In this paper, we firstly study a commutative algebra $E$ over a field $F$ of $Char(F) \neq 2$ that satisfying $\dim(E^2) = 1$. We show that, such an algebra is an evolution algebra. Afterwards, we pay attention to commutative duplicate of a commutative algebra $E$. We find necessary and sufficient condition in which the duplicate $D(E)$ is an evolution algebra. And, we finish by studying an evolution algebra that is a Bernstein algebra. We classify that algebras, up to isomorphism, in dimension $\leq 4$.

1. Introduction

Given a commutative field $F$ and a finite dimensional algebra $E$, we say that $E$ is an evolution algebra if it admits a basis $B = \{e_1, \ldots, e_n\}$ such that

$$e_ie_j = 0, \text{ for } 1 \leq i \neq j \leq n \text{ and } e_i^2 = \sum_{k=1}^{n} a_{ik}e_k, \text{ for } 1 \leq i \leq n. \quad (1)$$

Such a basis is called a natural basis of $E$. The matrix $M = (a_{ik})_{1 \leq i,k \leq n}$ is called the matrix of structural constants of $E$ relative to the natural basis $B$. Evolution algebras are commutative ([15]). The origin and the first study of the evolution algebras date from 1941 with the first formulation due to Etherington ([6,}

---

Received on August 31, 2020

AMS 2010 Subject Classification: Primary 17D92, 17A05, Secondary 17D99, 17A60

Keywords: Evolution algebras, Bernstein algebras, Duplicate, natural basis.
Page 34]) of strict self-fertilization in the absence of mutation. Subsequently, Holgate extended Etherington’s formulation to study the case of partial self-fertilization ([9]). It is from work of Tian ([14]) that these algebras were popularized and studied under the denomination of evolution algebras.

In section 2, we study $n$-dimensional commutative algebras $E$ satisfying $\dim(E^2) = 1$. We show that such algebras are evolution algebras, then we give a classification in dimension 2, 3 and 4.

In section 3, we exhibit a necessary and sufficient condition for a commutative duplicate of commutative algebra to be an evolution algebra.

In section 4, we characterize the baric algebras that are Bernstein algebras and we give a classification in dimension 2, 3 and 4.

2. Quadratic forms and evolution algebras

In this section, we study finite dimensional commutative algebra $E$ over a commutative field $F$ of $\text{Char}(F) \neq 2$ and satisfying $\dim(E^2) = 1$.

2.1. Case of dimensions 2 and 3

Example 2.1. Let $E$ be a commutative 2-dimensional algebra such that $\dim(E^2) = 1$. Then $E$ is an evolution algebra.

Proof. Let $E = Fe_1 \oplus Fe_2$ with $\dim(E^2) = 1$, i.e. $E^2 = Fc$ for a certain $c \in E$. The multiplication table of $E$ in the basis $\{e_1, e_2\}$ is given by $e_1^2 = \alpha c$, $e_2^2 = \beta c$ and $e_1 e_2 = \gamma c$. We set $x = x_1 e_1 + x_2 e_2 \in E$ and we have $x^2 = (\alpha x_1^2 + \beta x_2^2 + 2\gamma x_1 x_2)c$. For the reduction of the quadratic form $q(x) = \alpha x_1^2 + \beta x_2^2 + 2\gamma x_1 x_2$, we distinguish two cases

• $(\alpha, \beta) \neq 0$. Without loss of generality, we assume that $\alpha \neq 0$. Then $x^2 = (\alpha x_1^2 + \frac{2\gamma}{\alpha} x_1 x_2 + \beta x_2^2)c = (\alpha x_1 + \frac{\gamma}{\alpha} x_2)^2 + (\beta - \frac{\gamma^2}{\alpha}) x_2^2)c$. By taking $e_2' = -\frac{\gamma}{\alpha} e_1 + e_2$, we get $e_1 e_2' = 0$. Thus, $E$ is an evolution algebra in the natural basis $\{e_1, e_2'\}$.

• $\alpha = \beta = 0$. We have $x^2 = 2\gamma x_1 x_2 c = \frac{\gamma}{2} ((x_1 + x_2)^2 - (x_1 - x_2)^2)c$. By setting $e_1' = e_1 + e_2$ and $e_2' = e_1 - e_2$, we have $(e_1 + e_2)(e_1 - e_2) = 0$. Consequently, $E$ is an evolution algebra in the natural basis $\{e_1', e_2'\}$.

Example 2.2. Let $E$ be a commutative 3-dimensional algebra such that $\dim(E^2) = 1$. Then $E$ is an evolution algebra.

Proof. Let $E = Fe_1 \oplus Fe_2 \oplus Fe_3$ with $\dim(E^2) = 1$, i.e. $E^2 = Fc$ for a certain $c \in E$. The multiplication table of $E$ in the basis $\{e_1, e_2, e_3\}$ is given by $e_1^2 = \alpha c$, $e_2^2 = \beta c$, $e_3^2 = \gamma c$, $e_1 e_2 = \delta c$, $e_1 e_3 = \mu c$ and $e_2 e_3 = \lambda c$. Let $x = x_1 e_1 + x_2 e_2 + x_3 e_3 \in E$. 

we have \( x^2 = (\alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 + 2\delta x_1 x_2 + 2\mu x_1 x_3 + 2\lambda x_2 x_3) c \). For the reduction of the quadratic form \( q(x) = (\alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 + 2\delta x_1 x_2 + 2\mu x_1 x_3 + 2\lambda x_2 x_3) \), we distinguish the following cases

- \( (\alpha, \beta, \gamma) \neq 0 \). Without loss of generality, we assume that \( \alpha \neq 0 \). Then

\[
x^2 = \left( \alpha \left( x_1^2 + 2\left( \frac{\delta}{\alpha} x_2 + \frac{\mu}{\alpha} x_3 \right)x_1 \right) + \beta x_2^2 + \gamma x_3^2 + 2\lambda x_2 x_3 \right) c
\]

\[
= \left( \alpha \left( x_1 + \frac{\delta}{\alpha} x_2 + \frac{\mu}{\alpha} x_3 \right)^2 + \left( \beta - \frac{\delta^2}{\alpha} \right)x_2^2 + \left( \gamma - \frac{\mu^2}{\alpha} \right)x_3^2 + 
\frac{1}{\alpha} \left( \alpha \gamma - \mu^2 - \left( \frac{\alpha \lambda - \delta \mu}{\alpha \beta - \delta^2} \right)^2 \right)x_3^2 \right) c
\]

\[
i) \quad \delta^2 - \beta \alpha \neq 0 \text{ or } \mu^2 - \gamma \alpha \neq 0. \text{ We can take } \delta^2 - \beta \alpha \neq 0, \text{ without loss of generality.}
\]

\[
x^2 = \left( \alpha \left( x_1 + \frac{\delta}{\alpha} x_2 + \frac{\mu}{\alpha} x_3 \right)^2 + \left( \beta - \frac{\delta^2}{\alpha} \right)x_2^2 + \left( \gamma - \frac{\mu^2}{\alpha} \right)x_3^2 + 
\frac{1}{\alpha} \left( \alpha \gamma - \mu^2 - \left( \frac{\alpha \lambda - \delta \mu}{\alpha \beta - \delta^2} \right)^2 \right)x_3^2 \right) c
\]

By setting \( e_2' = -\frac{\delta}{\alpha} e_1 + e_2 \) and \( e_3' = \frac{1}{\alpha} \left( \frac{\lambda \delta}{\beta} - \frac{\delta^2 \mu}{\alpha \beta} - \mu \right)e_1 - \frac{\alpha \lambda - \delta \mu}{\alpha \beta - \delta^2} e_2 + e_3 \), we get \( e_1 e_3' = e_1 e_3' = e_2' e_3' = 0 \). So \( E \) is an evolution algebra in the natural basis \( \{e_1, e_2', e_3'\} \).

\[
ii) \quad \delta^2 - \beta \alpha = \mu^2 - \gamma \alpha = 0. \text{ Then}
\]

\[
x^2 = \left( \alpha \left( x_1 + \frac{\delta}{\alpha} x_2 + \frac{\mu}{\alpha} x_3 \right)^2 + \frac{1}{2} \left( \lambda - \frac{\delta \mu}{\alpha} \right)x_2 x_3 \right) c
\]

By taking \( e_2' = \frac{\delta + \mu}{2\alpha} e_1 + \frac{\delta}{2} e_2 + \frac{\delta}{2} e_3 \) and \( e_3' = \frac{\delta - \mu}{2\alpha} e_1 + \frac{\delta}{2} e_2 - \frac{\delta}{2} e_3 \), we obtain \( e_1 e_2' = e_1 e_3' = e_2' e_3' = 0 \). So \( E \) is an evolution algebra in the natural basis \( \{e_1, e_2', e_3'\} \).
• \( \alpha = \beta = \gamma = 0 \). Without loss of generality, we can take \( \delta \neq 0 \). Thus

\[
x^2 = 2\delta \left( x_1x_2 + \frac{\mu}{\delta} x_1x_3 + \frac{\lambda}{\delta} x_2x_3 \right) c
\]

\[
= 2\delta \left( \left( x_1 + \frac{\lambda}{\delta} x_3 \right) \left( x_2 + \frac{\mu}{\delta} x_3 \right) - \frac{\lambda \mu}{\delta^2} x_3^2 \right) c
\]

\[
= \left( \frac{\delta}{2} \left( x_1 + x_2 + \frac{\lambda + \mu}{\delta} x_3 \right)^2 - \frac{\delta}{2} \left( x_1 - x_2 + \frac{\lambda - \mu}{\delta} x_3 \right)^2 - \frac{2\lambda \mu}{\delta} x_3^2 \right) c
\]

By setting \( e_1' = e_1 + e_2, e_2' = e_1 - e_2 \) and \( e_3' = -\frac{\lambda}{\delta} e_1 - \frac{\mu}{\delta} e_2 + e_3 \), we get \( e_1'e_3' = e_2'e_3' = 0 \). So \( \mathcal{E} \) is an evolution algebra in the natural basis \( \{e_1', e_2', e_3'\} \).

\[
\square
\]

### 2.2. General case

Let \( (\mathcal{E}, b) \) be a bilinear space. A vector \( x \neq 0 \) of \( \mathcal{E} \) is said to be isotropic if \( b(x,x) = 0 \). Otherwise \( x \) is said to be anisotropic. If \( (\mathcal{E}, b) \) contains an isotropic vector, then \( (\mathcal{E}, b) \) is also called isotropic bilinear space. Otherwise \( (\mathcal{E}, b) \) is called anisotropic. A subspace \( W \) of \( \mathcal{E} \) is totally isotropic if \( b(W,W) = 0 \), i.e. \( b(x,y) = 0 \) for all \( x,y \in W \). The radical of a symmetric bilinear form \( b(x,y) \) is the set of all \( x \) such that \( b(x,y) = 0 \), for all \( y \in \mathcal{E} \).

**Theorem 2.3** ([10, Theorem 4.1, Witt’s Decomposition]). In characteristic \( \neq 2 \), any quadratic space \( (\mathcal{E}, q) \) admits orthogonal sum decomposition

\[
\mathcal{E} = \mathcal{E}_t \perp \mathcal{E}_{hyp} \perp \mathcal{E}_{an},
\]

called Witt’s decomposition, where \( \mathcal{E}_t = \text{rad}(q) \) is totally isotropic, \( \mathcal{E}_{hyp} = H_1 \perp \cdots \perp H_s \) is a hyperbolic space and \( \mathcal{E}_{an} \) is an anisotropic space.

**Proposition 2.4.** Any finite dimensional commutative algebra \( \mathcal{E} \) such that \( \dim(\mathcal{E}^2) = 1 \) is an evolution algebra. The natural basis being the orthogonal basis of Witt’s decomposition of the induced bilinear form.

**Proof.** Let \( \mathcal{E} \) be such an algebra. We choose \( c \in \mathcal{E} \) such that \( \mathcal{E}^2 = Fc \). For \( x,y \in \mathcal{E} \), \( xy = b(x,y)c \) where \( b : \mathcal{E} \times \mathcal{E} \to F \) is a non-zero symmetric bilinear form. The corresponding quadratic form \( q : \mathcal{E} \to F \) is defined by \( x^2 = q(x)c \). If another \( c' \) is chosen as the generator of \( \mathcal{E}^2 \), then \( c' = \lambda c \), for a certain \( \lambda \in F^* \). The corresponding bilinear form \( b' = \lambda^{-1} b \). Since \( q \) is a quadratic form, Theorem 2.3 tell us, algebra \( \mathcal{E} \) admits an orthogonal basis given by Witt’s decomposition. It follows that algebra \( \mathcal{E} \) is an evolution algebra and the natural basis being the orthogonal basis.

\[
\square
\]
2.3. Classification

Let \( \mathcal{E} = \mathcal{E}_t \perp \mathcal{E}_{\text{hyp}} \perp \mathcal{E}_{\text{an}} \) be Witt’s decomposition of the finite dimensional evolution algebra \( \mathcal{E} \) satisfying \( \dim(\mathcal{E}^2) = 1 \) over a commutative field \( F \) of \( \text{Char}(F) \neq 2 \). The Proof of Proposition 2.4 tells us, there are a non-zero symmetric bilinear \( b : \mathcal{E} \times \mathcal{E} \to F \) and \( c \in \mathcal{E} \) such that \( \mathcal{E}^2 = Fc \) and \( xy = b(x,y)c \) for all \( x,y \in \mathcal{E} \). Let \( q : \mathcal{E} \to F \) be the corresponding quadratic form of \( b \). We choose a basis \( \{u_1, \ldots, u_r\} \) of \( \mathcal{E}_{\text{an}} \) such that \( b(u_i, u_j) = 0 \), for \( i \neq j \), and \( q(u_i) = d_i \neq 0 \) \( (i = 1, \ldots, r) \). Then, we choose a basis \( \{x_i, y_i\} \) of \( H_i \) such that \( b(x_i, y_i) = 0 \), \( q(x_i) = -q(y_i) = 1 \) and finally, we choose a basis \( \{v_1, \ldots, v_t\} \) of \( \mathcal{E}_t = \text{rad}(b) \). Since \( x^2 = q(x)c \), it follows that \( x^3 = q(x)b(x,c)c \), \( \ldots \), \( x^{k+2} = q(x)b(x,c)^kc \). If \( \mathcal{E} \) is a nil-algebra, then \( b(x,c) = 0 \) for all \( x \in \mathcal{E} \); in this case \( c \in \mathcal{E}_t \). Let us suppose that \( \mathcal{E} \) is non-nil. There exists \( z \in \mathcal{E} \) such that \( b(z,c) \neq 0 \). Thus three cases are to be considered.

- \( c \) belongs to \( \mathcal{E}_t = \text{rad}(b) \), i.e. \( b(x,c) = 0 \) for all \( x \in \mathcal{E} \). The multiplication table of \( \mathcal{E} \) in the basis \( \{u_1, \ldots, u_r, v_1, \ldots, v_t\} \) is
  \[
  u_i^2 = d_i c \quad (i = 1, \ldots, r), \text{ the others products are zero.} \tag{3}
  \]

- \( c \) is isotropic, i.e. \( b(c,c) = 0 \) and \( c^2 = 0 \) but \( b(z,c) \neq 0 \), for some \( z \). So \( c \in \mathcal{E}_{\text{hyp}} \) and then there is an \( i \) such that \( c = x_i + y_i \). Without loss of generality, we can assume that \( i = 1 \). In this case \( \mathcal{E} = \mathcal{E}_{\text{hyp}} \perp \mathcal{E}_{\text{an}} \), where \( \mathcal{E}_{\text{hyp}} = H_1 \) and the multiplication table of \( \mathcal{E} \) in the basis \( \{u_1, \ldots, u_r, x_1, y_1, v_1, \ldots, v_t\} \) is
  \[
  u_i^2 = d_i(x_1 + y_1) \quad (i = 1, \ldots, r), \quad x_1^2 = -y_1^2 = x_1 + y_1, \quad \text{the others products are zero.} \tag{4}
  \]

- \( c \) is anisotropic, i.e. \( b(c,c) \neq 0 \). We have \( c^2 = q(c)c \) and by setting \( c' = q(c)^{-1}c \), it follows that \( c'^2 = c' \) is a non-zero idempotent. The multiplication table of \( \mathcal{E} \) in the basis \( \{v_1, \ldots, v_t, u_1, \ldots, u_r\} \) is
  \[
  u_1^2 = u_1, u_i^2 = d_i u_1 \quad (i = 2, \ldots, r), \text{ the others products are zero.} \tag{5}
  \]

Now, we give a low-dimensional classification of such algebras.

**Proposition 2.5.** [4, Theorem 4.1] Any 2-dimensional evolution algebra, over a commutative field \( F \) of \( \text{Char}(F) \neq 2 \), satisfying \( \dim_F(\mathcal{E}^2) = 1 \) is isomorphic to one of the following algebras:

- \( \mathcal{E}_1 : u_1^2 = u_2, u_2^2 = 0. \)
\[ \mathcal{E}_2 : u_1^2 = u_2^2 = u_1 + u_2. \]
\[ \mathcal{E}_3 : u_1^2 = u_1, u_2^2 = 0. \]
\[ \mathcal{E}_4(\alpha) : u_1^2 = u_1, u_2^2 = \alpha u_1, \text{ with } \alpha \in F^*. \]

**Proposition 2.6.** [3, Theorem 3.5(ii), Table 1] Any 3-dimensional evolution algebra, over a commutative field \( F \) of \( \text{Char}(F) \neq 2 \), satisfying \( \dim_F(\mathcal{E}^2) = 1 \) is isomorphic to one of the following algebras

\[ \mathcal{E}_1 : u_1^2 = u_1 + u_2, u_2^2 = -(u_1 + u_2), u_3^2 = 0. \]
\[ \mathcal{E}_2 : u_1^2 = u_1 + u_2, u_2^2 = -(u_1 + u_2), u_3^2 = u_1 + u_2. \]
\[ \mathcal{E}_3 : u_1^2 = u_3, u_2^2 = 0, u_3^2 = 0. \]
\[ \mathcal{E}_4(\alpha) : u_1^2 = u_3, u_2^2 = \alpha u_3, u_3^2 = 0, \text{ with } \alpha \in F^*. \]
\[ \mathcal{E}_5 : u_1^2 = u_1, u_2^2 = u_3^2 = 0. \]
\[ \mathcal{E}_6(\alpha) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = 0, \text{ with } \alpha \in F^*. \]
\[ \mathcal{E}_7(\alpha, \beta) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = \beta u_1 \text{ with } \alpha, \beta \in F^*. \]

With regard to dimension 4, by varying the dimension of \( \mathcal{E}_v \) from 0 to 3 in the equation (2) and taking account the three cases defined above, we have

**Proposition 2.7.** Any 4-dimensional evolution algebra, over a commutative field \( F \) of \( \text{Char}(F) \neq 2 \), satisfying \( \dim_F(\mathcal{E}^2) = 1 \) is isomorphic to one of the following algebras

\[ \mathcal{E}_1 : u_1^2 = v_3, v_1^2 = v_2^2 = v_3^2 = 0; \]
\[ \mathcal{E}_2 : x_1^2 = -y_1^2 = x_1 + y_1, v_1^2 = v_2^2 = 0; \]
\[ \mathcal{E}_3 : x_1^2 = -y_1^2 = x_1 + y_1, u_1^2 = x_1 + y_1, v_1^2 = 0; \]
\[ \mathcal{E}_4(\alpha) : x_1^2 = -y_1^2 = x_1 + y_1, u_1^2 = x_1 + y_1, v_1^2 = -\alpha(x_1 + y_1); \]
\[ \mathcal{E}_5(\alpha) : u_1^2 = v_2, u_2^2 = \alpha v_2, v_1^2 = v_2^2 = 0; \]
\[ \mathcal{E}_6 : u_1^2 = u_1, u_2^2 = u_3^2 = u_4^2 = 0; \]
\[ \mathcal{E}_7(\alpha) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = u_4^2 = 0; \]
\[ \mathcal{E}_8(\alpha, \beta) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = \beta u_1, u_4^2 = 0; \]
\[ \mathcal{E}_9(\alpha, \beta, \gamma) : u_1^2 = u_1, u_2^2 = \alpha u_1, u_3^2 = \beta u_1, u_4^2 = \gamma u_1; \]

with \( \alpha, \beta, \gamma \in F^* \).

**Remark 2.8.** If \( F \) is an algebraically closed field, in particular if any scalar \( \alpha \) of \( F \) is a square, i.e. \( F = F^2 \), the scalars \( \alpha, \beta \) and \( \gamma \) will be replaced by 1.

3. **Duplicate and evolution algebras**

Let \( \mathcal{E} \) be a commutative algebra over a commutative field of \( \text{Char}(F) \neq 2 \), not necessarily associative, nor having an unit element and let \( S_F^2(\mathcal{E}) \) be a second symmetric power of the \( F \)-linear space \( \mathcal{E} \). Let \( I \) and \( J \) be two countable parts. The multiplication \( \sum_{i\in I} (x_i, y_i) \sum_{j\in J} (x_j', y_j') = \sum_{i\in I} x_i y_i \sum_{j\in J} x_j y_j' \), where \( x_i, y_i, x_j', y_j' \) in \( \mathcal{E} \) and \( x_i, y_j \) denotes the symmetric product of \( x_i \) by \( y_j \), defines on \( S_F^2(\mathcal{E}) \) a commutative \( F \)-algebra structure called a **commutative duplicate** of \( \mathcal{E} \) [11].

The duplicate will be denoted by \( D(\mathcal{E}) \). The \( F \)-linear map \( \mu : D(\mathcal{E}) \rightarrow \mathcal{E}^2 \) defines by \( x, y \mapsto xy \) an onto \( F \)-algebra homomorphism called Etherington’s homomorphism. We have \( D(\mathcal{E}) \ker(\mu) = 0 \) and \( D(\mathcal{E}) = \mathcal{E}^2 \times \ker(\mu) \) (s.d. for semi-direct) algebras isomorphism. The semi-direct product is given by \( (x, x')(y, y') = (xy, \varphi(x, y)) \) for all \( x, y \in \mathcal{E}^2 ; x', y' \) in \( \ker(\mu) \) and \( \varphi : \mathcal{E}^2 \times \mathcal{E}^2 \rightarrow \ker(\mu) \) is a \( F \)-bilinear map. We set \( N_F(\mathcal{E}) = \ker(\mu) \). If the family \( \{e_1, \cdots, e_n\} \) is a basis of \( \mathcal{E} \), then \( \{e_i, e_j \mid 1 \leq i \leq j \leq n\} \) is a basis of \( D(\mathcal{E}) \), called the canonical basis of \( D(\mathcal{E}) \) and \( \dim(D(\mathcal{E})) = \frac{n(n+1)}{2} \).

Let \( \mathcal{E} \) be an evolution algebra in the natural basis \( \{e_1, \cdots, e_n\} \). We suppose that \( D(\mathcal{E}) \) is an evolution algebra with the canonical basis as the natural basis.

For \( i \neq j \), we have \( e_i e_j = 0 \), i.e. \( e_i e_j \in N_F(\mathcal{E}) \). For \( i \neq j \), we have \( 0 = (e_i, e_i)(e_j, e_j) = e_i^2, e_j^2 \). Either \( e_i^2 = 0 \) or \( e_j^2 = 0 \) or there exists \( i_0 \in \{1, \cdots, n\} \) such that \( e_i^2 \neq 0 \) and \( e_j^2 = 0 \) for all \( j \neq i_0 \). So either \( \mathcal{E}^2 = 0 \) or \( \mathcal{E}^2 = F e_{i_0}^2 \), i.e. \( \dim(\mathcal{E}^2) = 1 \). The multiplication table of \( D(\mathcal{E}) \) in natural basis \( \{e_i, e_j \mid 1 \leq i \leq j \leq n\} \) is given by \( (e_i, e_i)^2 = e_i^2, e_i^2 \), the others products are zero.

The canonical basis of \( D(\mathcal{E}) \) is not always a natural basis.

**Example 3.1.** Let \( \mathcal{E}_2 : e_1 e_1 = e_1, e_2 e_2 = e_1 \) be an evolution algebra. By taking \( e_{ij} := e_i e_j \), the multiplication table of \( D(\mathcal{E}_2) \) in the canonical basis is given by \( e_1^2 = e_1, e_1 e_2 = e_1, e_2 e_2 = e_1 \), the others products are zero. Since \( e_{11} e_{22} \neq 0 \), this basis is not a natural basis. By taking \( u = e_{22} - e_{11} \), we get \( e_1^2 = e_{11}, e_{11} e_{12} = e_{11} u = e_{12} e_{12} = e_{12} u = u^2 = 0 \). The duplicate algebra is an evolution algebra in the natural basis \( \{e_{11}, e_{12}, u\} \).
For $z$ and $w$ in $D(E)$, we notice that the product in $D(E)$ is given by $zw = \mu(z)\mu(w)$. So, if $E$ is a zero algebra, then for all $z, w \in D(E)$, we have $zw = \mu(z)\mu(w) = 0$ because $\mu(z) = \mu(w) = 0$. Consequently, $D(E)$ is an evolution algebra.

**Theorem 3.2.** Let $E$ be a $n$-dimensional non zero commutative $F$-algebra and $D(E)$ its commutative duplicate. Then $D(E)$ is an evolution algebra if and only if $\dim(E^2) = 1$.

**Proof.** Let us suppose that $D(E)$ is an evolution algebra in the natural basis $\{z_1, \ldots, z_s\}$, with $s = \frac{n(n+1)}{2}$. For $i \neq j$, the equality $z_iz_j = 0$ is equivalent to $\mu(z_i)\mu(z_j) = 0$. Since $E^2 \neq \{0\}$, it follows that there exists $i_0$ such that $\mu(z_{i_0}) \neq 0$. Thus, $\mu(z_j) = 0$ for all $j \neq i_0$, $z_j \in NF(E) = \{x \in D(E) \mid x \cdot D(E) = 0\} = \text{ann}(D(E))$, where $\text{ann}(D(E))$ is the annihilator of $D(E)$. So $\dim(NF(E)) = s - 1$ and $\dim(E^2) = 1$.

Conversely, let $E$ be a commutative $F$-algebra such that $\dim(E^2) = 1$. According to Proposition 2.4, such an algebra is an evolution algebra, the natural basis $\{e_1, e_2, \ldots, e_n\}$ being that orthogonal. Since $D(E)/NF(E) \cong E^2$, it follows that $\dim(NF(E)) = s - 1$. If $e_{i_0}^2 \neq 0$, then $(e_{i_0} \cdot e_{i_0})^2 = e_{i_0}^2 \cdot e_{i_0}^2 \neq 0$, generates $D(E)^2$ and we always deduce from Proposition 2.4 that $D(E)$ is an evolution algebra. 

\[ \Box \]

**4. Bernstein Algebra**

A finite dimensional commutative algebra $E$ over a commutative field $F$ is said to be **baric**, if there is nontrivial homomorphism $\omega : E \rightarrow F$ of algebras. The baric algebra $(E, \omega)$ is called **Bernstein algebra** if

\[ x^2 x^2 - \omega(x)^2 x^2 = 0, \text{ for all } x \in E. \tag{6} \]

Bernstein algebras have their origins in genetics ([2]). Holgate was the first to use the language of non-associative algebras to translate Bernstein’s problem ([8]). We defined inductively **plenary powers** of an element $x \in E$ by:

\[ x^{(1)} = x \text{ and } x^{(k+1)} = x^{(k)} x^{(k)}, \quad k \in \mathbb{N}, \]

while that of $E$ is defined by:

\[ E^{(1)} = E \text{ and } E^{(k+1)} = E^{(k)} E^{(k)}, \quad k \in \mathbb{N}. \]
The following results are well known ([16]).

4.1. Some properties of Bernstein algebras

Let \((\mathcal{E}, \omega)\) be a Bernstein algebra over a commutative field \(F\) of \(\text{Char}(F) \neq 2\). The following results are well known ([16]).

1) The homomorphism \(\omega : \mathcal{E} \rightarrow F\) is the unique weight function of \(\mathcal{E}\).

2) Algebra \(\mathcal{E}\) has at least one non-zero idempotent.

3) For an idempotent \(e\) of \(\mathcal{E}\), the algebra \(\mathcal{E}\) admits the following Peirce decomposition \(\mathcal{E} = Fe \oplus U_e \oplus V_e\), where \(U_e = \{x \in \mathcal{E} \mid ex = \frac{1}{2}x\}\) and \(V_e = \{x \in \mathcal{E} \mid ex = 0\}\). The subspaces \(U_e\) and \(V_e\) satisfy the relations

\[
U_eV_e \subseteq U_e, \quad V_e^2 \subseteq U_e, \quad U_e^2 \subseteq V_e \quad \text{and} \quad U_eV_e^2 = 0
\]

4) The set of idempotents of \(\mathcal{E}\) is given by \(I(\mathcal{E}) = \{e + \sigma + \sigma^2 \mid \sigma \in U_e\}\) for any idempotent \(e\) of \(\mathcal{E}\).

5) Let \(e_1 = e + \sigma + \sigma^2\), with \(\sigma \in U_e\), be another idempotent of \(\mathcal{E}\). We have the following relations \(U_{e_1} = \{u + \sigma u \mid u \in U_e\}\) and \(V_{e_1} = \{v - 2(\sigma + \sigma^2)v \mid v \in V_e\}\). It follows that although the decomposition of the Bernstein algebra depends on the choice of the idempotent \(e\), the dimension of the subspaces \(U_e\) and \(V_e\) of \(\mathcal{E}\) are invariants of \(\mathcal{E}\). If \(r = \dim U_e\) and \(s = \dim V_e\), the pair \((1 + r, s)\) is called the type of \(\mathcal{E}\). Also \(\dim_F(U_e^2)\) and \(\dim_F(U_eV_e + V_e^2)\) are invariants of the algebra \(\mathcal{E}\).

In ([1]), the authors obtain the identities (7) and (8) by linearizing (6).

\[
2x^2(xy) = \omega(xy)x^2 + \omega(x^2)(xy) \quad (7)
\]

\[
4(xz)(xy) + 2x^2(zy) = \omega(zy)x^2 + 2\omega(xy)(xz) + 2\omega(xz)(xy) + \omega(x^2)(zy) \quad (8)
\]

for all \(x, y, z \in \mathcal{E}\) and replacing \(y\) by \(z\) in (8), we get

\[
4(xz)^2 + 2x^2z^2 = \omega(z)^2x^2 + 4\omega(xz)(xz) + \omega(x^2)z^2 \quad (9)
\]

for all \(x, z \in \mathcal{E}\).

4.2. Characterization of Bernstein algebras that are evolution algebras

Let \(F\) be a commutative field of \(\text{Char}(F) \neq 2\).

**Theorem 4.1** ([13, Corollary 3.1.4]). A \(n\)-dimensional baric evolution algebra \((\mathcal{E}, \omega)\) admits a natural basis \(\{e_1, e_2, \ldots, e_n\}\) such that \(\omega(e_1) = 1\) and \(\omega(e_i) = 0\) for \(i > 1\). Moreover \(\mathcal{E} = Fe_1 \oplus \ker \omega\) with \(e_1 \ker \omega = 0\).

We deduce from Theorem 4.1 that the algebra \((\mathcal{E}, \omega)\) admits a natural basis \(\{e_1, e_2, \ldots, e_n\}\) which multiplication table is given by

\[
e_1^2 = e_1 + \sum_{k=2}^{n} a_{1k}e_k, \quad e_j^2 = \sum_{k=2}^{n} a_{jk}e_k \quad (10)
\]
with $\omega(e_1) = 1$, $\omega(e_j) = 0$ and $2 \leq j \leq n$.

In the following, any finite $n$-dimensional baric evolution algebra will be provided with such a natural basis.

**Theorem 4.2** (of characterization). A $n$-dimensional baric evolution algebra is a Bernstein algebra $(E, \omega)$ if and only if the following conditions are satisfying

i) $(e_i^2)^2 = e_i^2$;

ii) $e_i^2 e_j^2 = 0$, for $2 \leq i, j \leq n$;

iii) $e_i^2 e_j^2 = \frac{1}{2} e_i^2$, for $2 \leq i \leq n$.

**Proof.** Let us suppose that algebra $(E, \omega)$ is a Bernstein algebra. Then

(6) leads to i), we take $x = e_1$.

(9) gives ii), we set $x = e_i$ and $z = e_j$ with $i, j \neq 1$.

(9) gives iii), we take $x = e_1$ and $z = e_i$ with $i \neq 1$.

Conversely, it is assumed that conditions i), ii) and iii) are satisfied. Let $x = \sum_{k=1}^{n} x_k e_k$ be an element of $E$ with $\omega(x) = x_1$. We have the following equalities $x^2 = \sum_{k=1}^{n} x_k^2 e_k^2 = x_1^2 e_1^2 + \sum_{k=2}^{n} x_k^2 e_k^2$ and $x^2 x^2 = x_1^2 x_2^2 e_1^2 e_2^2 + 2x_1^2 \sum_{k=2}^{n} x_k^2 e_1^2 e_k^2 + \sum_{k, j=2}^{n} x_k^2 x_j^2 e_k^2 e_j^2 = x_1^2 (x_1 e_1^2 + \sum_{k=2}^{n} x_k^2 e_k^2) = \omega(x)^2 x^2$. So the baric evolution algebra $(E, \omega)$ is a Bernstein algebras.

We see that $e_i^2$ is a non-zero idempotent of $E$ and $e_i^2 \in U_{e_i^2}$ for $i \neq 1$. We deduce that $(\ker \omega)^2 \subseteq U_{e_i^2}$.

**Proposition 4.3.** If a $n$-dimensional baric evolution algebra $(E, \omega)$ is a Bernstein algebra, then

i) $U_{e_i^2} = \{x \in \ker \omega \mid e_i^2 x = \frac{1}{2} x\} = (\ker \omega)^2$ and

ii) $V_{e_i^2} = \{x \in \ker \omega \mid e_i^2 x = 0\} = \langle e_i - 2a_{i1} e_i^2 \mid 2 \leq i \leq n \rangle$.

**Proof.** i) Let us show that $(\ker \omega)^2 = U_{e_i^2}$. Since $(\ker \omega)^2 \subseteq U_{e_i^2}$, it remains to show that $U_{e_i^2} \subseteq (\ker \omega)^2$. Let $x = \sum_{i=2}^{n} x_i e_i \in U_{e_i^2}$, then $x = 2e_i^2 x = 2 \sum_{i=2}^{n} x_i (a_{i1} e_i^2) \in (\ker \omega)^2$. Hence $U_{e_i^2} \subseteq (\ker \omega)^2$ and $U_{e_i^2} = (\ker \omega)^2$.

ii) For $i \in \{2, \ldots, n\}$, we have $e_i^2 (e_i - 2a_{i1} e_i^2) = 0$; so $\langle e_i - 2a_{i1} e_i^2 \mid 2 \leq i \leq n \rangle \subseteq V_{e_i^2}$. Let $x = \sum_{i=2}^{n} x_i e_i \in V_{e_i^2}$, then $0 = e_i^2 x = \sum_{i=2}^{n} x_i a_{i1} e_i^2$. Thus $x = \sum_{i=2}^{n} x_i (e_i - 2a_{i1} e_i^2)$ and we have $V_{e_i^2} \subseteq \langle e_i - 2a_{i1} e_i^2 \mid 2 \leq i \leq n \rangle$. We deduce that $V_{e_i^2} = \langle e_i - 2a_{i1} e_i^2 \mid 2 \leq i \leq n \rangle$.

**Remark 4.4.** If the baric evolution algebra $(E, \omega)$ is a Bernstein algebra, then $U_{e_i^2} = (\ker \omega)^3 = (\ker \omega)^2 (\ker \omega)^2 = 0$, i.e. $E$ is a exceptional Bernstein algebra ([7]).
**Definition 4.5** ([17]). Let \((E, \omega)\) be a \((n + 1)\)-dimensional Bernstein algebra of type \((r + 1, s)\). If \(\ker \omega\) is a zero algebra, i.e. \((\ker \omega)^2 = 0\), then the algebra \(E\) is called a trivial Bernstein algebra of type \((r + 1, s)\).

**Remark 4.6.** In ([12]), the authors show that an algebra is a Jordan Bernstein algebra if and only if it is a train algebra of rank 3. We deduce that a finite dimensional evolution algebra \((E, \omega)\) is a Jordan Bernstein algebra if and only if \((\ker \omega)^2 = 0\) ([13, Theorem 3.2.3]). Thus, the only finite dimensional evolution algebras \((E, \omega)\), that are Jordan Bernstein algebras, are evolution algebras, that are trivial Bernstein algebras.

**Proposition 4.7.** If a baric evolution algebra \((E, \omega)\) is a 2-dimensional Bernstein algebra, then \(E\) is a trivial Bernstein algebra.

**Proof.** Since \(\ker \omega = \langle e_2 \rangle\), it follows that there are \(\alpha \in F\) such that \(e_2^2 = \alpha e_2\). 0 = \((\ker \omega)^3\) leads to \(0 = e_2^3 e_2^2 = \alpha^3 e_2\); hence \(\alpha^3 = 0\), i.e. \(\alpha = 0\). We deduce that \((\ker \omega)^2 = 0\) and the algebra \(E\) is a trivial Bernstein algebra. \(\square\)

**Proposition 4.8.** If a finite-dimensional baric evolution algebra \((E, \omega)\) is a Bernstein algebra, then \(U_{e_1^2} V_{e_1^2}\) is an invariant of \(E\). Moreover, if \(U_{e_1^2} \neq 0\), then \(U_{e_1^2} V_{e_1^2} \neq 0\).

**Proof.** Let \(e = e_1^2 + \sigma + \sigma^2 \in \mathcal{I}(E), u = u_1 + \sigma u_1 \in U_e\) and \(v = v_1 - 2(\sigma + \sigma^2)v_1 \in V_e\) with \(\sigma, u_1 \in U_{e_1^2}\) and \(v_1 \in V_{e_1^2}\). Since \(U_{e_1^2} = 0\), we have \(e = e_1^2 + \sigma, u = u_1\) and \(v = v_1 - 2\sigma v_1\). We also have \(uv = u_1(v_1 - 2\sigma v_1) = u_1 v_1 - 2u_1(\sigma v_1) = u_1 v_1\) because \(U_{e_1^2}(U_{e_1^2} V_{e_1^2}) \subseteq U_{e_1^2} = 0\). So \(U_e v = U_{e_1^2} V_{e_1^2}\).

We assume that \(\dim_F (\ker \omega)^2 = k \neq 0\). By renumbering the vectors of the family \(\{e_2, \ldots, e_n\}\), we can assume that the family \(\{e_j^2 \mid 2 \leq j \leq k + 1\}\) is a basis of \((\ker \omega)^2\). Set \(e_j^2 = \sum_{i=2}^{k+1} a_{ji} e_i^2\) with \(k + 2 \leq j \leq n\). If \(U_{e_1^2} V_{e_1^2} = 0\), then we would have \(e_j^2(e_j - 2a_{1j}e_j^2) = 0\) for \(2 \leq j \leq n\). What would result in the following system:
\[
\begin{align*}
  a_{2j} &= 0, \quad 2 \leq j \leq k + 1 \\
  a_{2j} \alpha_{ji} &= 0, \quad 2 + k \leq j \leq n \quad \text{and} \quad 2 \leq t \leq k + 1.
\end{align*}
\]

Thus, we would have \(\frac{1}{2} e_j^2 = e_1^2 e_j^2 = e_j^2 \sum_{j=2}^{n} a_{2j} e_j = e_j^2 \sum_{j=k+2}^{n} a_{2j} e_j = e_j^2 \sum_{j=k+2}^{n} a_{1j} (a_{2j} \alpha_{ji}) e_i^2 = 0\), so, \(e_j^2 = 0\). This would contradict linear independence of the family \(\{e_i^2 \mid 2 \leq i \leq k + 1\}\). We deduce that \(U_{e_1^2} V_{e_1^2} \neq 0\). \(\square\)

**Lemma 4.9.** If a \(n\)-dimensional baric evolution algebra \((E, \omega)\) is a Bernstein algebra, then the family \(\{e_i^2 \mid 2 \leq i \leq n\}\) is linear dependent.

**Proof.** We have \(\ker \omega = \langle e_2, \ldots, e_n \rangle\) and \((\ker \omega)^2 \subset \ker \omega\). We assume that the family is linear independent. Then \((\ker \omega)^2 = \ker \omega\); hence \(0 = (\ker \omega)^{(3)} =\)
(ker ω)^2 = ker ω, this is impossible. We deduce that the family is linear dependent.

**Theorem 4.10.** If a n-dimensional baric evolution algebra (E, ω) (with n > 2) is a non trivial Bernstein algebra, then dim_F(ker ω)^2 \leq \frac{1}{2}(n-1).

**Proof.** We have 0 \neq (ker ω)^2 \subseteq ker ω. We assume that p = dim_F(ker ω)^2. By renumbering the basis vectors, we can assume that (ker ω)^2 = (e_2, \ldots, e_{p+1}).

Let us show that the family \{e_2, \ldots, e_{p+1}, e_2^2, \ldots, e_{p+1}^2\} is linear independent. Let (α_k, β_k)_{2 \leq k \leq p+1} ∈ F^p × F^p such that

\[ \sum_{k=2}^{p+1} (α_k e_k + β_k e_k^2) = 0 \]  \hspace{1cm} (11)

By multiplying (11) by e_i, we obtain

\[ α_i e_i^2 + \sum_{k=2}^{p+1} β_k e_i e_k^2 = α_i e_i^2 + \sum_{k=2}^{p+1} β_k α_k e_i^2 = 0, \]  \hspace{1cm} (12)

By squaring (11), we get

\[ \sum_{k=2}^{p+1} (α_k^2 e_k^2 + \sum_{j=2}^{p+1} 2 α_k β_j e_k e_j) = \sum_{k=2}^{p+1} α_k (α_i + 2 \sum_{j=2}^{p+1} β_j a_{ji}) e_k^2 = 0, \]  \hspace{1cm} (13)

By multiplying (12) by 2α_i we get

\[ α_i (2α_i + \sum_{j=2}^{p+1} β_j a_{ji}) = 0, \]  \hspace{1cm} (14)

and by making the difference of (13) and (14), we have α_i^2 = 0. This leads to α_i = 0, for all i ∈ \{2, \ldots, p+1\}. Then (11) tell us that β_i = 0, ∀ i ∈ \{2, \ldots, p+1\}. We deduce that dim_F(ker ω)^2 \leq \frac{1}{2}(n-1). □

**Corollary 4.11.** If a finite n-dimensional baric evolution algebra (E, ω) is a Bernstein algebra such that dim(U_{e'_1}) = p, then dim(V_{e'_1}) ≥ p.

**Proof.** We assume that (ker ω)^2 = (e_2^2, \ldots, e_{p+1}^2) and let us show that the family \{e_2 - 2a_{12} e_2^2, \ldots, e_{p+1} - 2a_{1,p+1} e_{p+1}^2\} is linear independent. Let (α_k)_{2 \leq k \leq p+1} ∈ F^p such that \sum_{k=2}^{p+1} α_k (e_k - 2a_{1k} e_k^2) = 0.

We have \sum_{k=2}^{p+1} α_k (e_k - 2a_{1k} e_k^2) = \sum_{k=2}^{p+1} α_k e_k - 2a_{1k} α_k e_k^2 = 0. So α_k = 0, for all k ∈ \{2, \ldots, p+1\} because \{e_2, \ldots, e_{p+1}, e_2^2, \ldots, e_{p+1}^2\} is linear independent. Consequently, the family \{e_2 - 2a_{12} e_2^2, \ldots, e_{p+1} - 2a_{1,p+1} e_{p+1}^2\} is linear independent and dim(V_{e'_1}) ≥ p. □
4.3. Classification

Let \((E, \omega)\) be a Bernstein algebra that is evolution algebra in natural basis \(\{e_1, e_2, \ldots, e_n\}\) such that \(e_1^2 = e_1 + \sum_{k=2}^n a_{1k}e_k\) and \(e_j^2 = \sum_{k=2}^n a_{jk}e_k\). For \((a_{12}, \ldots, a_{1n}) = 0\), we have \(e_1^2 = e_1\), i.e. \(e_1\) is a non-zero idempotent of \(E\) and \(e_1\ker\omega = 0\) leads to \(E\) is of type \((1, n - 1)\), constant Bernstein algebra.

4.3.1. Three-dimensional Classification

**Theorem 4.12.** Let \((E, \omega)\) be an evolution algebra that is a 3-dimensional non trivial Bernstein algebra with canonical basis \(\{e, u, v\}\). Then, the algebra \(E\) is isomorphic to \(E_0: e^2 = e, eu = \frac{1}{2}u, uv = u, the others products are zero.

**Proof.** Let \((E, \omega)\) be a 3-dimensional non trivial Bernstein algebra that is evolution algebra in the natural basis \(\{e_1, e_2, e_3\}\). The multiplication table of \(E\) in the natural basis is given by \(e_1^2 = e_1 + a_{12}e_2 + a_{13}e_3\), \(e_2^2 = a_{22}e_2 + a_{23}e_3\) and \(e_3^2 = a_{32}e_2 + a_{33}e_3\) with \(\omega(e_1) = 1\) and \(\omega(e_2) = \omega(e_3) = 0\). We have \((\ker\omega)^2 \neq 0\) and \(1 \leq \dim(\ker\omega)^2 \leq \frac{1}{2}(3 - 1) = 1\). So \(dim(\ker\omega)^2 = 1\) and we set \((\ker\omega)^2 = Fe_1^2\). Then the vector \(e_2 - 2a_{12}e_2^2\) is a non-zero vector of \(V_{e_1}\) and we set \(e = e_1^2, u = e_2^2, v = e_2 - 2a_{12}e_2^2\). The multiplication table of \(E\) in the canonical basis \(\{e, u, v\}\) is \(e^2 = e, eu = \frac{1}{2}u, uv = e_3^2(e_2 - 2a_{12}e_2^2) = a_{22}u\) and \(v^2 = (e_2 - 2a_{12}e_2^2)^2 = e_2^2 - 4a_{12}e_2^2e_2 = (1 - 4a_{12}a_{22})u\). Since the algebra \(E\) is a non trivial Bernstein algebra, it follows that \(U_{e_1}V_{e_1} \neq 0\). Consequently, \(a_{12} \neq 0\). Let us find a canonical basis \(\{e', u', v'\}\) of \(E\) such that \(u'v' = u'\) and \(v'^2 = 0\). We set \(e' = e + au, u' = bu\) and \(v' = cv - 2au(cv) = c(v - 2aa_2u)\) with \(b, c \in F^*\). We have \(u' = bu, u'^1 = bcv = a_{22}bcu = a_{22}cu'\) leads to \(c = a_{22}^{-1}\) and \(0 = v'^2 = c^2(v - 2a_{22}u)^2 = a_{22}^{-2}(v^2 - 4aa_{22}uv) = a_{22}^{-2}((1 - 4a_{12}a_{22}) - 4aa_{22}^2)u = a_{22}^{-2}b^{-1}((1 - 4a_{12}a_{22}) - 4aa_{22}^2)u'\) implies \(0 = (1 - 4a_{12}a_{22}) - 4aa_{22}^2\), i.e. \(a = a_{22}^{-2}(\frac{1}{4} - a_{12}a_{22})\). We can take \(b = 1\) and we have \(e' = e + a_{22}^{-2}(\frac{1}{4} - a_{12}a_{22})u, u' = u\) and \(v' = a_{22}^{-1}((v - a_{22}^{-1}(\frac{1}{4} - 2a_{12}a_{22}))u). We deduce that algebra \(E\) is isomorphic to \(E_0\).

4.3.2. Four-dimensional Classification

**Theorem 4.13.** Let \((E, \omega)\) be an evolution algebra that is 4-dimensional non trivial Bernstein algebra with canonical basis \(\{e, u, v, w\}\). Then, \(E\) is isomorphic to one and only one of the following algebras \(E_1: uv = u, e^2 = e, eu = \frac{1}{2}u; E_2: uv = uv = vw = u, e^2 = e, eu = \frac{1}{2}u\) and the others products are zero.

The proof of the theorem uses the lemma below which follows from [5, Proof of Theorem, page 1435].
Lemma 4.14. Let $\mathcal{E}$ be a 4-dimensional Bernstein algebra with a canonical basis $\{e, u, v, w\}$ such that $e^2 = e$, $eu = \frac{1}{2}u$, $uv = u$, $v^2 = \gamma u$, $w^2 = \lambda u$, $vw = \mu u$ and the others products are zero.

- If $\lambda = \mu = 0$, then the algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_1$.
- If $\lambda \neq 0$, then the algebra $\mathcal{E}$ is isomorphic to $\mathcal{E}_2$.

Where the algebras $\mathcal{E}_1$ and $\mathcal{E}_2$ are defined in Theorem 4.13.

Proof of Theorem 4.13. Let $(\mathcal{E}, \omega)$ be a 4-dimensional non trivial Bernstein algebra that is evolution algebra with the natural basis $\{e_1, e_2, e_3, e_4\}$. The multiplication table of $\mathcal{E}$ in the natural basis is given by $e_1^2 = e_1 + \sum_{k=2}^{4} a_{1k} e_k$, $e_j^2 = \sum_{k=2}^{4} a_{jk} e_k$ with $\omega(e_1) = 1$ and $\omega(e_j) = 0$ where $2 \leq j \leq 4$. We have $(\ker \omega)^2 \neq 0$ and $1 \leq \dim(\ker \omega)^2 \leq \frac{1}{2}(4 - 1) = 1.5$. So $\dim(\ker \omega)^2 = 1$ and we set $(\ker \omega)^2 = Fe_2^2$. Then $e_2 - 2a_{12}e_2^2$ is a non-zero vector of $\mathcal{E}_{e_1}^2$ and there are scalars $\alpha$, $\alpha$, such that $e_2^2 = \alpha_3 e_2^2$ and $e_4^2 = \alpha_4 e_2^2$. We assume that $a_{23} = a_{24} = 0$, then the equality $0 = e_2^2 e_2^2 = a_{22}^2 e_2^2$ leads to $a_{22} = 0$. Thus $e_2^2 = 0$, this is impossible and we deduce that $(a_{23}, a_{24}) \neq 0$. For $V_{e_1}^2$ is generated by $(e_2 - 2a_{12}e_2^2), (e_3 - 2a_{13}e_3^2), (e_4 - 2a_{14}e_4^2)$, let us show that $\{e_2 - 2a_{12}e_2^2, e_3 - 2a_{13}e_3^2\}$ or $\{e_2 - 2a_{12}e_2^2, e_4 - 2a_{14}e_4^2\}$ is a basis of $V_{e_1}^2$. For this reason, consider the scalars $\alpha$, $\beta$, $\gamma$ such that $0 = \alpha(e_2 - 2a_{12}e_2^2) + \beta(e_3 - 2a_{13}e_3^2) + \gamma(e_4 - 2a_{14}e_4^2)$. Since $\alpha(e_2 - 2a_{12}e_2^2) + \beta(e_3 - 2a_{13}e_3^2) + \gamma(e_4 - 2a_{14}e_4^2) = (\alpha - 2(\alpha a_{12} + \beta a_{13} + \gamma a_{14} a_{22}) e_2 + (\beta - 2(\alpha a_{12} + \beta a_{13} + \gamma a_{14} a_{23}) e_3 + (\gamma - 2(\alpha a_{12} + \beta a_{13} + \gamma a_{14} a_{24}) e_4$, it follows that the equality $0 = \alpha(e_2 - 2a_{12}e_2) + \beta(e_3 - 2a_{13}e_3^2) + \gamma(e_4 - 2a_{14}e_4^2)$ gives $(\alpha - 2(\alpha a_{12} + \beta a_{13} + \gamma a_{14} a_{22}) = (\beta - 2(\alpha a_{12} + \beta a_{13} + \gamma a_{14} a_{23}) = (\gamma - 2(\alpha a_{12} + \beta a_{13} + \gamma a_{14} a_{24}) = 0$. If $a_{23} \neq 0$, for $\beta = 0$, we have $\alpha a_{12} + \beta a_{13} + \gamma a_{14} a_{24} = 0$ because $a_{23} \neq 0$. Hence $\alpha = \beta = 0$ and we similarly conclude that $\{e_2 - 2a_{12}e_2^2, e_3 - 2a_{13}e_3^2\}$ is a basis of $V_{e_1}^2$.

1) The multiplication table of $\mathcal{E}$ in the canonical basis $\{e_1^2, e_2, e_2 - 2a_{12}e_2^2, e_3 - 2a_{13}e_3^2\}$ is given by $e_1^2 = e_1^2$, $e_2^2 = e_2^2 = \frac{1}{2}e_2^2$, $e_2^2 = e_2^2 = e_2^2 - 2a_{12}e_2^2 = a_{22}e_2^2$, $e_2^2 = e_2^2 - 2a_{12}e_2^2 = a_{22}e_2^2$, $e_3 - 2a_{13}e_3^2 = e_3 - 2a_{13}e_3^2 = a_{23}e_3^2$, $(e_2 - 2a_{12}e_2^2)^2 = e_2^2 - 4a_{12}e_2^2 = (1 - 4a_{12}a_{22})e_2^2, (e_2 - 2a_{12}e_2^2)(e_3 - 2a_{13}e_3^2) = (e_2 - 2a_{12}e_2^2)(e_3 - 2a_{13}e_3^2) = -2a_{13}e_3^2 e_2 - 2a_{12}e_2^2 e_3 = -2a_{13}e_3^2 e_2 - 2a_{12}a_{23}e_3^2 = -2a_{13}a_{23}e_3^2 e_2 - 2a_{12}a_{23}e_3^2 = -2a_{13}a_{23}e_3^2 e_2 - 2a_{12}a_{23}e_3^2 = -2a_{13}a_{23}e_3^2 e_2 - 2a_{12}a_{23}e_3^2 = a_3(1 - 4a_{13}a_{23})e_2^2$ and the others products are zero. Since algebra $\mathcal{E}$ is a non trivial Bernstein algebra, we have $U_{e_1}^2 V_{e_1}^2 \neq 0$. Consequently, $(a_{22}, a_{23}a_3) \neq 0$ and we set $e = e_1^2, u = e_2^2$. We distinguish the following three cases

a) $a_{22} = 0$, then $a_{23}a_3 \neq 0$ and we set $v = a_{23}^{-1}a_3^{-1}(e_3 - 2a_{13}e_3^2), w = e_2^2$.
2a_{12}e_2^2. The multiplication table of $E$ in the canonical basis \{e, u, v, w\} is $e^2 = e$, $eu = \frac{1}{2}u$, $uv = u$, $v^2 = a_{23}^{-1}a_{3}^{-1}(1 - 4a_{13}a_{23}a_3)u$, $vw = -2a_{12}u$, $w^2 = u$ and the others products are zero. We deduce from Lemma 4.14 that the algebra $E$ is isomorphic to $E_2$.

b) $\alpha_3a_{23} = 0$, then $a_{22} \neq 0$, we set $v = a_{22}^{-1}(e_2 - 2a_{12}e_2^2)$ and $w = e_3 - 2a_{13}e_3^2$. The multiplication table of $E$ in the canonical basis \{e, u, v, w\} is given by $e^2 = e$, $eu = \frac{1}{2}u$, $uv = u$, $v^2 = a_{22}^{-1}(1 - 4a_{12}a_{22})u$, $vw = -2a_{13}a_3u$, $w^2 = \alpha_3u$ and the others products are zero. Lemma 4.14 tells us that, for $\alpha_3 = 0$ we get algebra $E_1$ and for $\alpha_3 \neq 0$, algebra $E$ is isomorphic to $E_2$.

c) $a_{22}a_{23}a_3 \neq 0$, then we set $v = a_{22}^{-1}(e_2 - 2a_{12}e_2^2)$ and $w = (e_3 - 2a_{13}e_3^2) - a_{22}^{-1}a_{3}^{-1}(e_2 - 2a_{12}e_2^2)$. The multiplication table of $E$ in the canonical basis \{e, u, v, w\} is $e^2 = e$, $eu = \frac{1}{2}u$, $uv = u$, $v^2 = a_{22}^{-1}(1 - 4a_{12}a_{22})u$, $vw = a_{22}^{-1}(e_2 - 2a_{12}e_2^2)(e_3 - 2a_{13}e_3^2) - a_{22}^{-1}a_{3}^{-1}(e_2 - 2a_{12}e_2^2)^2 = (-2a_{22}^{-1}a_{13}a_{22} + a_{13}a_{23}) - a_{22}^{-1}a_{3}^{-1}(1 - 4a_{12}a_{22}))u = a_{22}^{-1}a_{3}^{-1}(2a_{13}a_{22}a_3 - 1)u, w^2 = (e_3 - 2a_{13}e_3^2) + a_{22}^{-1}a_{3}^{-1}(e_2 - 2a_{12}e_2^2) - 2a_{22}^{-1}a_{3}^{-1}(e_3 - 2a_{13}e_3^2)(e_2 - 2a_{12}e_2^2) = (\alpha_3(1 - 4a_{13}a_{23}a_3) + a_{22}^{-1}a_{3}^{-1}(1 - 4a_{12}a_{22}) + 4a_{22}^{-1}a_{3}^{-1}(a_1a_{22} + a_{12}a_{23}))u = \alpha_3(1 + a_{22}^{-1}a_{3}^{-1})u$ and the others products are zero.

- For $1 + a_{22}^{-1}a_{3}^{-1}a_3 \neq 0$, algebra $E$ is isomorphic to $E_2$.
- For $1 + a_{22}^{-1}a_{3}^{-1}a_3 = 0$, we have $0 = e_2^2 = a_{22}^{-1}a_{3}^{-1}(a_1a_{22} + a_{12}a_{23}a_3)a_3u = 0$. We have $0 = e_2^2 = a_{22}^{-1}a_{3}^{-1}(a_1a_{22} + a_{12}a_{23}a_3)a_3 = 0$ because $a_{24} = 0$. So $e_2^2 = 0$ and we have $\frac{1}{2}e_2^2 = e_2^2 = a_{12}a_{22}^2 + a_{13}a_{23}a_3^2 = (a_1a_{22} - a_{13}a_{23}a_3^2)u$ gives $a_1a_{22} - a_{13}a_{23}a_3^2 = \frac{1}{2}$. Therefore $vw = -a_{22}^{-1}a_{3}^{-1}(2a_{13}a_{22}a_3 - 2a_{12}a_{22} + 1)u = 0$ and we deduce that algebra $E$ is isomorphic to $E_1$.

2) The multiplication table of $E$ in the canonical basis \{e_1^2, e_2^2, e_2 - 2a_{12}e_2^2, e_4 - 2a_{14}e_4^2\} is given by $e_2^2 = e_1^2$, $e_2^2 = \frac{1}{2}e_2^2$, $e_2^2(e_2 - 2a_{12}e_2^2) = e_2^2e_2 = a_{22}^2$, $e_2^2(e_4 - 2a_{14}e_4^2) = e_2^2e_4 = a_{24}^2a_4e_2^2$, $e_2^2(e_2 - 2a_{12}e_2^2)^2 = e_2^2 - 2a_{12}e_2^4 = (e_2 - 2a_{12}e_2^2)(e_4 - 2a_{14}e_4^2 = (e_2 - 2a_{12}e_2^2)(e_4 - 2a_{14}e_4^2) = -2a_{14}a_4e_2^2 - 2a_{12}e_2^4 = -2a_{14}a_4e_2^2 - 2a_{12}e_2^4 = -2a_{14}a_4e_2^2 - 2a_{12}e_2^4 = -2a_{14}a_4e_2^2 - 2a_{12}e_2^4 = -2a_{14}a_4e_2^2 - 2a_{12}e_2^4 = -2a_{14}a_4e_2^2 = (e_4 - 2a_{14}e_4^2)^2 = (e_2 - 2a_{12}e_2^2)^2 = e_2^2 - 4a_{14}a_4e_2^2e_4^2 = e_2^2 - 4a_{14}a_4e_2^2e_4^2 = e_2^2 - 4a_{14}a_4e_2^2e_4^2 = (1 - 4a_{13}a_{23}a_3)u = 0$. We obtain the multiplication table of the algebra defined in 1. We deduce that, for $a_{22} = 0$ or for $a_4 \neq 0$ and $a_{24} = 0$ or for $a_{22}a_{24}a_4(1 + a_{22}^{-1}a_{3}^{-1}) \neq 0$, algebra $E$ is isomorphic to $E_2$. It is isomorphic to algebra $E_1$ for $a_4 = 0$ or for $a_{22}a_{24}a_4 \neq 0$ and $1 + a_{22}^{-1}a_{3}^{-1}a_4 = 0$.\[\]
Acknowledgements

The second author thanks Professor Richard Varro for his remarks on the origin of evolution algebras.

REFERENCES


A. CONSEIBO  
*Université Norbert Zongo, BP 376 Koudougou, Burkina Faso*  
e-mail: andreconsebo@yahoo.fr

S. SAVADOYO  
*Université Norbert Zongo, BP 376 Koudougou, Burkina Faso*  
e-mail: sara01souley@yahoo.fr

M. OUATTARA  
*Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso*  
e-mail: ouatt_ken@yahoo.fr