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STABILITY OF DEFORMED OSCULATING HYPERRULED SURFACES

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1. Introduction.

The mean curvature vector plays an important role in the investigation of the crystallographie point of view of the minimal surfaces. Among them mainly those surfaces which are free of self-intersections seem to be of pratical significance, e.g. as biological membranes or amphiphilic films. These types of surfaces are related to crystal structures [6].

Many interestings problems in differential geometry deal with surfaces characterized by extremal properties. These problems may be formulated in terms of some variantional problems. The problem of the calculus of variations will be reduced to the solutions of a certain system of differential equations which are called the Euler equations of the calculus of variations.

Another important use of the mean curvatures is in the study of the graph of the equilibrium capillary surfaces. The classical problem of liquid in a capillary tube concerns with finding the minimum of a certain energy function, wich leads to surface of a prescribed mean curvature [16, 17]. The variational problem of the integrals of mean curvature describes equilibrium configurations in capillarity theory is introduced and studied in [15].

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One of the most interesting and profound aspects of classical differential geometry is its interplay with the calculus of variations. This phenomenon has its roots in the very origins of the subject, such as, for instance, in the theory of minimal surfaces. More recently, the variational principles which give rise to the field equations of the general theory of relativity have suggested the systematic investigation of a seemingly new type of variational problem. In the case of the earlier applications one is, at least implicitly, concerned with a multiple integral in the calculus of variations whose Lagrangian depends on the projection factors of some subspace, these projection factors being obtained by differentiation of the functions which appears in the parametric equations by means of which the subspace is represented; the corresponding Euler-Lagrange equations are supposed to yield appropriate functions of this kind. The variational problems of the second type referred to above are defined by Lagrangians whose arguments are the components of the metric tensor of a manifold, together with the first and second derivatives of these components, and the resulting Euler-Lagrange equations give rise to the required metric.

As is well known, the most useful method of studying the properties of a curve in a Euclidean space, from the standpoint of differential geometry, is making use of the Frenet formulas, in which the curvatures are the essential quantities for the curve. So, the motivation of the present work is to develop the variational problem in our work [1] by using auxiliary formulas of Frenet [2, 7, 8, 13]. Hence, the invariants of the hyperruled surfaces generated by the osculating space in E^{n+1} (osculating hyperruled surfaces) are interpreted in terms of the curvatures for the base curve. Furthermore, the variation of these invariants are calculated. The variation of the curvatures for the base curve and the Frenet-frame are obtained. The necessary and sufficient condition for the stability of the osculating hyperruled surfaces in terms of the curvatures for the base curve are derived. Finally, the solution of the differential equation which is produced from stability condition, for example in E^3 and E^4, is obtained.

Here, and in the sequel, we assume that the indices $\{\nu, \mu\}$, $\{\gamma, \lambda\}$ and $\{i, j, k\}$ run over the ranges $\{2, \dots, n-1\}$, $\{1, \dots n-1\}$ and $\{0, \dots n-1\}$ respectively unless otherwise stated.

Let M be an oriented n-dimensional hyperrruled surface in an Euclidean $(n+1)$-space E^{n+1}, with a base curve $r :]a, b[\to E^{n+1}, r = r(u^0)$ and u^0 is the arc length. The Frenet-frame is denoted by $\{e_i(u^0)\}$, $1 \le i \le n+1$, where $e_1(u^0)$ is the unit tangent vector. The $(n-1)$-dimensional linear osculating space of $r = r(u^0)$ is denoted by $OS_p(r(u^0))$ and generated by $\{e_\gamma(u^0)\}$. The n-dimensional hyperruled surface in E^{n+1} generated by $OS_p(r(u^0))$ can be represented locally by:

$$(1)\qquad X(u^i) = r(u^0) + u^1 e_1(u^0) + \sum_\nu u^\nu e_\nu\left(u^0\right), \quad u^0 \in]a, b[, u^\gamma \in R,$$

which is called an osculating n-hyperruled surface in E^{n+1}. The tangent space at a point p of the hyperruled surface is spanned by the generating $(n-1)$-dimensional linear osculating space $OS_p(r(u^0))$ through p and $X_0 := \frac{\partial X}{\partial u^0}$. The unit normal vector field on M at a point p is $e_{n+1}(u^0)$. Thus, the representation (1) is a regular parametrization for a regular base curve $r = r(u^0)$.

Throughout the rest of this section we would like to mention the following definition which are very important in the sequel [2, 7, 8, 13]:

Definition 1.1. If $r(u^0$ is a curve E^{n+1}, parametrized by arc length u^0 and unit tangent vector $e_1(u^0)$, we say that r is a Frenet curve of osculating order $n+1$ when there exist orthonormal vector fields $\{e_i(u^0)\}$ along r such that:

$$(2)\qquad \begin{cases} r(u^0) = e_1(u^0), \nabla_{e_1} e_1 = k_1 e_2, \nabla_{e_1} e_2 = -k_1 e_1 + k_2 e_3, \dots \\ \nabla_{e_1} e_n = -k_{n-1} e_{n-1} + k_n e_{n+1}, \nabla_{e_1} e_{n+1} = -k_n e_n, \end{cases}$$

where $k_1, \dots k_{n-1}$ are positive C^∞ functions of u^0 and $k_n \neq 0$ is C^∞ functions of u^0 and ∇ the Riemannian connection. The Eqs. (2) are the Frenet formulas in E^{n+1}.

2. The fundamental quantities.

In the following, the fundamental quantities g_{ij}, g^{ij}, h_{ij} and h_i^j of M are derived in terms of the curvatures k_i, $1 \leq i \leq n$, for the base curve $r(u^0)$.

Where

$$g_{ij} = \langle X_i, X_j \rangle, X_i = \frac{\partial X}{\partial u^i}, h_{ij} = \langle e_{n+1}, X_{ij} \rangle, h_i^j = \sum_k g^{jk} h_{ik},$$

and (g^{ij}) denote the inverse matrix of (g_{ij}).

From (1) and (2) we have:

$$(3)\qquad \begin{cases} \nabla_{e_1} X = X_0 = \frac{\partial X}{\partial u^0} = e_1 + u^1 k_1 e_2 + \sum_\nu u^\nu [-k_{\nu-1} e_{\nu-1} + k_\nu e_{\nu+1}], \\ X_\gamma = e_\gamma(u^0), \end{cases}$$

$$(4)\qquad \begin{cases} \nabla_{e_1} X_0 = -(u^1 k_1^2)e_1 + (k_1 + u^1 k_1)e_2 + (u^1 k_1 k_2)e_3 \\ \qquad + \sum_\nu u^\nu [(k_{\nu-1}k_{\nu-2})e_{\nu-2} - (k_{\nu-1})e_{\nu-1} \\ \qquad - \left(k_{\nu-1}^2 + k_\nu^2\right)e_\nu + (k_\nu)e_{\nu+1} + (k_\nu k_{\nu+1})e_{\nu+2} \\ \nabla_{e_1} X_\nu = -k_{\nu-1}e_{\nu-1} + k_\nu e_{\nu+1}, \\ \nabla_{e_1} X_1 = k_1 e_2, \quad X_{\gamma\gamma} = X_{\gamma\lambda} = 0, \forall \gamma, \lambda, \end{cases}$$

where $\nabla_{e_1} X_i = \dfrac{\partial X_i}{\partial u^0}, \dots = \dfrac{\partial}{\partial u^0}$.

From (3) we have:

$$(5)\qquad g_{00} = \langle \nabla_{e_1} X, \nabla_{e_1} X \rangle = 1 - 2u^2 k_1 + u^1 k_1 \left(u^1 k_1 - 2u^3 k_2\right) + \sum_\nu (u^\nu)^2 \left(k_{\nu-1}^2 + k_\nu^2\right),$$

$$(6)\qquad g_{01} = \langle \nabla_{e_1} X, X_1 \rangle = 1 - u^2 k_1,$$

$$(7)\qquad g_{0\nu} = \langle \nabla_{e_1} X, X_\nu \rangle = u^{\nu-1} k_{\nu-1} - u^{\nu+1} k_\nu,$$

$$(8)\qquad g_{\gamma\gamma} = 1, \quad g_{\gamma\nu} = g_{\nu\mu} = 0, \quad \forall \gamma, \nu, \mu.$$

Thus, using induction we can see that:

$$(9)\qquad g = \det(g_{ij}) = g_{00} - \sum_\gamma g_{0\gamma}^2.$$

From (5), (6) and (7) we have:

$$(10)\qquad g = \left[\left(u^1\right)^2 - \left(u^2\right)^2\right] k_1^2 - 2u^1 u^3 k_1 k_2 + \sum_\nu [(u^\nu)^2 \left(k_{\nu-1}^2 + k_\nu^2\right) - \left(u^{\nu-1} k_{\nu-1} - u^{\nu+1} k_\nu\right)^2].$$

After some calculations we can see that the inverse $(n \times n)$ matrix (g^{ij}) of $(n \times n)$ matrix (g_{ij}) is given by:

$$(11)\qquad g^{00} = \frac{1}{g}, g^{0\gamma} = g^{\gamma 0} = \frac{-g_{0\gamma}}{g}, g^{\nu\mu} = \frac{g_{0\nu}\, g_{0\mu}}{g}, g^{\gamma\gamma} = 1 - g^{0\gamma} g_{0\gamma},$$

where g_{ij} and g are given by (5), (6), (7), (8) and (10) respectively.

From (4), we have:

$$h_{00} = u^{n-1}k_n k_{n-1}, h_{i\gamma} = h_{\gamma\lambda} = 0, \forall i, \gamma, \lambda, \det(h_{ij}) = 0, \ \text{if } n \geq 2, \tag{12}$$

Thus, the $1\underline{st}$ and $2\underline{nd}$ fundamental forms of M are given by:

$$I = g_{00}\left(du^0\right)^2 + 2\sum_{\gamma} g_{0\gamma}du^0du^\gamma + \sum_{\nu} g_{\nu\nu}(du^\nu)^2, \tag{13}$$

and

$$II = h_{00}\left(du^0\right)^2, \tag{14}$$

respectively, where g_{ij} and h_{00} are given by (5), (6), (7), (8) and (12).

From Weingarten equations [1, 11], using (6), (7), (11) and (12) we have

$$\begin{cases} h_0^0 = g^{00}h_{00} = g^{00}(u^{n-1}k_nk_{n-1}), \\ h_0^1 = g^{01}h_{00} = -g^{00}(1 - u^2k_1)(u^{n-1}k_nk_{n-1}), \\ h_0^\nu = g^{0\nu}h_{00} = -g^{00}(u^{\nu-1} - u^{\nu+1}k_\nu)(u^{n-1}k_nk_{n-1}), \\ h_\gamma^i = h_\gamma^\gamma = h_\gamma^\lambda = 0, \forall i, \gamma, \lambda. \end{cases} \tag{15}$$

From (11) and (12) we have the following:

Corollary 2.1. *the mean curvature function H of M is given by:*

$$H = \frac{1}{n}g^{00}h_{00} = \frac{1}{n}g^{00}\left(u^{n-1}k_nk_{n-1}\right), \quad n \geq 2. \tag{16}$$

Corollary 2.2. *The Gaussian curvature G of M is given by:*

$$G = 0, \quad n \geq 2. \tag{17}$$

Using (12) and (16), we can see that:

Corollary 2.3. *The norm of the $2\underline{nd}$ fundamental form of M is given by:*

$$S^2 = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)^2 = \left(g^{00}h_{00}\right)^2 = n^2H^2 \tag{18}$$

$$= \left(g^{00}\right)^2\left(u^{n-1}k_nk_{n-1}\right)^2. \tag{19}$$

It is important to remark that the foregoing results are considered as a generalization of the well-known results for the 2-dimensional ruled surfaces in E^3 which confirm that the envelope of the osculating plane of a space curve is the tangential developable [9], [10], [14], [21].

3. The variation of the fundamental quantities.

In this section, the variation of the volume element,mean curvature and the norm of the second fundamental form are obtained in terms of the curvatures k_i for the base curve $r(u^0)$. For this purpose we give the following definition [1, 3, 12]:

Definition 3.1. Let M be a compact hyperrruled surface with piecewise smooth boundary ∂M, let $\varphi \in \wedge^o(M)$ be a continuous function vanishing identically on the boundary ∂M and satisfies the condition $\int_M \varphi(u^i) W du^0 \wedge du^1 \wedge \ldots du^{n-1} = 0$. We consider a smooth map $F : J \times M \to E^{n+1}$ such that for $t \in J = [0,1]$, the map $F_t : M \to E^{n+1}$, where $f_t(p) = F(t, p)$ for $p \in M$, is an immersion such that $F_0 = M$ with local representation (1) and $F_t = F_0$ on the boundary ∂M. The image $F(p)$ is representented by the parametrization:

$$\bar{X}(u^i, t) = X(u^i) + t\varphi(u^i) e_{n+1}(u^0). \tag{20}$$

This representation defines a normal variation of M in E^{n+1} associated with φ and the family of hyperruled surfaces representented by $\bar{X} = \bar{X}(u^i, t)$ is called a deformable hyperruled surfaces resulting from $X = X(u^i)$ by the normal variation such that the variation vector field $\delta X = \varphi(u^i) e_{n+1}(u^0)$ and the operator δ is defined as $(\partial/\partial t)|_{t=0}$.

From (13), the $1\underline{st}$ fundamental form of the variation $\bar{X}$ (deformed family of surfaces) is $\bar{I} = \sum_{ij} \bar{g}_{ij} du^i du^j$, where $\bar{g}_{ij} = \left\langle \bar{X}_i, \bar{X}_j \right\rangle$, using (2), (3) and (20) we have:

$$\begin{aligned}
\bar{g}_{00} &= g_{00} - 2t\varphi u^{n-1} k_n k_{n-1} + t^2 \Big[(\nabla_{e1}\varphi)^2 + k_n^2 \varphi^2 \Big], \\
\bar{g}_{0\gamma} &= g_{0\gamma} + t^2 (\varphi_\gamma \nabla_{e1}\varphi), \\
\bar{g}_{\gamma\gamma} &= 1 + t^2 \varphi_\gamma^2, \\
\bar{g}_{\gamma\lambda} &= t^2 \varphi_\gamma \varphi_\lambda, \forall \gamma \neq \lambda.
\end{aligned}$$

Thus, it is easy to see that the $1\underline{st}$ and $\underline{nd}$ variation of g_{ij} are:

$$\delta g_{00} = -2\varphi u^{n-1} k_n k_{n-1}, \tag{21}$$

$$\delta g_{i\gamma} = 0, \forall i, \lambda, \tag{22}$$

and

$$\delta^2 g_{00} = 2\big[(\nabla_{e1}\varphi)^2 + k_n^2 \varphi^2 \big] \succ 0, \tag{23}$$

$$\delta^2 g_{i\gamma} = 2\varphi_i \varphi_\gamma, \forall i, \gamma. \tag{24}$$

Thus, we have:

Lemma 3.1. *The* $1\underline{st}$ *and* $2\underline{nd}$ *varition of the metric tensor* g_{ij} *are given by* (21), (22), (23) *and* (24).

From (9), (21) and (22), we have:

Corollary 3.1.

$$\delta g = \delta g_{00} = -2\varphi u^{n-1} k_n k_{n-1}. \tag{25}$$

Using (6), (7), (8), (11), (22) and (25) we have the following:

$$\begin{cases} \delta g^{00} = 2\varphi (g^{00})^2 (u^{n-1} k_n k_{n-1}), \\ \delta g^{01} = -2\varphi (g^{00})^2 (1 - u^2 k_1)(u^{n-1} k_n k_{n-1}), \\ \delta g^{11} = 2\varphi (g^{00})^2 (1 - u^2 k_1)^2 (u^{n-1} k_n k_{n-1}), \\ \delta g^{0\nu} = -2\varphi (g^{00})^2 (u^{\nu-1} k_{\nu-1} - u^{\nu+1} k_\nu)(u^{n-1} k_n k_{n-1}), \\ \delta g^{\nu\mu} = 2\varphi (g^{00})^2 (u^{\nu-1} k_{\nu-1} - u^{\nu+1} k_\nu)(u^{\mu-1} k_n k_{\mu-1} - u^{\mu+1} k_\mu) \\ \qquad (u^{n-1} k_n k_{n-1}) \\ \delta g^{\nu\nu} = 2\varphi (g^{00})^2 (u^{\nu-1} k_{\nu-1} - u^{\nu+1} k_\nu)^2 (u^{n-1} k_n k_{n-1}), \end{cases} \tag{26}$$

From the foregoing results, we have:

Lemma 3.2. *The* $1\underline{st}$ *variations of the metric tensors* g^{ij} *are given by* (26).

The volume element $\bar{d}A$ of the variation $\bar{X} = \bar{X}(u^i, t)$ is:

$$\bar{d}A = \bar{W}\frac{dA}{W}, \; W = \sqrt{g}, \; \bar{W} = \sqrt{\bar{g}} \text{ and } \bar{g} = Det\left(\bar{g}_{ij}\right).$$

Thus and using (25) we have:

Corollary 3.2. *The variation of the volume element* dA *is given by:*

$$\delta(dA) = -\left(\varphi g^{00} u^{n-1} k_n k_{n-1}\right) dA \tag{27}$$

From [1], the variations of the $2\underline{nd}$ fundamental quantities h_{ij} are given by:

$$\delta h_{ij} = \nabla_i \nabla_j \varphi - \varphi \sum_k h_i^k h_{kj}. \tag{28}$$

Thus, using (12) and (15), we have in more explicitly:

$$\delta h_{ij} = \nabla_i \nabla_j \varphi - \delta_0^i \varphi g^{00} \left(u^{n-1} k_n k_{n-1}\right)^2, \; \delta_0^i = \begin{cases} 1, i = 0 \\ 0, i \neq 0 \end{cases}. \tag{29}$$

From [1], using (11) and (16) we have:

$$n\delta h = \Delta\varphi + \varphi\left(g^{00}\right)^2 \left(u^{n-1} k_n k_{n-1}\right)^2. \tag{30}$$

Thus, we have:

Theorem 3.1. *The variation of the mean curvature function H is given by* (30).

Using (16), (22), (25), (26) and (30) one can easily obtain $\delta^2 g, \delta^2 g^{00}$, $\delta^2 g^{01}, \delta^2 g^{11}, \delta^2 g^{0\nu}, \delta^2 g^{\nu\mu}$ and $\delta^2 g^{\nu\nu}$ in terms of the Laplace operator of φ and the curvatures for the base curve $r(u^0)$, as follows:

$$(31)\qquad \begin{cases} \delta^2 g = \frac{-2}{3}\varphi g(4\Delta\varphi - \xi), \\ \delta^2 g^{00} = 2\varphi g^{00}\xi, \\ \delta^2 g^{01} = -2\varphi g^{00}(1 - u^2 k_1)\xi, \\ \delta^2 g^{11} = -2\varphi g^{00}(1 - u^2 k_1)^2\xi, \\ \delta^2 g^{0\nu} = -2\varphi g^{00}(u^{\nu-1}k_{\nu-1} - u^{\nu+1}k_\nu)\xi, \\ \delta^2 g^{\nu\mu} = -2\varphi g^{00}(u^{\nu-1}k_{\nu-1} - u^{\nu+1}k_\nu)(u^{\nu-1}k_{\nu-1} - u^{\nu+1}k_\nu)\xi, \\ \delta g^{\nu\nu} = -2\varphi g^{00}(u^{\nu-1}k_{\nu-1} - u^{\nu+1}k_\nu)^2\xi, \end{cases}$$

where, $\xi = \Delta\varphi + 3\varphi(g^{00})^2(u^{n-1}k_n k_{n-1})^2$.

From the foregoing results, we have:

Corollary 3.3. *The* 2<u>nd</u> *variations of the determinant* g *and the metric tensor* g^{ij} *are given by* (31).

The variation of the norm S of the 2<u>nd</u> fundamental form (18) is given by using (16) and (30) as:

Corollary 3.4.

$$(32)\qquad \delta S^2 = 2g^{00}\left(u^{n-1}k_n k_{n-1}\right)\left[\Delta\varphi + \varphi\left(g^{00}\right)^2\left(u^{n-1}k_n k_{n-1}\right)^2\right].$$

4. The variation of the curvatures for the base curve and the Frenet-frame.

Here, the 1<u>st</u> and 2<u>nd</u> variations of the curvatures k_i for the base curve are derived. The variations of the Frenet-frame for the base curve are obtained.

From (7) and (22), we have:

$$(33)\qquad \delta k_1 = 0.$$

From (7) and (22), we have:

$$(34)\qquad u^{\nu+1}\delta k_\nu = u^{\nu-1}\delta k_{\nu-1}.$$

Using (33), we have:

$$(35)\qquad \delta k_\nu = 0.$$

From (16), (26), (30), and (35), we can obtain:

$$(36)\qquad \delta k_n = \frac{1}{(g^{00}u^{n-1}k_{n-1})}\left[\Delta\varphi - \varphi\left(g^{00}\right)^2\left(u^{n-1}k_n k_{n-1}\right)^2\right].$$

From the foregoing results, we have:

Corollary 4.1. The $1\underline{st}$ variation of the curvatures k_i for the base curve are given by (33), (35) and (36).

From (6) and (24), we have:

$$\delta^2 k_1 = \frac{-2}{u^2}\varphi_0\varphi_1. \tag{37}$$

From (7) and (24), we have:

$$\delta^2 k_\nu = \frac{1}{u^{\nu+1}}\left[u^{\nu-1}\delta^2 k_{\nu-1} - 2\varphi_0\varphi_\nu\right]. \tag{38}$$

From (37) and (38), we have:

$$\delta^2 k_2 = \frac{-2\varphi_0}{u^2u^3}\left[u^1\varphi_1 + u^2\varphi_2\right]. \tag{39}$$

From (38) and (39), we have:

$$\delta^2 k_3 = \frac{-2\varphi_0}{u^3u^4}\left[u^1\varphi_1 + u^2\varphi_2 + u^3\varphi_3\right]. \tag{40}$$

From (38) and (40), we have:

$$\delta^2 k_4 = \frac{-2\varphi_0}{u^4u^5}\left[u^1\varphi_1 + u^2\varphi_2 + u^3\varphi_3 u^4\varphi_4\right]. \tag{41}$$

$$\vdots$$

$$\delta^2 k_n = \frac{-2\varphi_0}{u^n u^{n+1}}\left[u^1\varphi_1 + u^2\varphi_2 + \ldots + u^n\varphi_n\right]. \tag{42}$$

From the foregoing results, we have:

Corollary 4.2. The $2\underline{nd}$ variation of the curvatures k_i for the base curve are given by:

$$\delta^2 k_i = \frac{-2\varphi_0}{u^i u^{i+1}}\sum_{i=1}^{n} u^i\varphi_i, \quad \varphi_i = \frac{\partial\varphi}{\partial u^i}. \tag{43}$$

From (3) and (20), we have:

$$\delta e_\gamma = \varphi_\gamma e_{n+1}. \tag{44}$$

From (2), (4), (20), (35) and (44), we can see that:

$$\delta e_{\nu+1} = \frac{1}{k_\nu}[(\varphi_{0\nu} + \varphi_{\nu-1}k_{\nu-1})e_{n+1} - \varphi_\nu k_n e_n]. \tag{45}$$

From the foregoing results, we have:

Corollary 4.3. *The variation of the Frenet-frame for the base curve is given by (44) and (45).*

5. Stability condition.

Here, the necessary and sufficient condition for the stability of an immersion X in terms of the curvatures for the base curve, with respect to $I = \int_M H^c dA, c \geq 0$, are derived. The solution of the differential equation which is produced from the stability condition, for example in E^3 and E^4 is obtained. For this purpose we give the following definition [1, 3, 4, 5, 12, 18, 19, 20]:

Definition 5.1. A closed hyperruled surface M^n in E^{n+1} is called an S-hyperruled surface if it is stable with respect to the integral $I(M^n) = \int_M H^n dA$, i.e., for any normal variation of M^n in E^{n+1}, we have $\delta(I(M)) = O$.

Thus and using (16), we have:

$$I(M) = \int_M \left(\frac{1}{n} g^{00} u^{n-1} k_n k_{n-1}\right)^c dA, \quad c \geq 0. \tag{46}$$

From (27), we have:

$$\delta I(M) = \int_M [\frac{c}{n^c}(g^{00}u^{n-1}k_n k_{n-1})^{c-1} u^{n-1}(g^{00}k_n \delta k_{n-1} + g^{00}k_{n-1}\delta k_n$$
$$+ k_n k_{n-1}\delta g^{00})]dA - \int_M \frac{\varphi}{n^c}(g^{00}u^{n-1}k_n k_{n-1})^{c+1} dA.$$

Using (26), (35) and (36), we have:

$$\delta I(M) = \int_M [\frac{c}{n^c}(g^{00}u^{n-1}k_n k_{n-1})^{c-1}\Big(\Delta\varphi + \varphi(g^{00}u^{n-1}k_n k_{n-1})^2\Big)]dA$$
$$- \int_M \frac{\varphi}{n^c}(g^{00}u^{n-1}k_n k_{n-1})^{c+1} dA.$$

Suppose that M is closed; then, by Greens theorem, we have:

$$\delta I(M) = \frac{\varphi}{n^c}\int_M [c\Delta(g^{00}u^{n-1}k_n k_{n-1})^{c-1} + \tag{47}$$
$$+ (c-1)(g^{00}u^{n-1}k_n k_{n-1})^{c-1}]dA.$$

From (47), we see that M is stable with respect to the integral (46), if and only if the right-hand side of (47) is identically zero for all differentiable functions φ on M, that is:

$$S_c : c\Delta(g^{00}u^{n-1}k_n k_{n-1})^{c-1} + (c-1)(g^{00}u^{n-1}k_n k_{n-1})^{c+1} = 0. \tag{48}$$

Thus, we obtain the proof of the main theorem:

Theorem 5.1. *The oriented closed osculating hyperruled surface* $X : M \to E^{n+1}$ *is stable with respect to the integral (46) if and only if the condition* (S_c) *is valid for the curvatures* k_i *of the base curve.*

Now we shall try to find a solution of the differential equation (48) in *case when n=2 and n=3.*

(I) We put c=a $\geq$ *2, n=2* in (48), using (10) and (11), we have:

$$(49)\quad S_a : a\Delta\left[\frac{k_2(u^0)}{u^1 k_1(u^0)}\right]^{a-1} + (a-1)\left[\frac{k_2(u^0)}{u^1 k_1(u^0)}\right]^{a-1} + (a-1) = 0, \quad k_1 \neq 0.$$

Let us denote:

$$(50)\qquad F(u^0, u^1) = \left[\frac{k_2(u^0)}{u^1 k_1(u^0)}\right].$$

Using the definition of Laplace operator of any differentiable function on M [1] we have:

$$(51)\qquad \Delta F^{a-1} = \frac{-1}{\sqrt{g}}\Big\{\big(\sqrt{g}g^{00}\big)\frac{\partial^2 F^{a-1}}{\partial (u^0)^2} + \frac{\partial^2 F^{a-1}}{\partial u^0}\frac{\partial(\sqrt{g}g^{00})}{\partial u^0} +$$
$$+ \big(\sqrt{g}g^{11}\big)\frac{\partial^2 F^{a-1}}{\partial (u^1)^2} + \frac{\partial^2 F^{a-1}}{\partial u^1}\frac{\partial(\sqrt{g}g^{11})}{\partial u^1} + 2\big(\sqrt{g}g^{01}\big)\frac{\partial^2 F^{a-1}}{\partial u^0 \partial u^1} +$$
$$+ \frac{\partial F^{a-1}}{\partial (u^1}\frac{\partial(\sqrt{g}g^{01})}{\partial u^0} + \frac{\partial F^{a-1}}{\partial u^0}\frac{\partial(\sqrt{g}g^{01})}{\partial u^1}\Big\},$$

where, from (6), (10), (11) and (50), we have:

$$(52)\qquad \begin{cases} \dfrac{\partial F^{a-1}}{\partial u^0} = -(a-1)\left(\dfrac{u^1}{k_2^2}\right)(k_2 k_1 - k_1 k_2)F^a, \\ \dfrac{\partial F^{a-1}}{\partial u^1} = -(a-1)\left(\dfrac{1}{u^1}\right)F^{a-1}, \\ \dfrac{\partial^2 F^{a-1}}{\partial (u^0)^2} = (a-1)\left(\dfrac{u^1}{k_2^2}\right)^2\begin{bmatrix} (a - 2k_1^2 k_2^2 + a k_2^2 k_1^2 - k_2^2 k_1 k_1 \\ +(2-2a)k_1 k_2 k_1 k_2 + k_1^2 k_2 k_2 \end{bmatrix}F^{a+1}, \\ \dfrac{\partial^2 F^{a-1}}{\partial (u^1)^2} = a(a-1)\left(\dfrac{1}{u^1}\right)^2 F^{a-1}, \\ \dfrac{\partial^2 F^{a-1}}{\partial u^0 \partial u^1} = (a-1)^2\left(\dfrac{1}{k_2}\right)^2 [k_2 k_1 - k_1 k_2]F^a, \\ \dfrac{\partial(\sqrt{g}g^{00})}{\partial u^0} = \dfrac{\partial(\sqrt{g}g^{01})}{\partial u^0} = \dfrac{-k_1}{u^1 k_1^2}, \\ \dfrac{\partial(\sqrt{g}g^{11})}{\partial u^1} = \dfrac{(u^1)^2 k_1^2 - 1}{(u^1)^2 k_1}, \quad \dfrac{\partial(\sqrt{g}g^{01})}{\partial u^1} = \dfrac{1}{(u^1)^2 k_1}. \end{cases}$$

Using (50) and (51), the condition (S_a) takes the form:

$$(a+a^2)k_1^2k_2^2+u^1[(2a^2-a)k_1^2k_2\frac{\partial k_2}{\partial u^0}-2a^2k_1k_2^2\frac{\partial k_1}{\partial u^0}]+(u^1)^2[(a^2-a)k_1^4k_2^2$$

$$-k_1^2k_2^4+(a+a^2)k_2^2(\frac{\partial k_1}{\partial u^0})^2+(a-2a^2)k_1k_2\frac{\partial k_1}{\partial u^0}\frac{\partial k_2}{\partial u^0}+(a^2-2a)k_1^2(\frac{\partial k_2}{\partial u^0})^2$$

$$-ak_1k_2^2\frac{\partial^2 k_1}{\partial (u^0)^2}+ak_1^2k_2\frac{\partial^2 k_2}{\partial (u^0)^2}]=0.$$

Thus, $\forall u^1 \in R$, we have:

(53) $$(a+a^2)k_1^2k_2^2=0,$$

(54) $$(2a^2-a)k_1^2k_2\frac{\partial k_2}{\partial u^0}-2a^2k_1k_2^2\frac{\partial k_1}{\partial u^0}=0,$$

(55) $$(a^2+a)k_1^4k_2^2-k_1^2k_2^4+(a+a^2)k_2^2(\frac{\partial k_1}{\partial u^0})^2+(a-2a^2)k_1k_2\frac{\partial k_1}{\partial u^0}\frac{\partial k_2}{\partial u^0}$$

$$+(a^2-a)k_1^2(\frac{\partial k_2}{\partial u^0})^2-ak_1k_2^2\frac{\partial^2 k_1}{\partial (u^0)^2}+ak_1^2k_2\frac{\partial^2 k_2}{\partial (u^0)^2}=0.$$

From (53), we have $a \neq 0 (a \geq 2)$, $k_1 \neq 0$, thus:

(56) $$k_2=0.$$

Thus, we see that (54) and (55) vanish automatically. So, we have the proof of the following theorem:

Theorem 5.2. *The condition of stability (S_a) is valid for the closed osculating ruled surface, for which the base curve is a plane curve with respect to the integral (46) in $E^3 (c=a\geq 2)$.*

(II) We put c=a $\geq$ 2, n=3 in (48), using (10) and (11), we have:

(57) $$S_a : a\Delta\left[\frac{k_3(u^0)}{u^2k_2(u^0)}\right]^{a-1}+(a-1)\left[\frac{k_3(u^0)}{u^2k_2(u^0)}\right]^{a+1}=0, \quad k_2\neq 0.$$

After a long straight-forward computation similar to (49), we can see that the condition (57) s splits into four conditions:

$$(58)\qquad \begin{cases} ak_1k_2^2k_3^2 = 0, \\ (a+a^2)k_1^2k_2^2k_3^2 = 0, \end{cases}$$

$$(59)\quad ak_1^2k_2^2k_3^2+(a^2-a)k_2^4k_3^2-k_2^2k_3^4+(a+a^2)k_3^2\left(\frac{\partial k_2}{\partial u^0}\right)^2+(a-2a^2)k_2k_3\frac{\partial k_2}{\partial u^0}\frac{\partial k_3}{\partial u^0}$$

$$+(a^2-2a)k_2^2\left(\frac{\partial k_3}{\partial u^0}\right)^2-ak_2k_3^2\frac{\partial^2 k_2}{\partial (u^0)^2}+ak_2^2k_3\frac{\partial^2 k_3}{\partial (u^0)^2}=0.$$

$$(60)\qquad ak^2k^3\frac{\partial k_1}{\partial u^0}-2a^2k_1k_3\frac{\partial k_2}{\partial u^0}+\left(a+2a^2\right)k_1k_2\frac{\partial k_3}{\partial u^0}=0.$$

Since a$\neq 0 (a \geq 2)$, $k_1 \neq 0$, $k_2 \neq 0$ and from (58) we have:

$$(61)\qquad k_3 = 0.$$

Thus, also (59) and (60) are satisfied. So, we have the proof of the following theorem:

Theorem 5.3. *The closed osculating hyperruled surface in E^4, for which the base curve is a hyperplanar curve, is stable with $c = a \geq 2$.*

Using (10) and (11), the condition (S_c) takes the form:

$$S_c : c\Delta\left[\frac{k_n(u^0)}{u^{n-1}k_{n-1}(u^0)}\right]^{c-1}+(c-1)\left[\frac{k_n(u^0)}{u^{n-1}k_{n-1}(u^0)}\right]^{c+1}=0,\ k_{n-1}\neq 0,$$

where $c \geq 0, n \geq 2$.

Thus, *in the case when c=0*, the condition (S_c) is degenerate to $k_n = 0$. So, we have:

Corollary 5.1. *The differential equation of stability (S_0) $(c = 0)$ has a solution within an osculating hyperruled surface in E^{n+1}, for which the base curve is a hyperplanar curve.*

In the case when c=1, we have:

Corollary 5.2. *The condition (S_1) $(c = 1)$ is valid for an osculating hyperrruled surface in E^{n+1}, for which the base curve is not a hyperplanar curve.*

It is important to remark that the foregoing results are a confirmation of our preceding examples in [1], whereas well-known for any plane curve in $E^3, k_2 = 0, k_1 \neq 0$ and for any hyperplanar curve in E^4, $(k_1 k_2) \neq (0, 0)$ and $k_3 = 0$.

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