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(R,P)-ABSOLUTELY SUMMING DUAL OPERATORS ON THE PROJECTIVE TENSOR PRODUCTS OF SPACES

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For $U \in L(X\tilde{\otimes}_\pi Y, Z)$ we consider the operator $U^\# : X \to L(Y, Z)$ defined by $(U^\# x)(y) = U(x \otimes y)$, for $x \in X, y \in Y$. We prove that, if $U \in L(X\tilde{\otimes}_\pi Y, Z)$ has the property that $U^\# \in As^{dual}_{r,p}(X, As^{dual}_{p,q}(Y, Z))$, then the dual operator $U^* \in As_{r,q}(Z^*, As^{dual}_{r,p}(X, Y^*))$, from which we deduce that $As^{dual}_{r,p} \otimes_\pi As^{dual}_{p,q} \subset As^{dual}_{r,q}$, in particular, we obtain a result first proved by B. Carl, A. Defant, M. S. Ramanujan that the normed ideal of the p-absolutely summing dual operators is stable under projective tensor products. Also, if $L(X, Y^*) = As_p(X, Y^*$, then for any Banach space Z, if $U \in As_p(X\tilde{\otimes}_\pi Y, Z)$, we have $U^\# \in As_p(X, As_p(Y, Z))$.

For X and Y Banach spaces we denote by $L(X, Y)$ the Banach space of all linear and continuous operators from X to Y equipped with the operator norm, by $X\tilde{\otimes}_\pi Y$ the projective tensor product of X and Y i.e. the completion of the algebraic tensor product $X \otimes Y$ with respect to the projective crossnorm:

$$\pi(u) = \inf \{\sum_{i=1}^n \|x_i\| \, \|y_i\| \mid u \in X \otimes Y, u = \sum_{i=1}^n x_i \otimes y_i\}.$$

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Also if $U \in L(X, X_1)$, $V \in L(Y, Y_1)$ we denote by $U\tilde{\otimes}_\pi V : X\tilde{\otimes}_\pi Y \to X_1\tilde{\otimes}_\pi Y_1$ the projective tensor product of the operators U and V. For $1 \leq r < \infty$ and $x_1, \ldots, x_n \in X$ we write

$$l_r(x_i| \leq i \leq n) = \left(\sum_{i=1}^{n} \|x_i\|^r\right)^{\frac{1}{r}}$$

and

$$w_r(x_i|1 \leq i \leq n) = \sup \{(\sum_{i=1}^{n} |x^*(x_i|^r)^{\frac{1}{r}} | x^* \in X^*, \|x^*\| \leq 1\}.$$

Let us observe that using the *weak**−denseness of the closed unit ball B_X of X in $B_{X^{**}}$ we have

$$w_r(x_i^*|1 \leq i \leq n) = \sup \{(\sum_{i=1}^{n} |x_i^*(x)|^r)^{\frac{1}{r}} | \|x\| \leq 1\},$$

for each $x_1^*, \ldots, x_n^* \in X^*$.

We will use this observation in the sequel without explicit reference.

Given $1 \leq p \leq r < \infty$, $U \in L(X, Y)$ is called (r, p)−absolutely summing if there is some $C > 0$ such that if $x_1, \ldots, x_n \in X$ then

$$l_r(Ux_i|1 \leq i \leq n) \leq Cw_p(x_i|1 \leq i \leq n).$$

The (r, p)−absolutely summing norm of U is $\|U\|_{r,p} = \inf C$.

We denote by $As_{r,p}(X, Y)$ the Banach space of all (r, p)−absolutely summing operators from X into Y equipped with the (r, p)−absolutely summing norm. As is well known $(As_{r,p}, \| \|_{r,p})$ is a normed ideal of operators in the sense of A. Pietsch, see [4] or [7]; instead of $(As_{p,p}, \| \|_{p,p})$ we write simply $(As_p, \| \|_p)$. Also $As_{r,p}^{dual}(X, Y) = \{U \in L(X, Y)|U^* \in As_{r,p}(Y^*, X^*)\}$ and for $U \in As_{r,p}^{dual}(X, Y)$ we denote $\|U\|_{r,p,dual} = \|U^*\|_{r,p}$. Let us observe that $(As_{r,p}^{dual}, \| \|_{r,p,dual})$ is also a normed ideal of operators in the sense of A. Pietsch, see [4] or [7].

For other notations and notions used and not defined we refer the reader to [3] or [7].

For $U \in L(X\tilde{\otimes}_\pi Y, Z)$ and each $x \in X$ we consider the operator $U^\# x : Y \to Z$ given by $(U^\# x)(y) = U(x \otimes y)$, for $y \in Y$; evidently, $U^\# : X \to L(Y, Z)$ is linear and continuous.

A natural problem is the connection between the operators U and $U^\#$ for some normed ideals of operators; see [9] and [11] for the operators on injective tensor products. In the sequel we study this problem for the normed ideal of the (r, p)−absolutely summing dual operators.

Theorem 1. *Let* $1 \leq q \leq p \leq r < \infty$ *and* $U \in L(X\tilde{\otimes}_\pi Y, Z)$. *If* $U^{\#}x \in As_{p,q}^{dual}(Y, Z)$ *for each* $x \in X$ *and* $U^{\#} \in As_{r,p}^{dual}(X, As_{p,q}^{dual}(Y, Z))$, *then* $U^*(z^*) \in As_{r,p}^{dual}(X, Y^*)$, *for each* $z^* \in Z^*$ *and* $U^* \in As_{r,q}(Z^*, As_{r,p}^{dual}(X, Y^*))$. *In addition:* $\|U^*\|_{r,q} \leq \|U^{\#}\|_{r,p,dual}$. *In particular* $U \in As_{r,q}^{dual}(X\tilde{\otimes}_\pi Y, Z)$.

Proof. We have that $U^* : Z^* \to (X\tilde{\otimes}_\pi Y)^* = L(X, Y^*)$ satisfies $U^*(z^*) = S_{z^*} \circ U^{\#}$, for $z^* \in Z^*$ where $S_{z^*} : L(Y, Z) \to Y^*$ is given by $S_{z^*}(V) = V^*(z^*)$, for $V \in L(Y, Z)$; (use the relation: $[U^*(z^*)](x)(y) = (z^* \circ U^{\#}x)(y)$, for each $x \in X, y \in Y$).

Since for $z^* \in Z^*$, we may consider the operator $S_{z^*} : As_{p,q}^{dual}(Y, Z) \to Y^*$, given by $S_{z^*}(V) = V^*(z^*)$, for $V \in As_{p,q}^{dual}(Y, Z)$ and, by hypothesis, $U^{\#} : X \to As_{p,q}^{dual}(Y, Z)$ is an $(r, p)-$absolutely summing dual operator, from the ideal properties of $(r, p)-$absolutely summing dual operators it follows that: $U^*(z^*) \in As_{r,p}^{dual}(X, Y^*)$. Take now $z_1^*, \ldots, z_n^* \in Z^*$ and $\varepsilon > 0$. Then from the definition of the $(r, p)-$absolutely summing norm it follows that there exist $\sigma_i \subset \mathbb{N}$, $(1 \leq i \leq n)$, σ_i finite and $(y_{ij}^{**})_{j\in\sigma_i} \subset Y^{**}$ such that

$$w_p(y_{ij}^{**}|j \in \sigma_i) \leq 1 \text{ and } \|U^*(z_i^*)\|_{r,p,dual} - \varepsilon < l_r((U^*(z_i^*))^*(y_{ij}^{**})|j \in \sigma_i).$$

It is easy to prove (see [8]) that, for y^{**} and $x \in X$, we have

$$[y^{**} \circ U^*(z^*)](x) = y^{**}(z^* \circ U^{\#}x)$$

hence, $T_{y^{**},z^*} : As_{p,q}^{dual}(Y, Z) \to \mathbb{R}$ (or $\mathbb{C}$) defined by $T_{y^{**},z^*}(V) = y^{**}(V^*(z^*))$ is a linear and continuous functional on $As_{p,q}^{dual}(Y, Z)$ and the above relation shows that

$$T_{y^{**},z^*} \circ U^{\#} = y^{**} \circ U^*(z^*).$$

Then using the fact that the dual of $U^{\#} : X \to As_{p,q}^{dual}(Y, Z)$ is $(r, p)-$absolutely summing, we obtain

$$[\sum_{i=1}^{n}(\|U^*(z_i^*)\|_{r,p,dual} - \varepsilon)^r]^{\frac{1}{r}} < l_r(y_{ij}^{**} \circ U^*(z_i^*)|1 \leq i \leq n; j \in \sigma_i =$$

$$l_r(T_{y_{ij}^{**},z_i^*} \circ U^{\#}|1 \leq i \leq n; j \in \sigma_i) =$$

$$l_r((U^{\#})^*(T_{y_{ij}^{**},z_i^*})|1 \leq i \leq n; j \in \sigma_i) \leq$$

$$\|(U^{\#})^*\|_{r,p} w_p(T_{y_{ij}^{**},z_i^*}|1 \leq i \leq n; j \in \sigma_i) =$$

$$\|U^{\#}\|_{r,p,dual} w_p(T_{y_{ij}^{**},z_i^*}|1 \leq i \leq n; j \in \sigma_i).$$

But for $V \in As^{dual}_{p,q}(Y, Z)$, with $\|V\|_{p,q,dual} \leq 1$ we have:

$$\sum_{j\in\sigma_i} |T_{y^{**}_{ij},z^*_i}(V)|^p = \sum_{j\in\sigma_i} |y^{**}_{ij}(V^*(z^*_i))|^p \leq$$

$$\|V^*(z^*_i\|^p \sup\{\sum_{j\in\sigma_i} |y^{**}_{ij}(y^*)|^p; \|y^*\| \leq 1\} =$$

$$\|V^*(z^*_i\|^p[w_p(y^{**}_{ij}|j\in\sigma_i)]^p \leq \|V^*(z^*_i)\|^p$$

and hence:

$$(\sum_{i=1}^{n}\sum_{j\in\sigma_i} |T_{y^{**}_{ij},z^*_i}(V)|^p)^{\frac{1}{p}} \leq (\sum_{i=1}^{n} \|V^*(z^*_i)\|^p)^{\frac{1}{p}} =$$

$$l_p(V^*(z^*_i)|1 \leq i \leq n) \leq \|V^*\|_{p,q} w_q(z^*_i|1 \leq i \leq n) =$$

$$\|V\|_{p,q,dual} w_q(z^*_i|1 \leq i \leq n) \leq w_q(z^*_i|1 \leq i \leq n),$$

from where:

$$w_q(T_{y^{**}_{ij},z^*_i}|1 \leq i \leq n;\ j\in\sigma_i) =$$

$$\sup\{(\sum_{i=1}^{n}\sum_{j\in\sigma_i} |T_{y^{**}_{ij},z^*_i}(V)|^p)^{\frac{1}{p}} | V \in As^{dual}_{p,q}(Y, Z), \|V\|_{p,q,dual} \leq 1\}$$

$$\leq w_q(z^*_i|1 \leq i \leq n).$$

From this we obtain that:

$$[\sum_{i=1}(\|U^*(z^*_i)\|_{r,p,dual} - \varepsilon)^r]^{\frac{1}{r}} < \|U^\#\|_{r,p,dual} w_q(z^*_i|1 \leq i \leq n)$$

and so:

$$[\sum_{i=1}(\|U^*(z^*_i)\|^r_{r,p,dual}]^{\frac{1}{r}} \leq \|U^\#\|_{r,p,dual} w_q(z^*_i|1 \leq i \leq n)$$

i.e.

$$U^* \in As_{r,q}(Z^*, As^{dual}_{r,p}(X, Y^*)) \text{ and } \|U^*\|_{r,q} \leq \|U^\#\|_{r,p,dual}. \qquad \square$$

In [1] Theorem 2.1, or in the book [2] chapter III, p. 445–466, some stability results for a large class of normed ideals of operators are proved. In particular it is proved that the normed ideal of the $p-$absolutely summing dual operators is stable under projective tensor products, i.e. if $U \in As^{dual}_p(X, X_1)$, $V \in As^{dual}_p(Y, Y_1)$, then $(U\tilde{\otimes}_\pi V)^* : L(X_1, Y^*_1) \to L(X, Y^*)$ is $p-$absolutely summing and so a natural question is: if $U \in As^{dual}_p(X, X_1)$, $V \in As^{dual}_p(Y, Y_1)$, then is $(U\tilde{\otimes}_\pi V)^* : L(X_1 Y^*_1 \to As_p(X, Y^*)$ $p-$absolutely summing.

We will prove in the next theorem a more general result which improves that from [1] or [2].

Theorem 2. *Let* $1 \leq q \leq p \leq r < \infty$. *If* $U \in As_{p,q}^{dual}(X, X_1)$, $V \in As_{r,p}^{dual}(Y, Y_1)$, *then the dual of the projective tensor product* $(U\tilde{\otimes}_\pi V)^* \in As_{r,q}(L(X_1, Y_1^*), As_{r,p}(X, Y^*))$, *and* $\|(U\tilde{\otimes}_\pi V)^*\|_{r,q} \leq \|U\|_{p,q,dual}\|V\|_{r,p,dual}$. *In particular:*

$As_{r,p}^{dual} \otimes_\pi As_{p,q}^{dual} \subset As_{r,q}^{dual}$ *and the normed ideal of operators* $(As_p^{dual}, \| \|_{p,dual})$ *is tensor stable with respect to the projective tensor product i.e. if* $U \in As_p^{dual}(X, X_1)$, $V \in As_p^{dual}(Y, Y_1)$, *then the projective tensor product* $U\tilde{\otimes}_\pi V \in As_p^{dual}(X\tilde{\otimes}_\pi Y, X_1\tilde{\otimes}_\pi Y_1)$ *and* $\|U\tilde{\otimes}_\pi V\|_{p,dual} \leq \|U\|_{p,dual}\|V\|_{p,dual}$.

Proof. Let $S = V\tilde{\otimes}_\pi U : Y\tilde{\otimes}_\pi X \to Y_1\tilde{\otimes}_\pi X_1$. For $y \in Y$, let $A_y : X_1 \to Y_1\tilde{\otimes}_\pi X_1$ be given by $A_y(x_1) = (Vy) \otimes x_1$. Evidently $S^\# y = A_y \circ U$ and, because $U \in As_{p,q}^{dual}(X, X_1)$, by the ideal property of the $(p, q)-$absolutely summing dual operators, $S^\# y \in As_{p,q}^{dual}(X, Y_1\tilde{\otimes}_\pi X_1)$.

For $y_1 \in Y_1$, Let $B_{y_1} : X_1 \to Y_1\tilde{\otimes}_\pi X_1$ be the operator given by $B_{y_1}(x_1) = y_1 \otimes x_1$ and $B : Y_1 \to L(X, X_1\tilde{\otimes}_\pi X_1)$ defined by $B(y_1)(x) = y_1 \otimes (Ux)$. We have: $B(y_1) = B_{y_1} \circ U$. Since U has $(p, q)-$absolutely summing dual, we obtain: $B(y_1) \in As_{p,q}^{dual}(X, Y_1\tilde{\otimes}_\pi X_1)$ and $\|B(y_1)\|_{p,q,dual} \leq \|B_{y_1}\| \; \|U\|_{p,q,dual} \leq \|y_1\| \; \|U\|_{p,q,dual}$, for each $y_1 \in Y_1$ and so

$$\|B : Y_1 \to As_{p,q}^{dual}(X, Y_1\tilde{\otimes}_\pi X_1)\|_{op} \leq \|U\|_{p,q,dual}.$$

Now as it is easy to see $S^\# = B \circ V$ and, since $V \in As_{r,p}^{dual}(Y, Y_1)$, the ideal property of $(r, p)-$absolutely summing dual operators shows that:

$$S^\# \in As_{r,p}^{dual}(Y, As_{p,q}^{dual}(X, Y_1\tilde{\otimes}_\pi X_1))$$

and

$$\|S^\# : Y \to As_{p,q}^{dual}(X, Y_1\tilde{\otimes}_\pi X_1)\|_{r,p,dual} \leq$$

$$\|B : Y_1 \to As_{p,q}^{dual}(X, Y_1\tilde{\otimes}_\pi X_1)\|_{op}\| \; V \; \|_{r,p,dual}.$$

From the above inequalities we will obtain

$$\|S^\# : Y \to As_{p,q}^{dual}(X, Y_1\tilde{\otimes}_\pi X_1)\|_{r,p,dual} \leq \| \; U \; \|_{p,q,dual}\| \; V \; \|_{r,p,dual}.$$

Using theorem 1 we obtain that $S^* : L(Y_1, X_1^*) \to As_{r,p}^{dual}(Y, X^*)$ is $(r, q)-$absolutely summing and

$$\|S^* : L(Y_1, X_1^*) \to As_{r,p}^{dual}(Y, X^*)\|_{r,q} \leq$$

$$\|S^\# : Y \to As_{p,q}^{dual}(X, Y_1\tilde{\otimes}_\pi X_1)\|_{r,p,dual}.$$

Hence:

$$\|S^* : L(Y_1, X_1^*) \to As_{r,p}^{dual}(X, Y^*)\|_{r,q} \le \| U \|_{p,q,dual}\| V \|_{r,p,dual}.$$

Let us consider now two natural isometries: $h : L(X_1, Y_1^*) \to L(Y_1, X_1^*), h(\hat{\psi}$, where $[\psi(x_1)](y_1) = [\hat{\psi}(y_1)](x_1)$ and $g : As_{r,p}^{dual}(Y, X^*) \to As_{r,p}(X, Y^*)$, $g(T) = T^* \circ J_X$, where J_X is the canonical embedding of X into the bidual. Then a simple calculation shows: $(U\tilde{\otimes}_\pi V)^* = g \circ S^* \circ h$ and by the ideal property of $(r, q)-$absolutely summing operators we obtain that:

$$(U\tilde{\otimes}_\pi V)^* \in As_{r,q}(L(X_1, Y_1^*), As_{r,p}(X, Y^*))$$

and

$$\|(U\tilde{\otimes}_\pi V)^* : L(X_1, Y_1^*) \to As_{r,p}(X, Y^*))\|_{r,q} \le$$
$$\|g\| \; \|S^* : L(Y_1, X_1^*) \to As_{r,p}^{dual}(Y, X^*)\|_{r,q}\| h \|.$$

From these last inequalities we obtain

$$\|(U\tilde{\otimes}_\pi V)^* : L(X_1, Y_1^*) \to As_{r,p}(X, Y^*))\|_{r,q} \le \| U \|_{p,q,dual}\| V \|_{r,p,dual}.$$

□

To have some examples, let $j : l_2 \to c_0$ be the canonical injection, whose dual $j^* : l_1 \to l_2$ is 1-absolutely summing and the identity map $i : c_0 \to c_0$ whose dual $i^* : l_1 \to l_1$ is (2,1)-absolutely summing, see [7] for these classical results. By theorem 2, we obtain that the restriction mapping $R : L(c_0, l_1) \to As_{2,1}(l_2, l_1)$ is a (2,1)-absolutely summing operator. In the same way the restriction mapping $R : L(c_0, l_1) \to As(l_2, l_2)$ is an absolutely summing operator. This example is interesting since in [5] it is proved that the restriction map $R : K(c_0, l_1) \to As_2(l_2, l_2)$ is an absolutely summing operator which does not factor through any $L_1(\mu)$.

The composition operator that we consider in the next proposition has been studies in [1], [6], [10], [12], [13] for some ideals of operators. Also in the paper [1] Proposition 3.3, it is proved that a certain composition operator is $p-$absolutely summing with respect to the operator norm, more precisely: if $A \in As_p^{dual}(X, Y)$, $B \in As_p(Z, T)$, then $h : L(Y, Z) \to L(X, T), h(U) = BUA$ is a $p-$absolutely summing with respect to the operator norm. Again a natural question is: if $A \in As_p^{dual}(X, Y)$, $B \in As_p(Z, T)$, then is $h : L(Y, Z) \to As_p(X, T), h(U) = BUA$, $p-$ absolutely summing.

In our next two proposition we will prove also more general results in this direction.

Proposition 3. *Let X, Y, Z, T be Banach spaces, $1 \leq q \leq p \leq r < \infty$, $A \in As^{dual}_{p,q}(X,Y)$, $B \in As_{r,p}(Z,T)$, and $h : L(Y,Z) \to As_{r,p}(X,T), h(U) = BUA$. Then h is an $(r,q)-$absolutely summing operator and $\|h\|_{r,q} \leq \|B\|_{r,p}\|A\|_{p,q,dual}$.*

Proof. Choose $U_1, \ldots, U_n \in L(Y,Z)$ with $0 < \varepsilon < \|h(U_i)\|_{r,p}$. From the definition of the $(r,p)-$absolutely summing norm it follows that there exists $\sigma_i \subset \mathbb{N}, \sigma_i$ finite $(1 \leq i \leq n)$ and $(x_{ij})_{j\in\sigma_i} \subset X$ such that $\|h(U_i)\|_{r,p} - \varepsilon < l_r(h(U_i)(x_{ij}) \mid j \in \sigma_i)$ and $w_p(x_{ij} \mid j \in \sigma_i) \leq 1$ for each $i = 1, \ldots, n$. Then

$$[\sum_{i=1}^{n}(\|h(U_i)\|_{r,p} - \varepsilon)^r]^{\frac{1}{r}} < l_r((BU_iA)(x_{ij}) \mid 1 \leq i \leq n; j \in \sigma_i) \leq$$

$$\|B\|_{r,p} w_p((U_iA)(x_{ij}) \mid 1 \leq i \leq n; j \in \sigma_i),$$

since $B \in As_{r,p}(Z,T)$. For $z^* \in Z^*$, $\|z^*\| \leq 1$,

$$\sum_{i=1}^{n}\sum_{j\in\sigma_i} \mid z^*[(U_iA)(x_{ij})] \mid^p = \sum_{i=1}^{n}\sum_{j\in\sigma_i} \mid [A^*(U_i^*(z^*))](x_{ij} \mid^p \leq$$

$$\sum_{i=1}^{n} \| A^*(U_i^*(z^*))\|^p [w_p(x_{ij} \mid j \in \sigma_i)]^p \leq \sum_{i=1}^{n} \|A^*(U_i^*(z^*))\|^p \leq$$

$$\|A^*\|^p_{p,q}[w_q(U_i^*(z^*) \mid 1 \leq i \leq n)]^p,$$

where we have used that $A \in As^{dual}_{p,q}(X,Y)$. Hence

$$w_p((U_iA)(x_{ij}) \mid 1 \leq i \leq n; j \in \sigma_i) = \sup_{\|z^*\|\leq 1} (\sum_{i=1}^{n}\sum_{j\in\sigma_i} \mid z^*[(U_iA)(x_{ij})] \mid^p)^{\frac{1}{p}} \leq$$

$$\|A\|_{p,q,dual} \sup_{\|z^*\|\leq 1} w_q(U_i^*(z^*) \mid 1 \leq i \leq n).$$

But

$$\sup_{\|z^*\|\leq 1} w_q(U_i^*(z^*) \mid 1 \leq i \leq n) = \sup_{\|z^*\|\leq 1, \|y\|\leq 1} (\sum_{i=1}^{n} \mid \langle y, U_i^*(z^*) \rangle \mid^q)^{\frac{1}{q}} \leq$$

$$\sup\{(\sum_{i=1}^{n} \mid \langle U_i, \psi \rangle \mid^q)^{\frac{1}{q}} \mid \psi \in (L(Y,Z))^*, \|\psi\| \leq 1\} = w_q(U_i \mid 1 \leq i \leq n).$$

Summarizing the above inequalities we obtain

$$[\sum_{i=1}^{n}(\|h(U_i)\|_{r,p}-\varepsilon)^r]^{\frac{1}{r}} < \|B\|_{r,p}\|A\|_{p,q,dual}w_q(U_i \mid 1 \le i \le n),$$

$$(\sum_{i=1}^{n}(\|h(U_i)\|_{r,p}^r)^{\frac{1}{r}} \le \|B\|_{r,p}\|A\|_{p,q,dual}w_q(U_i \mid 1 \le i \le n)$$

and the proposition is proved . □

The ideal property of the $(r, p)-$absolutely summing dual operators shows that h takes its values also in $As_{p,q}^{dual}(X, T)$. For $p = q = r \ge 1$ we can prove the following.

Proposition 4. *Let X, Y, Z, T be Banach spaces, $p \ge 1$, $A \in As_p^{dual}(X, Y)$, $B \in As_p(Z, T)$, and $h : L(Y, Z) \to As_p^{dual}(X, T)$, $h(U) = BUA$. Then h is a $p-$absolutely summing operator and $\|h\|_p \le \|B\|_p\|A\|_{p,dual}$.*

Proof. Choose $U_1, \ldots, U_n \in L(Y, Z)$ with $0 < \varepsilon < \|h(U_i)\|_{p,dual}$. From the definition of the $p-$absolutely summing dual norm it follows that there exist $\sigma_i \subset \mathbb{N}$, σ_i finite, $(1 \le i \le n)$ and $(t_{ij}^*)_{j\in\sigma_i} \subset T^*$ such that

$$\|h(U_i)\|_{p,dual} - \varepsilon < l_p([h(U_i)]^*(t_{ij}^*) \mid j \in \sigma_i) \text{ and } w_p(t_{ij}^* \mid j \in \sigma_i) \le 1$$

for each $i = 1, \ldots, n$.

Then

$$[\sum_{i=1}^{n}(\|h(U_i\|_{p,dual}-\varepsilon)^p]^{\frac{1}{p}} < l_p((A^* U_i^* B^*)(t_{ij}^*) \mid 1 \le i \le n;\ j \in \sigma_i) \le$$

$$\|A^*\|_p w_p((U_i^* B^*)(t_{ij}^*) \mid 1 \le i \le n;\ j \in \sigma_i),$$

since $A \in As_p^{dual}(X, Y)$. For $y \in Y$, $\|y\| \le 1$,

$$\sum_{i=1}^{n}\sum_{j\in\sigma_i} \mid [(U_i^* B^*)(t_{ij}^*)](y) \mid^p = \sum_{i=1}^{n}\sum_{j\in\sigma_i} \mid (t_{ij}^* \circ B \circ U_i)(y) \mid^p \le$$

$$\sum_{i=1}^{n} \|B(U_i(y))\|^p [w_p(t_{ij}^*) \mid j \in \sigma_i)]^p \le$$

$$\sum_{i=1}^{n} \|B(U_i(y))\|^p \leq \|B\|_p^p [w_p(U_i(y) \mid 1 \leq i \leq n)]^p.$$

where we have used that $B \in As_p(Z, T)$.

Hence

$$w_p((U_i^* B^*)(t_{ij}^*) \mid 1 \leq i \leq n; j \in \sigma_i) = \sup_{\|y\| \leq 1} (\sum_{i=1}^{n} \sum_{j \in \sigma_i} | [(U_i^* B^*)(t_{ij}^*)](y)|^p)^{\frac{1}{p}} \leq$$

$$\|B\|_p \sup_{\|y\| \leq 1} w_p(U_i(y) \mid 1 \leq i \leq n).$$

But

$$\sup_{\|y\| \leq 1} w_p((U_i(y) \mid 1 \leq i \leq n) = \sup_{\|z^*\| \leq 1, \|y\| \leq 1} (\sum_{i=1}^{n} | \langle U_i(y), z^* \rangle |^p)^{\frac{1}{p}} \leq$$

$$\sup\{(\sum_{i=1}^{n} | \langle U_i, \psi \rangle |^p)^{\frac{1}{p}} \mid \psi \in (L(Y, Z))^*, \|\psi\| \leq 1\} = w_p(U_i \mid 1 \leq i \leq n).$$

Summarizing the above inequalities we obtain:

$$[\sum_{i=1}^{n} (\|h(U_i)\|_{p,dual} - \varepsilon)^p]^{\frac{1}{p}} < \|B\|_p \|A\|_{p,dual} w_p(U_i \mid 1 \leq i \leq n),$$

$$(\sum_{i=1}^{n} \|h(U_i)\|_{p,dual}^p)^{\frac{1}{p}} \leq \|B\|_p \|A\|_{p,dual} w_p(U_i \mid 1 \leq i \leq n)$$

and the proposition is proved. □

Proposition 5. *Let X and Y be Banach spaces and $1 \leq p < \infty$. Then the following conditions are equivalent:*

a) $L(X, Y^*) = As_p(X, Y^*)$.

b) *For any Banach space Z, and $U \in As_p(X \tilde{\otimes}_\pi Y, Z)$ we have $U^\# \in As_p(X, As_p(Y, Z))$.*

Proof. a)→b) If a) is true, we can prove a result much more generalthan b), namely: If $U \in As_{r,p}(X \tilde{\otimes}_\pi Y, Z)$, then: $U^\# \in As_{r,p}(X, As_{r,p}(Y, Z))$. Indeed, by the ideal property of $(r, p)-$absolutely summing operators it follows that $U^\# x \in As_{r,p}(Y, Z)$, for each $x \in X$. Choose now $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$.

Then by the definition of the (r, p)−absolutely summing norm it follows that there exist: $\sigma_i \subset \mathbb{N}$, σ_i finite $(1 \leq i \leq n)$ and $(y_{ij})_{j \in \sigma_i} \subset Y$ such that

$$w_p(y_{ij} \mid j \in \sigma_i) \leq 1 \text{ and } \|U^{\#}(x_i)\|_{r,p} - \varepsilon < l_r((U^{\#}x_i)(y_{ij}) \mid j \in \sigma_i)$$

for each $i = 1, \ldots, n$.

Hence using the fact that $U \in As_{r,p}(X\tilde{\otimes}_\pi Y, Z)$ we obtain

$$(\sum_{i=1}^{n}[\|U^{\#}(x_i)\|_{r,p} - \varepsilon]^r)^{\frac{1}{r}} < l_r((U^{\#}x_i)(y_{ij}) \mid 1 \leq i \leq n; j \in \sigma_i) =$$

$$l_r(U(x_i \otimes y_{ij}) \mid 1 \leq i \leq n; j \in \sigma_i) \leq \|U\|_{r,p} w_p(x_i \otimes y_{ij} \mid 1 \leq i \leq n; j \in \sigma_i).$$

But, the hypothesis a) implies that there exists a constant $C > 0$ such that for $\psi \in (X\tilde{\otimes}_\pi Y)^* = L(X, Y^*) = As_p(X, Y^*)$ we have: $\|\psi\|_p \leq C\|\psi\|$.

Now

$$\sum_{i=1}^{n}\sum_{j\in\sigma_i} \mid \psi(x_i \otimes y_{ij}) \mid^p = \sum_{i=1}^{n}\sum_{j\in\sigma_i} \mid \psi(x_i)(y_{ij}) \mid^p \leq$$

$$\sum_{i=1}^{n} \|\psi(x_i)\|^p [w_p(y_{ij} \mid j \in \sigma_i)]^p \leq$$

$$\sum_{i=1}^{n} \| \psi(x_i)\|^p \leq \| \psi\|_p^p [w_p(x_i \mid 1 \leq i \leq n)]^p \leq$$

$$C^p \| \psi \|^p [w_p(x_i \mid 1 \leq i \leq n)]^p;$$

Hence

$$w_p(x_i \otimes y_{ij} \mid 1 \leq i \leq n; j \in \sigma_i) \leq C w_p(x_i \mid 1 \leq i \leq n)$$

i.e.

$$(\sum_{i=1}^{n}[\|U^{\#}(x_i)\|_{r,p} - \varepsilon]^r)^{\frac{1}{r}} < C\| U\|_{r,p} w_p(x_i \mid 1 \leq i \leq n),$$

$$(\sum_{i=1}^{n}[\|U^{\#}(x_i)\|_{r,p}^r)^{\frac{1}{r}} \leq C\| U\|_{r,p} w_p(x_i \mid 1 \leq i \leq n)$$

i.e. $U^{\#} \in As_{r,p}(Y, Z)$).

b) $\Rightarrow$ a) Let $T \in L(X, Y^*) = (X\tilde{\otimes}_\pi Y)^*$ and let $U : X\tilde{\otimes}_\pi Y \to \mathbb{K}$ be the canonical functional associated to T, where $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$). We have

$U \in As_p(X\tilde{\otimes}_\pi Y, \mathbb{K})$ and from b) taking $Z = \mathbb{K}$ we have that $U^\# : X \to As_p(Y, \mathbb{K}) = Y^*$ is $p-$absolutely summing. But $U^\# = T \in As_p(X, Y^*)$ and a) is fulfilled. □

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REFERENCES

[1] B. Carl - A. Defant - M.S. Ramanujan, *On tensor stable ideals,* Michigan Math. Journ., 36 (1989), pp. 63–75.

[2] A. Defant - K. Floret, *Tensor norms and operator ideals,* North-Holland Math. Studies, 176 (1993).

[3] J. Diestel - J.J. Uhl, *Vector measures,* Math. Surv. n. 15 A.M.S., 1977.

[4] J. Diestel - H. Jarchow - A. Tonge, *Absolutely Summing Operators,* Cambridge Stud. Adv. Math. 43, Cambridge University Press, Cambridge, 1995.

[5] Y. Gordon - D.R. Lewis, *Absolutely summing operators and local unconditional structures,* Acta Math., 133 (1974), pp. 27–48.

[6] M. Lindstrom - G.Schluchtermann, *Composition of operators ideals,* Math. Scand., 84 (1999), pp. 284–296.

[7] A. Pietsch, *operator ideals,* North-Holland, Berlin, 1978.

[8] D. Popa, *Grothendieck operators on the projective tensor products of spaces,* Le Matematiche, 50 (1995), pp. 83–85.

[9] D. Popa, *Pietsch integral operators on the injective tensor products of spaces,* Glasgow Math. Journ., 39 (1997), pp. 227–230.

[10] G. Racher, *On tensor product of weakly compact operators,* Math. Annalen, 294 (1992), pp. 267–275.

[11] P. Saab - S.M. Smith, *$p-$summing operators on injective tensor product of spaces,* Proc. of the Royal Society of Edinburgh, 120 A (1992), pp. 283–296.

[12] E. Saksmann - H.O. Tylli, *Weak compactness of multiplication operators on spaces of bounded linear operators,* Math. Scand., 70 (1992), pp. 91–111.

[13] E. Saksmann - H.O. Tylli, *Weak essential spectra of multiplication operators on spaces of bounded linear operators,* Math. Annalen, 299 (1994), pp. 299–304.

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