (3.24),

$$\int_0^T < u', \eta > dt + \int_0^T (\nabla u, \nabla \eta)dt = \int_0^T \int_\Omega \rho(u)|\nabla h|^2 \eta dx dt. \tag{3.28}$$

Since functions of the form (3.27) are dense in $L^2(0, T; H_0^1(\Omega))$, the identity (3.28) holds for all $\eta \in L^2(0, T; H_0^1(\Omega))$ and thus, in particular, we have (3.8). This proves the existence of a weak solution to problem (3.3)-(3.9).

Final Remark. Since $u_t - \Delta u = \rho(u)|\nabla h|^2 \in L^2(Q)$, we have

$$u \in H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega)), \quad and \quad h \in H^1(0, T; H_0^1 \cap H^2(\Omega)).$$

Taking into account this result of regularity, we can prove that the solution is unique proceeding, with minor changes, as in [2].

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