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ON SERIES INVOLVING ZEROS OF TRANSCENDENTAL FUNCTIONS ARISING FROM VOLTERRA INTEGRAL EQUATIONS

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Series arising from Volterra integral equations of the second kind are summed. The series involve inverse powers of roots of the characteristic equation. It is shown how previous similar series obtained from differential-difference equations are particular cases of the present development. A number of novel and interesting results are obtained. The techniques are demonstrated through illustrative examples.

1. Introduction.

Silberstein [13] found the sums of two series arising from the differential-difference equation

$$u'(x) = u(x - \eta).$$

Cerone and Keane [2] generalized the results to obtain the sum of the series $\sum(p_j)^{-k}$ and $\sum(1+\eta p_j)^{-k}$ where p_j are the roots of $p = e^{-\eta p}$ and summation is over all p_j.

The current paper examines summing series of roots of transcendental equations arising from integral equations. The development is at first based on

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a Volterra integral equation of the second kind describing the births of a single-sex population. This was the initial motivation for the work and it is felt to be instructive and illuminating.

In particular, using Laplace transform and residue techniques on integral equations, sums of series of the form

$$\sum \frac{1}{(p_j - \alpha)^n \mu_j}, \alpha \neq p_j, n \text{ a positive integer}$$

are obtained where, the summation is over all the roots p_j of the characteristic equation $\phi^*(p) = 1$ and $\mu_j = -[\frac{d}{dp}\phi^*(p)]_{p=p_j}$. The sum is obtained in closed form *without* explicitly determining the roots p_j.

The sum of the above series is given in terms of a recurrence relation which facilitates the evaluation. The technique is believed by the author to be novel and does not appear to be in the literature. The approach provides a great and varied range of new results.

2. Basic Equation and Results.

The renewal integral equation has been studied by many authors (including Feller [5], Cox [4] and Tijms [15]) and was introduced to the field of population dynamics by Sharpe and Lotka [12]. The single-sex deterministic model representing the births $B(t)$ at a time t is given by the Volterra integral equation of the second kind (see Lotka [10], Keyfitz [7])

$$B(t) = F(t) + \int_0^t B(t-u)\phi(u)\,du \tag{2.1}$$

where $F(t)$ is the contribution of those alive at the origin of time, and $\phi(u)$ is the net maternity function which is of compact support and bounded.

If $\phi(u)$ were a probability density and $F(u)$ its distribution function then (2.1) would be a renewal integral equation with $B(t)$ being the renewal function. Equation (2.1) is more general since $\phi(u)\,du$ is the chance of living to age u and giving birth in the next interval of length du and so $\phi(u)$ is not necessarily a density.

The integral equation with which we will at first be interested is (2.1) with (Keyfitz [7])

$$F(t) = \frac{\phi(x+t)}{l(x)} \tag{2.2}$$

which represents the situation where there is only one ancestor aged x at our chosen origin. Here in (2.2) , $l(x)$ is the survivor function which gives the probability of surviving to age x of a newborn.

The solution of (2.1) has been extensively examined in the past and a rigorous methodology is presented by Feller [5] using Laplace transform techniques. The asymptotic behaviour has been studied by Lopez [9] in relation to population modelling and in general by Bellman and Cooke [1]. Chandhry [3] investigates a related problem, that of the number of renewals.

We will also use Laplace transform techniques here so that from (2.1) and (2.2) we obtain after minor manipulation

$$B_x(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\frac{V(p,x)}{1-\phi^*(p)}dp \tag{2.3}$$

where

$$V(p,x) = \frac{e^{px}\int_x^\infty e^{-pu}\phi(u)du}{l(x)} = \frac{v(p,x)}{l(x)} \tag{2.4}$$

and $\phi^*(p)$ is the Laplace transform of $\phi(x)$ with γ being chosen in such a manner as to ensure convergence. Assuming that the roots of the denominator of (2.3) are the only poles of the integrand and are simple then

$$B_x(t) = \sum \frac{V(p_j,x)e^{p_j t}}{\mu_j}, t > 0, \tag{2.5}$$

where

$$\mu_j = -[\frac{d}{dp}\phi^*(p)]_{p=p_j} = \int_0^\infty e^{-p_j u}u\phi(u)\,du. \tag{2.6}$$

Lopez [9] shows that the real root of $\phi^*(p) = 1$ has the greatest real part and the rest occur in complex conjugate pairs (Pollard [11]) for $\phi(u)$ positive. In realistic population dynamics applications $\phi(u)$ is also bounded and of compact support.

If we now allow $t \to 0+$ then, since the Laplace transform gives the mean value at a discontinuity (Bellman and Cooke [1], Widder [17]), we obtain from (2.3) and (2.4)

$$\frac{1}{2}\phi(x+) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{v(p,x)}{1-\phi^*(p)}dp, \tag{2.7}$$

where $\phi(x+) = \lim_{\varepsilon \to 0} \phi(x+\varepsilon)$, $\varepsilon > 0$.

Assuming that the roots of $\phi^*(p) = 1$ are the only poles of (2.7) (which has been shown to be the case by Lopez [9] for population dynamics applications) then

$$\frac{1}{2}\phi(x+) = \sum \frac{v(p_j, x)}{\mu_j} \tag{2.8}$$

and in particular with $x \to 0+$, with $v(p_j, x)$ defined in (2.4), gives

$$S_0 = \sum \frac{1}{\mu_j} = \frac{\phi(0+)}{2}. \tag{2.9}$$

Integration of (2.7) from t to ∞ , gives upon interchanging the order of integration, which is permissible since ϕ is positive and exponentially bounded,

$$\frac{v(0,t)}{2} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{v(0,t) - v(p,t)}{p[1-\phi^*(p)]} dp, t > 0, \tag{2.10}$$

where $v(p, t)$ is as defined in (2.4) .

Theorem 1. *Let p_j be the simple and non-zero roots of $\phi^*(p) = 1$, $\phi^*(p) \neq 0(\frac{1}{p})$,*

$$M_0 = \int_0^\infty \phi(u)\, du = \phi^*(0) < \infty, M_0 \neq 1$$

and μ_j is as given in equation (2.6), then

$$S_1 = \sum \frac{1}{p_j \mu_j} = \frac{1}{2} \cdot \frac{M_0 + 1}{M_0 - 1}, \tag{2.11}$$

where the summation is over all p_j.

Proof. Allowing $t \to 0+$ in equation (2.10) gives

$$\frac{M_0}{2} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{M_0 - \phi^*(p)}{p[1-\phi^*(p)]} dp. \tag{2.12}$$

We note that if $\phi^*(0) = 1$ so that $M_0 = 1$ then we obtain the degenerate result that was also obtained by Cerone and Keane [2] viz.,

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} \frac{dp}{p} = \frac{1}{2}, \tag{2.13}$$

since there is a contribution of $\frac{1}{2}$ from integration in an anticlockwise direction along a semicircular contour to the left of the line integral and a contribution of 1 from the residue.

From (2.12) we have

$$\frac{M_0}{2} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{M_0 - 1}{p[1-\phi^*(p)]}dp + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{dp}{p}. \tag{2.14}$$

We may evaluate the integrals using the theory of residues to give

$$\frac{M_0}{2} = -1 + \sum \frac{M_0 - 1}{p_j \mu_j} + \frac{1}{2} \tag{2.15}$$

where the terms on the right are contributions from the pole at zero, the simple poles p_j of $\phi^*(p) = 1$ for the first integral and the last term is as given by (2.13). A simple rearrangement of (2.15) gives the desired result (2.11). □

It is a straightforward matter to deduce from (2.14) upon using (2.13) that

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dp}{p[1-\phi^*(p)]} = \frac{1}{2} \tag{2.16}$$

and so from (2.10)

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{v(p,t)}{p[1-\phi^*(p)]}dp = 0 \tag{2.17}$$

where $v(p,t)$ is given by (2.4).
We note that putting $t = 0$ in (2.17) gives

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\phi^*(p)}{p[1-\phi^*(p)]}dp = 0 \tag{2.18}$$

which agrees with the results (2.13) and (2.16) since (2.18) is equation (2.13) minus (2.16) .

Lemma 1. *Let $|\phi(t)| \le Ke^{-\lambda t}$ for $K, \lambda \ge 0$, constants.*
Further, let $I_0(t) = \phi(t)$, $J_0(p,t) = v(p,t)$, as given in (2.4),
and $I_n(t) = \int_t^\infty I_{n-1}(x)dx$, $J_n(p,t) = \int_t^\infty J_{n-1}(p,x)dx$, $n = 1, 2, \ldots$.
Then

$$I_n(t) = \int_t^\infty \phi(x)\frac{(x-t)^{n-1}}{(n-1)}dx, n = 1, 2, \ldots \tag{2.19}$$

and

$$J_n(p,t) = \frac{1}{p^n}\left[\sum_{k=1}^{n}(-1)^{k-1}p^{n-k}I_{n-k+1}(t) + (-1)^n J_0(p,t)\right] \tag{2.20}$$

for $n = 1, 2, \ldots$

Proof. A straight forward induction argument and a change of order of integration, permissible from the postulates, produces the desired results (2.19) and (2.20). □

Theorem 2. *Let $|\phi(t)| \le Ke^{-\lambda t}$ for $K, \lambda \ge 0$, constants. Then*

$$S_n = \sum \frac{1}{p_j^n \mu_j} \tag{2.21}$$

for $n \ge 3$, satisfies the recurrence relation

$$(1 - M_0)S_n = \sum_{k=2}^{n-1} (-1)^{n+k} \frac{M_{n-k}}{(n-k)} S_k + \frac{(-1)^n}{(n-1)} \frac{M_{n-1}}{(1-M_0)}, \tag{2.22}$$

$$\text{where } M_n = \int_0^\infty u^n \phi(u)\, du < \infty \text{ is the } n^{th} \text{ moment of } \phi(u) \text{ and } M_0 \ne 1. \tag{2.23}$$

Moreover, we have

$$S_2 = \sum \frac{1}{p_j^2 \mu_j} = \frac{M_1}{(1-M_0)^2}. \tag{2.24}$$

Proof. From equation (2.17) we have

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{J_{n-1}(p,t)}{p[1 - \phi^*(p)]}\, dp = 0, n = 2, 3, \ldots \tag{2.25}$$

where from equation (2.20)

$$J_{n-1}(p,t) = \frac{1}{p^{n-1}} \left[\sum_{j=0}^{n-2} (-1)^j p^{n-j-2} I_{n-j-1}(t) + (-1)^{n-1} v(p,t) \right] \tag{2.26}$$

with $I_n(t)$ being given by (2.19) and $v(p,t)$ by (2.4).
Thus from (2.25) and (2.26) we have

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\sum_{j=0}^{n-2} (-1)^j p^{n-j-2} I_{n-j-1}(t) + (-1)^{n-1} v(p,t)}{p^n [1 - \phi^*(p)]}\, dp = 0 \tag{2.27}$$

$$n = 2, 3, \ldots.$$

Now, there is a simple pole at $p = 0$ in (2.27) since

$$v(p,t) = \sum_{r=0}^{\infty} (-1)^r p^r I_{r+1}(t) \tag{2.28}$$

and $\phi^*(0) \neq 1$. The expansion (2.28) is allowed since $I_n(t)$ can be easily demonstrated to be exponentially bounded given that ϕ is.

The fact that there is a simple pole at $p = 0$ may be more easily seen from (2.27) and (2.28) from which,

$$\begin{aligned}
&\sum_{j=0}^{n-2} (-1)^j p^{n-j-2} I_{n-j-1}(t) + (-1)^{n-1} v(p,t) \\
&= \sum_{k=0}^{n-2} (-1)^{n-2-k} p^k I_{k+1}(t) + (-1)^{n-1} \sum_{r=0}^{\infty} (-1)^r p^r I_{r+1}(t) \\
&= \sum_{k=0}^{n-2} \left[(-1)^{n-2-k} + (-1)^{n-1} \cdot (-1)^k\right] p^k I_{k+1}(t) + (-1)^{n-1} \sum_{r=n-1}^{\infty} (-1)^r p^r I_{r+1}(t).
\end{aligned}$$

Now, using the fact that

$$(-1)^{n-2-k} + (-1)^{n-1} \cdot (-1)^k = (-1)^{n-2-k}[1 + (-1)^1] = 0,$$

then

$$\begin{aligned}
&\sum_{j=0}^{n-2} (-1)^j p^{n-j-2} I_{n-j-1}(t) + (-1)^{n-1} v(p,t) \\
&= (-1)^{2(n-1)} p^{n-1} I_n(t) + (-1)^{n-1} \sum_{r=n}^{\infty} (-1)^r p^r I_{r+1}(t).
\end{aligned} \tag{2.29}$$

The contribution from the pole at $p = 0$ is thus, from (2.27) and using (2.29)

$$\frac{I_n(t)}{1 - \phi^*(0)}. \tag{2.30}$$

Further, the contribution from the roots of $\phi^*(p) = 1$ gives for $n \geq 3$

$$\sum_{j=0}^{n-3} (-1)^j S_{j+2} I_{n-1-j}(t) + (-1)^{n-2} S_n I_1(t) + (-1)^{n-1} \sum \frac{v(p_j,t)}{p_j^n \mu_j}. \tag{2.31}$$

Hence, combining (2.30) and (2.31) results in

$$\sum \frac{v(p_j,t)}{p_j^n \mu_j} - S_n I_1(t) = \sum_{j=0}^{n-3} (-1)^{n+j} I_{n-1-j}(t) S_{j+2} + (-1)^n \frac{I_n(t)}{1-\phi^*(0)}. \tag{2.32}$$

Evaluation of (2.32) at $t=0$ and using the facts from (2.19) , (2.4) and (2.23), that

$$I_{n+1}(0) = \frac{M_n}{n!},\ v(p_j,0)=1 \text{ and } M_0 = \phi^*(0) \neq 1$$

then

$$(1-M_0)S_n = \sum_{j=0}^{n-3} (-1)^{n+j} \frac{M_{n-j-2}}{(n-j-2)!} S_{j+2} + \frac{(-1)^n M_{n-1}}{(n-1)!(1-M_0)}. \tag{2.33}$$

The substitution $k=j+2$ in (2.33) gives the desired result (2.22) .

Now, from (2.27) we have for $n=2$

$$\frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{I_1(t) - v(p,t)}{p^2[1-\phi^*(p)]}\, dp = 0 \tag{2.34}$$

so that the contribution from the simple pole at $p=0$ is

$$\frac{I_2(t)}{1-\phi^*(0)} \tag{2.35}$$

since from (2.28),

$$v(p,t) = I_1(t) - pI_2(t) + \sum_{r=2}^{\infty} (-1)^r p^r I_{r+1}(t).$$

The contribution from the roots of $\phi^*(p)=1$ gives from (2.34)

$$S_2 I_1(t) - \sum \frac{v(p_j,t)}{p_j^2 \mu_j}. \tag{2.36}$$

Combining (2.35) and (2.36) then gives

$$S_2 I_1(t) - \sum \frac{v(p_j,t)}{p_j^2 \mu_j} + \frac{I_2(t)}{1-\phi^*(0)} = 0$$

and evaluation at $t = 0$ produces the stated result (2.24).

The theorem is now completely proven. □

Evaluation of (2.27) at $t = 0$ gives on using the facts that $I_n(0) = \frac{M_{n-1}}{(n-1)!}$, and $v(p,0) = \phi^*(p)$,

$$\sum_{k=2}^{n}(-1)^k \frac{M_{n-k}}{(n-k)!} A_k + (-1)^{n-1} B_n = 0; \; n = 2, 3, \ldots \tag{2.37}$$

where

$$A_n = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dp}{p^n[1-\phi^*(p)]} \tag{2.38}$$

and

$$B_n = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\phi^*(p)}{p^n[1-\phi^*(p)]}\, dp. \tag{2.39}$$

Now,

$$B_n = A_n - \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dp}{p^n} = A_n, \tag{2.40}$$

since the integral shown is zero for $n > 1$.

Taking $n = 2$ in (2.37) gives

$$M_0 A_2 - B_2 = 0$$

and so $(M_0 - 1)A_2 = 0$ since $B_2 = A_2$ from (2.40). Hence, since $M_0 \neq 1$, $A_2 = 0$.

Now from (2.37) we have, on using (2.40),

$$\sum_{k=2}^{n}(-1)^k \frac{M_{n-k}}{(n-k)!} A_k + (-1)^{n-1} A_n = 0; \text{ for } n \geq 3. \tag{2.41}$$

An inductive argument on (2.41) gives since $M_0 \neq 1$, $A_n = 0$ for $n > 2$ and hence from (2.40), $B_n = 0$ for $n > 2$. Thus, from (2.38) and (2.39)

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dp}{p^n[1-\phi^*(p)]} = 0, n = 2, 3, \ldots \tag{2.42}$$

and

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\phi^*(p)}{p^n[1-\phi^*(p)]}\,dp = 0, n = 2, 3, \ldots. \tag{2.43}$$

It follows from ((2.42) that,

$$S_n = \sum \frac{1}{p_j^n \mu_j} = -\text{Res}_{p=0}^{(n)} \tag{2.44}$$

where

$$\text{Res}_{p=0}^{(n)} = \frac{1}{(n-1)!}\Big[\frac{d^{n-1}}{dp^{n-1}}(\frac{1}{1-\phi^*(p)})\Big]_{p=0}, \tag{2.45}$$

is the contribution from a pole of order n at $p = 0$.

Theorem 3. *Equations* (2.44) *and* (2.45) *give a different, although equivalent, representation for* S_n *than that given by equations* (2.22) *and* (2.24). *These expressions hold for* $n > 1$.

Proof. Firstly, the sum S_0 and S_1 are given by (2.9) and (2.11) respectively. To prove the theorem it is sufficient to show that, for $n = 3, 4, \ldots,$

$$(1-\phi^*(p))\frac{d^{n-1}}{dp^{n-1}}\Big(\frac{1}{1-\phi^*(p)}\Big) \tag{2.46}$$

$$= \sum_{k=2}^{n-1}\binom{n-1}{k-1}\frac{d^{n-k}}{dp^{n-k}}\phi^*(p)\frac{d^{k-1}}{dp^{k-1}}(\frac{1}{1-\phi^*(p)}) + (\frac{1}{1-\phi^*(p)})\frac{d^{n-1}}{dp^{n-1}}\phi^*(p),$$

since $M_n = (-1)^n\big[\frac{d^n}{dp^n}\phi^*(p)\big]_{p=0}$.

In addition, for $n = 2$, obviously,

$$(1-\phi^*(p))\frac{d}{dp}\Big(\frac{1}{1-\phi^*(p)}\Big) = \frac{1}{1-\phi^*(p)}\frac{d}{dp}\phi^*(p). \tag{2.47}$$

Evaluation of (2.46) and (2.47) at $p = 0$ would give the required result.

Now,

$$\frac{d^{n-1}}{dp^{n-1}}\Big(\frac{1}{1-\phi^*(p)}\Big) = \frac{d^{n-1}}{dp^{n-1}}\Big(1+\frac{\phi^*(p)}{1-\phi^*(p)}\Big)$$

$$= \frac{d^{n-1}}{dp^{n-1}}\Big(\frac{\phi^*(p)}{1-\phi^*(p)}\Big)$$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{d^{n-1-k}}{dp^{n-1-k}} \phi^*(p) \frac{d^k}{dp^k}\Big(\frac{1}{1-\phi^*(p)}\Big)$$

$$= \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{d^{n-k}}{dp^{n-k}} \phi^*(p) \frac{d^{k-1}}{dp^{k-1}}\Big(\frac{1}{1-\phi^*(p)}\Big)$$

$$= \sum_{k=2}^{n-1} \binom{n-1}{k-1} \frac{d^{n-k}}{dp^{n-k}} \phi^*(p) \frac{d^{k-1}}{dp^{k-1}}\Big(\frac{1}{1-\phi^*(p)}\Big)$$

$$+ \binom{n-1}{0}\Big(\frac{1}{1-\phi^*(p)}\Big)\frac{d^{n-1}}{dp^{n-1}}\phi^*(p)$$

$$+ \binom{n-1}{n-1}\phi^*(p)\frac{d^{n-1}}{dp^{n-1}}\Big(\frac{1}{1-\phi^*(p)}\Big).$$

A simple rearrangement produces result (2.46) and hence the theorem is proved. □

It is important to note that although Theorem 3 shows the sum of the series (2.21) to be equivalently given by (2.22) and (2.44) - (2.45) , the recurrence relation representation (2.22) is much easier to apply in practice.

Theorem 3 effectively shows that the recurrence relation (2.22) could be obtained from taking (2.43) instead of (2.42) leading to

$$S_n = \sum \frac{1}{p_j^n \mu_j} = \frac{-1}{(n-1)!}\Big[\frac{d^{n-1}}{dp^{n-1}}\big(\frac{\phi^*(p)}{1-\phi^*(p)}\big)\Big]_{p=0}.$$

It is further of interest to note that series of the general form

$$\sigma_n(\alpha) = \sum \frac{1}{(p_j-\alpha)^n \mu_j} \tag{2.48}$$

can be summed by the above arguments where $\phi^*(\alpha) \neq 1$. This may be accomplished by multiplying equation (2.7) by $e^{-\alpha x}x^{n-1}$, $n \geq 1$ before integration. The $\sigma_n(\alpha)$ of equation (2.48) then satisfy expressions similar to those obtained for S_n if M_n and $\mathrm{Res}_{p=0}^{(n)}$ are replaced by

$$L_n(\alpha) = \int_0^\infty e^{-\alpha x} x^n \phi(x)dx \tag{2.49}$$

and

$$\text{Res}^{(n)}_{p=\alpha} = \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dp^{n-1}}\left(\frac{1}{1-\phi^*(p)}\right)\right]_{p=\alpha}, \tag{2.50}$$

respectively. Contrarily allowing $\alpha \to 0$ gives the previous results since $S_n = \sigma_n(0)$. The technique will be used subsequently of working with $\sigma_n(\alpha)$ to obtain results even when moments are not finite.

Before proceeding to some simple derivations of the results which will be followed by examples, modifications to the above procedures will be discussed when $M_0 = \phi^*(0) = 1$. A similar argument would follow for $L_0(\alpha) = \phi^*(\alpha) = 1$.

Theorem 4. *For the conditions as in Theorem 2 with $\phi^*(0) = M_0 = 1$ then*

$$\bar{S}_n = \sum_{p_j \neq 0} \frac{1}{p_j^n \mu_j} \tag{2.51}$$

for $n \geq 3$ satisfies the recurrence relation

$$M_1\bar{S}_n = \sum_{k=3}^{n}(-1)^{n+k}\frac{M_{n-k+2}}{(n-k+2)!}\bar{S}_{k-1} + \frac{(-1)^n}{M_1^2}\left(M_1 \cdot M_{n+1}(n+1)! - \frac{M_2}{2}\cdot\frac{M_n}{n!}\right). \tag{2.52}$$

Furthermore,

$$M_1\bar{S}_2 = \frac{1}{3!}\frac{M_3}{M_1} - \frac{1}{4}\left(\frac{M_2}{M_1}\right)^2 \tag{2.53}$$

and

$$\bar{S}_1 = \frac{1}{2} - \frac{M_2}{2M_1^2}. \tag{2.54}$$

Proof. From (2.16) with $\phi^*(0) = 1$ there is a double pole at $p = 0$ giving a contribution $2M_1^2$. The contribution from the non-zero roots of the characteristic equation give $\bar{S}_1$. A rearrangement produces (2.54) .

For $n = 2, 3, \ldots$ it may be noticed from (2.27) , (2.28) and (2.29) that

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(-1)^{2(n-1)}p^{n-1}I_n(t) + (-1)^{2n-1}p^n I_{n+1}(t)}{p^n[1-\phi^*(p)]} \tag{2.55}$$

$$+ \frac{(-1)^{n-1}\sum\limits_{r=n+1}^{\infty}(-1)^r p^r I_{r+1}(t)}{p^n[1-\phi^*(p)]}\,dp = 0.$$

Thus, from the coefficient of $I_{n+1}(t)$ there is a simple pole at $p = 0$ since $\phi^*(0) = 1$ and a double pole from the coefficient of $I_n(t)$. The contribution from the pole at $p = 0$ from (2.55) is

$$\frac{-I_{n+1}(t)}{M_1} \text{ and } \frac{M_2}{2M_1^2}I_n(t). \tag{2.56}$$

There is no further contribution for zero from the remainder of the terms in (2.55) since there is no pole at zero. Evaluation of the residues from the poles $p_j \neq 0$ from $\phi^*(p) = 1$ from (2.27) gives for $n \geq 3$

$$\sum_{j=0}^{n-3}(-1)^j\bar{S}_{j+2}I_{n-j-1}(t) + (-1)^{n-1}\left[\sum_{p_j\neq 0}\frac{v(p_j,t)}{p_j^n\mu_j} - I_1(t)\bar{S}_n\right]. \tag{2.57}$$

Combining (2.56) and (2.57) and evaluation at $t = 0$ gives

$$0 = (1-M_0)\bar{S}_n = \sum_{k=2}^{n-1}(-1)^{n+k}\frac{M_{n-k}}{(n-k)!}\bar{S}_k +$$

$$\frac{(-1)^{n-1}}{M_1^2}\left[\frac{M_1\cdot M_n}{n!} - \frac{M_2}{2}\cdot\frac{M_{n-1}}{(n-1)!}\right]$$

and hence, since $M_0 = 1$

$$M_1\bar{S}_n = \sum_{k=2}^{n-1}(-1)^{n+k+1}\frac{M_{n-k+1}}{(n-k+1)!}\bar{S}_k + \frac{(-1)^n}{M_1^2}\left[\frac{M_1\cdot M_{n+1}}{(n+1)!} - \frac{M_2}{2}\cdot\frac{M_n}{n!}\right].$$

Adjusting the summation index by 1 produces (2.52) .

Now for $\bar{S}_2$. Evaluation of (2.56) and (2.57) gives the contribution from the pole at $p = 0$ and, for $p_j \neq 0$ where $\phi^*(p_j) = 1$ as

$$-\frac{I_4(t)}{M_1} + \frac{M_2}{2M_1^2}I_3(t) \tag{2.58}$$

and

$$S_2 I_2(t) + \sum_{p_j \neq 0} \frac{v(p_j, t)}{p_j^3 \mu_j} - I_1(t)\bar{S}_3 \tag{2.59}$$

respectively. Thus, combining (2.58) and (2.59) and equating to zero since these were derived from (2.27) , gives

$$I_2(t)\bar{S}_2 = I_1(t)\bar{S}_3 - \sum_{p_j \neq 0} \frac{v(p_j, t)}{p_j^3 \mu_j} + \frac{I_4(t)}{M_1} - \frac{M_2}{2M_1^2} I_3(t). \tag{2.60}$$

Evaluation of (2.60) at $t = 0$ and using the facts that

$$I_{n+1}(0) = \frac{M_n}{n!} \quad \text{and} \quad v(p_j, 0) = 1$$

produces the result (2.53) as stated. □

An alternative representation for $\bar{S}_n$ may be obtained from (2.42) as

$$\bar{S}_n = -\overline{\text{Res}}_{p=0}^{(n+1)}, \; n = 2, 3, \ldots, \tag{2.61}$$

where

$$\overline{\text{Res}}_{p=0}^{(n+1)} = \frac{1}{n!}\left[\frac{d^n}{dp^n}\left(\frac{p}{1 - \phi^*(p)}\right)\right]_{p=0}$$

is the contribution from a pole of order $n + 1$ at $p = 0$.

A similar argument to the one followed in the proof of Theorem 3 shows that (2.52) - (2.53) and (2.61) are equivalent representations of $\bar{S}_n$.

3. Some Simple Derivations of the Results of Section 2.

Consider the Volterra integral equation

$$B(t) = F(t) + \int_0^t B(t - x)\phi(x)dx. \tag{3.1}$$

With $F(t) = e^{\alpha t}$ then we may readily obtain, using Laplace Transform Techniques, that

$$B(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{pt}dp}{(p - \alpha)[1 - \phi^*(p)]}, \tag{3.2}$$

where γ is chosen to the right of α and the roots of $\phi^*(p) = 1$.

That is, evaluating (3.2) using the theory of residues gives

$$B(t) = \frac{e^{\alpha t}}{1 - \phi^*(\alpha)} + \sum \frac{e^{p_j t}}{(p_j - \alpha)\mu_j} \tag{3.3}$$

where we are assuming that the roots of $\phi^*(p) = 1$ are simple and that $\phi^*(\alpha) \neq 1$.

Since the Laplace transform gives the mean value at a discontinuity (Widder [17]), evaluation of (3.2) and (3.3) at $t = 0$ produces,

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dp}{(p - \alpha)[1 - \phi^*(p)]} = \frac{F(0+)}{2} = \frac{1}{2} \tag{3.4}$$

and

$$\sigma_1(\alpha) = \frac{1}{2} - \frac{1}{1 - \phi^*(\alpha)}. \tag{3.5}$$

Equation (3.5) agrees with (2.11) on putting $\alpha = 0$ and noting $\sigma_1(0) = S_1$.

Differentiation of (3.4) with respect to α produces

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dp}{(p - \alpha)^n[1 - \phi^*(p)]} = 0, n = 2, 3, \ldots \tag{3.6}$$

from which the result

$$\sigma_n(\alpha) = \sum \frac{1}{(p_j - \alpha)^n \mu_j} = -\text{Res}_{p=\alpha}^{(n)} \tag{3.7}$$

is obtained on using (2.48) , (2.50) and (2.44) , (2.45) . The differentiation of (3.4) is permissible since if (3.4) exists then so does (3.6) .

Equation (3.7) could have been obtained directly from (3.5) by differentiation with respect to α and using the result

$$\sigma_n(\alpha) = \frac{1}{n-1} \sigma_{n-1}'(\alpha), n = 2, 3, \ldots. \tag{3.8}$$

We note that $|\sigma_n(\alpha)| < |\sigma_1(\alpha)|$ and so differentiation is justified. As discussed previously, the $\sigma_n(\alpha)$ also satisfy (2.22) with M_n being replaced by $L_n(\alpha)$ as given by (2.49) .

Further, equations (3.6) and (3.7) can be obtained from (3.1) by taking

$$F(t) = e^{\alpha t} t^{n-1}$$

and noting that

$$F(0+) = \begin{cases} 0, & n > 1 \\ 1, & n = 1 \end{cases}.$$

An alternate way to derive the sums of the series would be to take F in equation (3.1) as

$$e^{\alpha t} f(x+t).$$

Thus, with the integral equation

$$b(t) = e^{\alpha t} f(x+t) + \int_0^t b(t-x)\phi(x)dx$$

and assuming f to have a Taylor series expansion about $t = 0$ we would obtain

$$(3.9) \qquad \frac{1}{2} f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \sum_{m=0}^{\infty} \frac{f^{(m)}(x)}{(p-\alpha)^{m+1}[1 - \phi^*(p)]} dp.$$

Hence using residues we get from equation (3.9)

$$(3.10) \qquad \frac{1}{2} f(x) = \sum_{m=0}^{\infty} f^{(m)}(x) \left[\sum \frac{1}{(p_j - \alpha)^{m+1} \mu_j} + \mathrm{Res}_{p=\alpha}^{(m+1)} \right]$$

where $\mathrm{Res}_{p=\alpha}^{(m+1)}$ is the residue at $p = \alpha$ from a pole of order $m+1$ of the integrand in (3.9) .

Since $f(x)$ is an arbitrary function then, equating coefficients of $f^{(m)}(x)$ we obtain from equation (3.10) , $\sigma_n(\alpha)$ as given by equation (3.5) and (3.7) with $n = m + 1$.

It is important to emphasize that although the results could have been obtained directly through the techniques outlined in the present section, the insights gained from Section 2 that led to the recurrence relations (2.22) , (2.52) (and their generalizations for $\sigma_n(\alpha)$) would not have been possible. The current section's results may indicate a relaxation of some of the postulates of Section 2.

A number of examples will now be presented to highlight and elucidate the results obtained.

4. Particular Results.

A. Examples Involving Heaviside Functions

Consider

$$\phi(x) = H(\gamma - x), \tag{4.1}$$

where $H(x) = 1$ for $x > 1$ and zero otherwise, giving

$$\phi^*(p) = \frac{1 - e^{-\gamma p}}{p}$$

and so

$$M_n = \frac{\gamma^{n+1}}{n+1}, \mu_j = \frac{\gamma p_j + 1 - \gamma}{p_j}.$$

Now, from (2.21) , for $\gamma \neq 1$,

$$S_n = \sum \frac{1}{p_j^{n-1}(\gamma p_j + 1 - \gamma)}, \tag{4.2}$$

with p_j the roots of $p = 1 - e^{-\gamma p}$, satisfies (2.11) and (2.22) giving, for example

$$S_1 = \sum \frac{1}{\gamma p_j + 1 - \gamma} = \frac{1}{2} \frac{\gamma + 1}{\gamma - 1}$$

and

$$S_2 = \sum \frac{1}{p_j(\gamma p_j + 1 - \gamma)} = \frac{1}{2}\left(\frac{\gamma}{1 - \gamma}\right)^2.$$

Further, breaking (4.2) into partial fractions would produce sums of series of the form

$$\sum \frac{1}{p_j^k}.$$

In particular from the above expressions for S_1 and S_2,

$$\sum \frac{1}{p_j} = \frac{\gamma}{2(\gamma - 1)}.$$

Taking $\phi(x)$ to be represented by a histogram would give a generalization of the results obtained from $\phi(x)$ given by equation (4.1) .

Thus if

$$\phi(x) = \sum_{r=0}^{R-1} \alpha_r H(x - b_r) H(b_{r+1} - x) \tag{4.3}$$

then series of the form

$$S_n = \sum \frac{1}{p_j^{n-1}\left[1 + \sum_{r=0}^{R} \gamma_r b_r e^{-p_j b_r}\right]}$$

would satisfy (2.11) and (2.22) for $M_0 = \phi^*(0) \neq 1$ where

$$\gamma_r = \begin{cases} \alpha_r, & r = 0 \\ \alpha_r - \alpha_{r-1}, & 0 < r < R \\ -\alpha_{r-1}, & r = R \end{cases}.$$

Taking $R = 1$, $b_0 = 0$, $b_1 = \gamma$ in (4.3) would reproduce the results obtained for $\phi(x)$ as given by (4.1) .

Another special case of (4.3) would be if

$$\phi(x) = H(x - \eta) \tag{4.4}$$

in which instance we note that the moments are not finite.

As envisaged in the previous section we need to work with $\sigma_n(\alpha)$ and $L_n(\alpha) < \infty$. Allowing $\alpha \to 0$ will produce the required results.

From (2.11) (on substitution of $\sigma_1(\alpha)$ for S_1 and $L_0(\alpha)$ for M_0) or from (3.5) we have

$$\sigma_1(\alpha) = \sum \frac{p_j}{(p_j - \alpha)(1 + \eta p_j)} = \frac{1}{2}\frac{L_0(\alpha) + 1}{L_0(\alpha) - 1} \tag{4.5}$$

where

$$L_0(\alpha) = \frac{e^{-\alpha\eta}}{\alpha}.$$

On taking $\alpha \to 0$ in (4.5) reproduces the result of Silberstein [13] and Cerone and Keane [2] namely

$$S_1 = \sum \frac{1}{1 + \eta p_j} = \frac{1}{2} \tag{4.6}$$

where the summation is over all the roots p_j of $pe^{\eta p} = 1$.

From a modified form of (2.24)

$$S_2 = \lim_{\alpha\to 0}\sum \frac{p_j}{(p_j-\alpha)^2(1+\eta p_j)} = \lim_{\alpha\to 0}\frac{L_1(\alpha)}{(1-L_0(\alpha))^2} \tag{4.7}$$

where from (2.49) and (4.4)

$$L_1(\alpha) = \frac{e^{-\alpha\eta}}{\alpha^2}(1+\alpha\eta).$$

Thus, from (4.7)

$$S_2 = \sum \frac{1}{p_j(1+\eta p_j)} = 1. \tag{4.8}$$

Further generalisations to both (4.6) and (4.8) were obtained by Cerone and Keane [2].

B. Exponential ϕ

The case provides both a simple and an instructive example.

Consider

$$\phi(x) = \lambda e^{-\mu x}, \lambda, \mu > 0 \tag{4.9}$$

and so $\phi^*(p) = \frac{\lambda}{p+\mu}$ giving from (2.6) and (2.23)

$$\mu_j = \frac{\lambda}{(p_j+\mu)^2} = \frac{1}{\lambda} \tag{4.10}$$

and

$$M_n = \lambda\frac{n!}{\mu^{n+1}}, \tag{4.11}$$

respectively.

It should be noted that this example gives only one root of the characteristic equation $\phi^*(p) = 1$, namely, $p_j = \lambda - \mu$.

Further, from (2.21) and (4.11) ,

$$S_n = \sum \frac{1}{p_j^n \mu_j} = \frac{\lambda}{(\lambda-\mu)^n}. \tag{4.12}$$

A straight forward induction argument shows that (4.12) satisfies the recurrence relation (2.22) where the M_n are as given by (4.11) . The degenerate case of $n = 1$ requires special mention. It may be seen from (2.18) that the contribution from $p = 0$ and $p_j = \lambda - \mu$ cancel to give a degenerate case. Also from (2.12) the singularity at $p = 0$ can be seen to be removable and so a relationship between the residue at $p = 0$ and that at $p = p_j$ is not possible.

C. Polynomial ϕ

Let $\phi(x) = \frac{x^m}{m!}$ then $\phi^*(p) = p^{-(m+1)}$ and the moments M_n are not finite. However, from (2.49) $L_n(\alpha) < \infty$ for $\alpha > 0$ and so $\sigma_n(\alpha)$ will satisfy (2.22) with M_n replaced by $L_n(\alpha)$ where

$$\sigma_n(\alpha) = \frac{1}{m+1}\sum_{j=0}^{m}\frac{p_j}{(p_j-\alpha)^n} \tag{4.13}$$

and

$$p_j = e^{\frac{2\pi i}{m+1}j},\ j = 0, 1, 2, \ldots, m. \tag{4.14}$$

Further from (2.50) , (3.7) , (4.13) and (4.14)

$$\sum_{j=0}^{m}\frac{p_j}{(p_j-\alpha)^n} = -\frac{(m+1)}{(n-1)!}\left[\frac{d^{n-1}}{dp^{n-1}}\left(\frac{1}{p^{(n+1)}-1}\right)\right]_{p=\alpha}. \tag{4.15}$$

For $m = 0$, $p_0 = 1$, and so on using (4.13) and (4.15) , $\sigma_n(\alpha) = \frac{1}{(1-\alpha)^n}$.

D. Dirac Delta ϕ

Consider the example where $\phi(x)$ is a Dirac delta (see for example Kreyszig [8]), namely

$$\phi(x) = a\delta(x-b), \tag{4.16}$$

defined as zero everywhere except at $x = b$.

The results of Section 2 are derived more simply in Section 3 from the Volterra integral equation (3.1) . For $\phi(x)$ a Dirac delta, then (3.1) would be equivalent to a difference equation.

Results from difference equations were treated in [2]. Moreover, Theorem 3 shows that the recurrence relation developed under strict conditions is equivalent to (2.44) - (2.45) which holds under the weaker requirement that a Laplace transform exists. The Laplace transform of a Dirac delta is given by

$$\phi^*(p) = ae^{-bp} \text{ and } M_n = ab^n. \tag{4.17}$$

We notice that the roots of the characteristic equation are given explicitly as

$$p_j = \frac{\ln a - (2\pi i) j}{b}, \; j = 0, \pm 1, \pm 2, \ldots \tag{4.18}$$

and

$$\mu_j = b. \tag{4.19}$$

Hence from (2.21) , (4.18) and (4.19)

$$S_n = \sum_{j=-\infty}^{\infty} \frac{b^{n-1}}{[\ln a - (2\pi i) j]^n}$$

and so

$$S_n = \frac{b^{n-1}}{(2\pi)^n} \left\{ \frac{1}{\alpha^n} + 2 \sum_{j=1}^{\infty} \frac{\sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} \alpha^{n-2k} j^{2k}}{[\alpha^2 + j^2]^n} \right\} \tag{4.20}$$

where

$$\alpha = \frac{\ln a}{2\pi} \tag{4.21}$$

and $[x]$ represents the integer part of x.

In particular for $n = 1$ in (4.20) , using (2.11) , (4.17) and (4.21) gives on rearrangement

$$\sum_{j=1}^{\infty} \frac{1}{\alpha^2 + j^2} = \frac{\pi}{2\alpha} \coth \pi\alpha - \frac{1}{2\alpha^2}, \tag{4.22}$$

agreeing with the result in Whittaker and Watson [16]. .

Allowing $\alpha \to 0$ in (4.22) gives $\sum_{j=1}^{\infty} j^{-2} = \frac{\pi^2}{6}$ which has been obtained many times previously, and in particular, using Parseval's Theorem by Titchmarsh [14] (and originally by Euler using results from the theory of equations). See also Kalman [6] who gives a variety of ways of summing $\zeta(2) = \sum_{j=1}^{\infty} j^{-2}$.

The Riemann zeta function $\zeta(s)$ arises in may areas, in particular, in number theory.

Taking $n = 2$ in (4.20) and using (2.24) , (4.17) gives

$$\sum_{j=1}^{\infty} \frac{\alpha^2 - j^2}{[\alpha^2 + j^2]^2} = \frac{\pi^2}{2}\text{cosech}^2\pi\alpha - \frac{1}{2\alpha^2} \tag{4.23}$$

and on using (4.22)

$$\sum_{j=1}^{\infty} \frac{j^2}{[\alpha^2 + j^2]^2} = \left(\frac{\pi}{2}\right)^2\left[\frac{\coth\pi\alpha}{\pi\alpha} - \text{cosech}^2\pi\alpha\right]. \tag{4.24}$$

Substitution of (4.24) into (4.23) produces

$$\sum_{j=1}^{\infty} \frac{1}{[\alpha^2 + j^2]^2} = \left(\frac{\pi}{2\alpha}\right)^2\left[\frac{\coth\pi\alpha}{\pi\alpha} + \text{cosech}^2\pi\alpha\right] - \frac{1}{2\alpha^4}. \tag{4.25}$$

The result in (4.25) could be obtained directly from (4.22) by differentiation with respect to α. Also taking the limit as $\alpha \to 0$ in (4.25) gives

$$\sum_{j=1}^{\infty} j^{-4} = \frac{\pi^4}{90}.$$

A similar procedure with $n = 3$ would give from (4.20), (2.22) , and (4.17)

$$\sum_{j=1}^{\infty} \frac{\alpha^2 - 3j^2}{[\alpha^2 + j^2]^3} = \frac{\pi^3}{2\alpha}\text{cosech}^2\pi\alpha\coth\pi\alpha - \frac{1}{2\alpha^4}$$

leading to

$$\sum_{j=1}^{\infty} \frac{j^2}{[\alpha^2 + j^2]^3} = \left(\frac{\pi}{4\alpha}\right)^2\left\{\frac{\coth\pi\alpha}{\pi\alpha} + (1 - 2\pi\alpha\coth\pi\alpha)\text{cosech}^2\pi\alpha\right\}$$

and

$$\sum_{j=1}^{\infty} \frac{1}{[\alpha^2 + j^2]^3} = \frac{\pi^2}{(2\alpha)^4}\left\{3\frac{\coth\pi\alpha}{\pi\alpha} + (3 + 2\pi\alpha\coth\pi\alpha)\text{cosech}^2\pi\alpha - \frac{1}{2\alpha^6}\right\}.$$

Further such series may be obtained from (4.20) , (2.22) and (4.17) (together with a lot of perseverance!).

The above series could have been obtained by the alternate procedure of differentiating (4.22) with respect to α . This can be done systematically using a suitable computer algebra package. The recurrence relation represented by (2.22) gives the series in a straight forward way as demonstrated.

5. Conclusion.

Series of roots of transcendental equations have been summed using residue theory. Previous results have been shown to be special cases of the current development. The insights and analysis that led to the recurrence relation (2.22) for summing the series would not, it is believed, have been possible if the approach of Section 3 had been followed. Examples have been provided to elaborate and elucidate the techniques developed.

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