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ON TWO REVISED NODES OF S.N. BERNSTEIN INTERPOLATION PROCESS

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In this work we study the well-known problem put forward by S. N. Bernstein in 1930. We construct an operator $P_{n,a}(f; x)$ that converges uniformly to $f \in C^j[-1, 1](0 \leq j \leq a)$. For any $f \in C^j[-1, 1]$ and for any $x \in [-1, 1]$, we prove the best convergence order of $P_{n,a}(f; x)$ cannot exceed $\frac{1}{n^{a+1}}$, where a is an odd integer.

1. Introduction.

Let $f \in C[-1, 1]$ and $U_n(x) = \frac{\sin (n+1)\theta}{\sin \theta}$ $(x = \cos\theta, 0 \leq \theta \leq \pi)$ (where U_n is the Chebyshev polynomial of second kind);

The roots of U_n are

$$x_k = \cos \theta_k = \cos \frac{k\pi}{n+1}, k = 1, 2, \cdots, n. \tag{1.1}$$

We denote by

$$q_k(x) = (-1)^{k+1} \frac{(1 - x_k^2)U_n(x)}{(n+1)(x - x_k)}, \quad k = 1, 2, \cdots, n \tag{1.2}$$

the fundamental polynomials of Langrange interpolation based on the nodes $\{x_k\}_1^n$.

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The famous Langrange interpolation polynomial $L_n(f; x)$ is as follows.

$$L_n(f; x) = \sum_{k=1}^{n} f(x_k) q_k(x). \tag{1.3}$$

We all know Lagrange interpolation polynomials do not converge uniformly to all $f \in C[-1, 1]$. In order to improve its uniform convergence, S. N. Bernstein [1] put forward the following question: for a given $\lambda(1 < \lambda < 2)$ and all $f \in C[-1, 1]$ and $N := \{1, 2, \cdots\}$, is it possible to construct an interpolation polynomial with degree M with $M < \lambda N$ such that it obtains the same as $f(x)$ at given N points and it converges to $f \in C[-1, 1]$ uniformly when $N \to \infty$? We named this problem as "S. N. Bernstein problem".

In reference [1], Bernstein constructed an interpolation polynomial $Q_n(f; x)$ and answered the above question. $Q_n(f; x)$ is constructed as follows:

Given the nodes $x_1, x_2, \cdots, x_n$, these nodes are still Chebyshev knots, let $y = f(x), y_k = f(x_k), k = 1, 2, \cdots, n$. for given even $2l (l \in N)$, the nodes $x_1 > x_2 > \cdots > x_n$ are divided into $s (s \in N)$ groups according to $2l, n = 2ls + d, 0 \le d \le 2l$. The constructed interpolation polynomial $Q_n(f; x) = f(x)$ at $2l - 1$ nodes of every group and $Q_n(f; x) = A_{2lt}, (t = 1, 2, \cdots, s)$ at the $2lt - th$ node, where

$$A_{2lt} = y_{2lt} + \sum_{p=1}^{l} (y_{2l(t-1)+2p-1} - y_{2l(t-1)+2p}). \tag{1.4}$$

$Q_n(f; x)$ is given by

$$Q_n(f; x) = \sum_{k=1}^{n} A_k q_k(x), \tag{1.5}$$

here $A_k = f(x_k) = y_k$ when $k \neq 2lt$.

In reference [1], he proved that $Q_n(f; x)$ is a polynomial of degree $M = n - 1$ and it fits to the conditions given by S. N. Bernstein in 1930 and it converges uniformly to $f \in C[-1, 1]$.

In 1993, Laiyi Zhu gave an estimation of the rate of convergence of $Q_n(f; x)$ in reference [2]. The results are as follows:

If $f \in C[-1, 1]$, then for all $x \in [-1, 1]$,

$$|f(x) - Q_n(f; x)| \le c \Bigg\{ \omega(f, \frac{|x|}{2} |\theta - \theta_{k_0}|^2 + (1 - x^2)^{1/2} |\theta - \theta_{k_0}|) \tag{1.6}$$

$$+ |T_n(x)|\omega(f, (1-x^2)^{1/2}/n + 1/n^2$$

$$+ \left|\frac{T_n(x)}{n}\right| \int_{1/n}^{1} \frac{\omega(f, |x|t^2 + (1-x^2)^{1/2}t)}{t^2} dt \Bigg\},$$

here $x_{k_0} = \cos\theta_{k_0}$ is the nearest node to $x = \cos\theta, \theta_k = k\pi/(n+1)$. c is a constant which dependes only on $l, \omega(f, \delta)$ is the modulus of continuity of $f \in C[-1, 1], T_n = \cos n\theta = \cos n \arccos x, (x \in [-1, 1])$ is the Chebyshev polynomial of first kind.

Observe that the statement shows that the convergence order of $Q_n(f; x)$ does not reach the best approximation order and for any continuous function the highest convergence order cannot exceed $1/n$.

In reference [5], Jiaxing He and Jichang Ye constructed a new polynomial $H_n(f; x)$ of interpolation that fits to the conditions given by S. N. Bernstein in 1930 and its convergence order can reach the best for any $f \in C^j[-1, 1] (0 \leq j \leq 3)$. The main results is as follows.

If $f \in C^1[-1, 1]$, then for all $x \in [-1, 1]$,

$$|f(x) - H_n(f; x)| = O\left(\frac{1}{n}\omega(f, \frac{1}{n}) + \frac{1}{n^2}\|f'\|\right), \tag{1.7}$$

here "expression"$=$ O ("something") means $\lim\limits_{n\to\infty} \frac{\text{"expression"}}{\text{"something"}} = C \neq 0$.

Naturally, what will be interesting is how to construct an interpolation polynomial satisfying the S. N. Bernstein's requirement mentioned as above so that the convergence order is the best for function with arbitrary continuous derivative.

In this work we construct a new interpolation $P_{n,a}(f; x)$ (where a is an odd integer) and answer S. N. Bernstein problem satisfactorily in a new method.

Let $f \in C^j[-1, 1], (0 \leq j \leq a), P_{n,a}(f; x)$ is constructed as follows:

For given even $2l (2 \leq l \in N)$, the nodes $\{x_k\}_1^n$ are divided into $s (s \in N)$ group according to $2l, n = 2ls + d (0 \leq d < 2l)$. The constructed polynomial satisfies $P_{n,a}(f; x) = f(x)$ at $2l - 2$ nodes of every group, $P_{n,a}(f; x) = A_{2lt-1} (t = 1, 2, \cdots, s)$ at the $(2lt - 1) - th$ nodes and $P_{n,a}(f; x) = A_{2lt}$ at the $2lt - th (t = 1, 2, \cdots s)$ nodes, here

$$A_{2lt-1} = f(x_{2lt-1}) + \sum_{p=1}^{l} \Delta_h^{a+1} f(x_{2l(t-1)+2p-1}) = f(x_{2lt-1}) + A^*_{2lt-1}; \tag{1.8}$$

$$A_{2lt} = f(x_{2lt}) + \sum_{p=1}^{l} \Delta_h^{a+1} f(x_{2l(t-1)+2p}) = f(x_{2lt}) + A^*_{2lt}, \tag{1.9}$$

where

$$\Delta_h^{a+1} f(x_k) = \frac{1}{2^{a+1}} \sum_{i=0}^{a+1} (-1)^{\beta+l} \binom{a+1}{i} f(x_{k+l+1-\beta}), \tag{1.10}$$

$\beta = \dfrac{a+3}{2}$, $\dbinom{a+1}{i} = \dfrac{(a+1)!}{i!(a+1-i)!}$, $x_k = \cos\theta_k = \cos\dfrac{k\pi}{n+1}$, $h = \dfrac{\pi}{n+1}$.

Let

(i) $f(x_{-j}) = f(x_j)$ when $k + 1 + i - \beta = -j (j = 1, 2, \cdots, \beta - 1)$ according to $\cos(\theta_j) = \cos(\theta_{-j})$;

(ii) $f(x_{n+j+1}) = f(x_{n-j+1})$ when $k + 1 + i - \beta = n + j (j = 1, 2, \cdots, \beta - 1)$ according to $\cos(\theta_{n+j+1}) = \cos(\theta_{n-j+1})$;

Therefore

$$P_{n,a}(f; x) = \sum_{k=1}^{n} A_k q_k(x), \tag{1.11}$$

where A_k is given by (1.8) when $k = 2lt - 1$, A_k is given by (1.9) when $k = 2lt (t = 1, 2, \cdots s)$, the remaining A_k are equal to $f(x_k)$, $q_k(x)$ are the fundamental polynomials of Langrange interpolation.

From (1.11), we know that $P_{n,a}(f; x)$ is a polynomial of degree $n - 1$. It is equal to $f(x)$ at $N \geq s(2l - 2)$ zero nodes and

(1.12)

$$\frac{M}{N} \leq \frac{n-1}{s(2l-2)} = \frac{2ls + d - 1}{s(2l-2)} = 1 + \frac{2s + d - 1}{s(2l-2)} = \lambda, \quad (1 < \lambda < 2).$$

For operator $P_{n,a}(f; x)$, we get

Theorem 1. *Let $f \in C[-1, 1]$, then for any $x \in [-1, 1]$.*

$$\lim_{n\to\infty} P_{n,a}(f; x) = f(x). \tag{1.13}$$

Theorem 2. *Let $f \in C^j[-1, 1]$, $(0 \leq j \leq a)$, then*

$$|P_{n,a}(f; x) - f(x)| = O\Big(\frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n}) + \frac{1}{n^{j+1}}\Big), \tag{1.14}$$

here "expression" $= O$ ("something") means

$$\lim_{n\to\infty} \frac{|P_{n,a}(f;x) - f(x)|}{\frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n}) + \frac{1}{n^{j+1}}} = C \neq 0,$$

$\omega(f^{(j)}, \delta)$ is the modulus of continuity of $f^{(j)}(x)$ on $[-1, 1]$.

2. Lemmas.

Lemma 1. (Cf. [5]) *$q_k(x)$ have the following properties:*

$$\sum_{t=1}^{s}\sum_{p=1}^{l-1} \left|q_m(x) - q_j(x)\right| = O(1), \tag{2.1}$$

$$\sum_{t=1}^{s}\sum_{p=1}^{l-1} \left|q_{m+1}(x) - q_{j+1}(x)\right| = O(1), \tag{2.2}$$

$$\sum_{k=\beta}^{n-a-2+p} \left(\frac{1}{2^{a+1}}\sum_{i=0}^{a+1}\binom{a+1}{i}\left|q_k(x) + (-1)^{\beta+i}q_{k+l+1-\beta}(x)\right|\right) = O(1), \tag{2.3}$$

where $m = 2lt - 1$, $j = 2l(t-1) + 2p - 1$.

Proof. Using the same method in [5] we can prove this Lemma.

Lemma 2. [5] *For $f(x) \in C^j[-1,1]$, $0 \leq j \leq a$, let $u_{n-1}(x)$ polynomials with degree $n-1$. We have*

$$\left|u_{n-1}(x) - f(x)\right| = O\left(\frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n})\right). \tag{2.4}$$

3. Proof of Theorem 2.

Proof of theorem 2. For $n = 2ls + d, 0 \leq d < 2l, h = \frac{\pi}{n+1}$, we have

$$
\begin{aligned}
(3.1) \qquad P_{n,a}(f;x) - f(x) &= \sum_{k=1}^{n} A_k q_k(x) - f(x) \\
&= \left(\sum_{k=1}^{n} f(x_k) q_k(x) - f(x) \right) + \sum_{t=1}^{s} \left(A_{2lt-1}^{*} q_{2lt-1}(x) + A_{2lt}^{*} q_{2lt}(x) \right) \\
&= \left\{ \sum_{k=1}^{n} \left(f(x_k) + \Delta_h^{a+1} f(x_k) \right) q_k(x) - f(x) \right\} \\
&\quad + \sum_{t=1}^{s} \sum_{p=1}^{l-1} \Delta_h^{a+1} f(x_j) \left(q_m(x) - q_j(x) \right) \\
&\quad + \sum_{t=1}^{s} \sum_{p=1}^{l-1} \Delta_h^{a+1} f(x_{j+1}) \left(q_{m+1}(x) - q_{j+1}(x) \right) \\
&\quad - \sum_{p=2ls+1}^{n} \Delta_h^{a+1} f(x_p) q_p(x) \\
&= \sum_{v=1}^{4} B_v.
\end{aligned}
$$

Using Lemma 2, by the property of Lagrange interpolation polynomial, we have

$$
(3.2) \qquad \sum_{k=1}^{n} \left(u_{n-1}(x_k) + \Delta_h^{a+1} u_{n-1}(x_k) \right) q_k(x) = u_{n-1}(x) + \Delta_h^{a+1} u_{n-1}(x),
$$

and

$$
\begin{aligned}
(3.3) \qquad B_1 &= \sum_{k=1}^{n} \left(f(x_k) + \Delta_h^{a+1} f(x_k) \right) \cdot q_k(x) - f(x) \\
&= \sum_{k=1}^{n} \left(f(x_k) - u_{n-1}(x_k) + \Delta_h^{a+1} f(x_k) - \Delta_h^{a+1} u_{n-1}(x_k) \right) \cdot q_k(x)
\end{aligned}
$$

$$+\sum_{k=1}^{n}\big(u_{n-1}(x_k)+\Delta_h^{a+1}u_{n-1}(x_k)\big)\cdot q_k(x)-f(x)$$

$$=\sum_{k=1}^{n}\big(f(x_k)-u_{n-1}(x_k)\big)\left(\frac{1}{2^{a+1}}\sum_{i=0}^{a+1}\binom{a+1}{i}\big(q_k(x)+(-1)^{\beta+1}q_{k+i+1-p}(x)\big)\right)$$

$$+\left\{\big(u_{n-1}(x)-f(x)\big)+\big(\Delta_h^{a+1}u_{n-1}(x)-\Delta_h^{a+1}f(x)\big)\right\}+\Delta_h^{a+1}f(x)$$

$$=D_1+D_2+D_2$$

For D_1, since

$$D_1=\sum_{k=1}^{\beta-1}\big(f(x_k)-u_{n-1}(x_k)+\Delta_h^{a+1}f(x_k)-\Delta_h^{a+1}u_{n-1}(x_k)\big)q_k(x)$$

$$+\sum_{k=\beta}^{n-a-2+\beta}\big(f(x_k)-u_{n-1}(x_k)\big)\left(\frac{1}{2^{a+1}}\sum_{i=0}^{a+1}\binom{a+1}{i}\big(q_k(x)+(-1)^{\beta+i}\,q_{k+i+1-\beta}(x)\big)\right)$$

$$+\sum_{k=n-a-1+\beta}^{n}\big(f(x_k)-U_{n-1}(x_k)+\Delta_h^{a+1}u_{n-1}(x_k)\big)q_k(x)$$

$$=g_1+g_2+g_3$$

By (2.3) in Lemma 1 and Lemma 2, we have

$$g_2=O\left(\frac{1}{n^j}\omega\,(f^{(j)},\frac{1}{n})\right),$$

By $q_k(x)=O(1)$[4], $(k=1,2,\cdots,n)$ and Lemma 2 ; it follows that

$$g_1=O\left(\frac{1}{n^j}\omega\,(f^{(j)},\frac{1}{n})\right),\quad g_3=O\left(\frac{1}{n^j}\omega\,(f^{(j)},\frac{1}{n})\right).$$

Thus

$$D_1=O\left(\frac{1}{n^j}\omega\,(f^{(j)},\frac{1}{n})\right).$$

In the same way, we see that

$$D_2=O\left(\frac{1}{n^j}\omega\,(f^{(j)},\frac{1}{n})\right).$$

For D_3, let $W(\theta) = f(\cos\theta)$, we have

$$W^{(j)}(\theta) = \frac{d^j W(\theta)}{d\theta^j} = \sum_{c=1}^{j} \sigma_c f^{(c)} \cdot \cos^{c_1}\theta \triangleq \sum_{c=1}^{j} \sigma_c e_c(\theta), \tag{3.4}$$

where $c_1 = c_1(j, c), c_2 = c_2(j, c)$ with $c = c_1 + c_2, \sigma_c$ are constants, and

$$\begin{aligned}
|D_3| &= \frac{1}{2^{a+1}} \left| (\frac{\pi}{n+1})^j \sum_{i=0}^{a+1-j} (-1)^i \binom{a+1-j}{i} W^{(j)}(\theta_{\zeta i}) \right| \\
&= \frac{1}{2^{a+1}} (\frac{\pi}{n+1})^j \left| \sum_{i=0}^{a-j} (-1)^i \binom{a-j}{i} (W^{(j)}(\theta_{\xi i}) - W^{(j)}(\theta_{\xi i+1})) \right| \\
&\leq \frac{\sigma}{2^{a+1}} (\frac{\pi}{n+1})^j \sum_{i=0}^{a-j} \binom{a-i}{i} \left(\sum_{c=1}^{j} \left| e_c(\theta_{\xi i}) - e_c(\theta_{\xi i+1}) \right| \right),
\end{aligned} \tag{3.5}$$

where $\sigma = \max_{1 \leq e \leq j} |\sigma_l|, \theta + (i+1-\beta)\frac{\pi}{n+1} \leq \theta_{\xi i} \leq \theta + (i+j+1-\beta)\frac{\pi}{n+1}$.
Since

$$\left| e_c(\theta_{\xi i}) - e_c(\theta_{\xi i+1}) \right| = O\left(\frac{1}{n}\right), \quad c = 1, 2, \cdots, j-1,$$

$$\left| e_c(\theta_{\xi i}) - e_c(\theta_{\xi i+1}) \right| = O\left(\frac{1}{n} + \omega(f^{(j)}, \frac{1}{n})\right), \quad c = j,$$

we have

$$D_3 = O\left(\frac{1}{n^{j+1}} + \frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n})\right).$$

Thus

$$B_1 = O\left(\frac{1}{n^{j+1}} + \frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n})\right).$$

Moreover, in the same way, using (2.1) and (2.2), we have

$$B_\nu = O\left(\frac{1}{n^{j+1}} + \frac{1}{n^j}\omega(f^{(j)}, \frac{1}{n})\right), (\nu = 2, 3, 4).$$

Theorem 2 is proved.

By Theorem 2, Theorem 1 is correct.

4. Other result.

Theorem 3. *for an arbitrary continuous function on* $[-1, 1]$*, the highest convergence order of operator* $P_{n,a}(f; x)$ *is* $\frac{1}{n^{a+1}}$.

Proof. In fact, according to the properties of Langrangè, we have

$$P_{n,a}(f; , x_{2lt-1}) = A^*_{2lt-1}, \quad t = 1, 2, \cdots, s$$

Let $f_0(x) = x, \; x = \cos\theta \;, l = 1$, we have

$$\left|P_{n,a}(f_0; x_{2t-1}) - f_0(x_{2t-1})\right| = \left|A^*_{2t-1}\right|$$

$$= \left|\Delta_h^{a+1}\cos\theta_{2t-1}\right|$$

$$= \left(\sin\frac{\pi}{2n+1}\right)^{a+1} \cdot \left|\sin\left(\theta_{2t-1} + \frac{\pi}{2n+1}\right)\right|$$

When $n \to \infty$, we know Theorem 4 is correct.

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